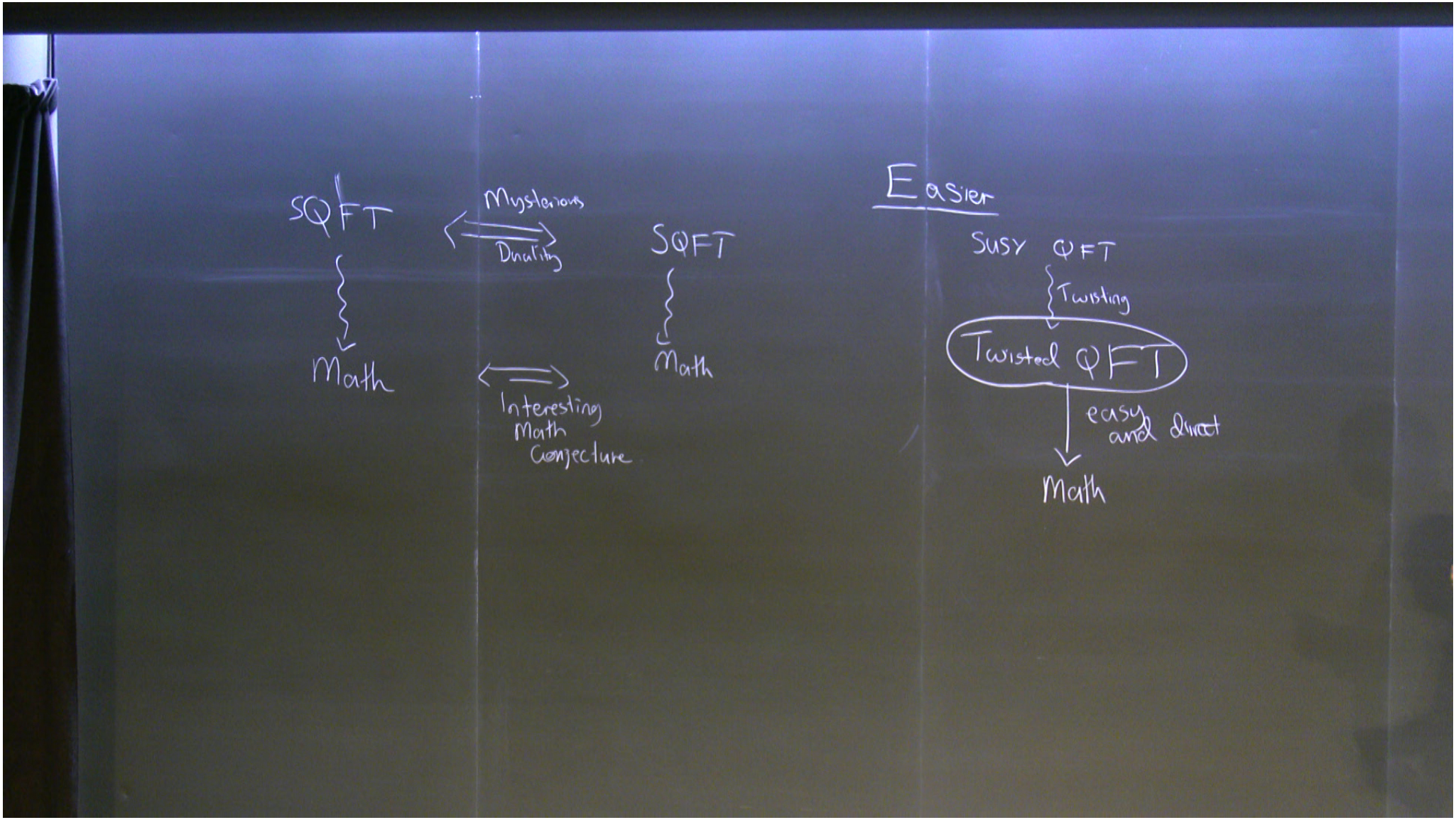
Title: A and B Models in 3d and 4d II

Speakers: Kevin Costello

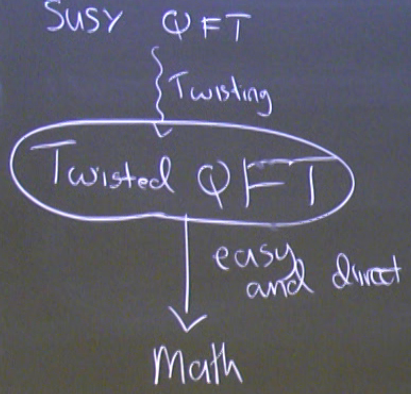
Collection: QFT for Mathematicians 2022

Date: June 20, 2022 - 9:00 AM

URL: https://pirsa.org/22060063


SQFT
Mysterious
Duality
SQFT
Math
Math
Interesting
Math
Conjecture
Easier
Susy QFT
Twisting
Twisted QFT
easy
and direct
Math

Easier



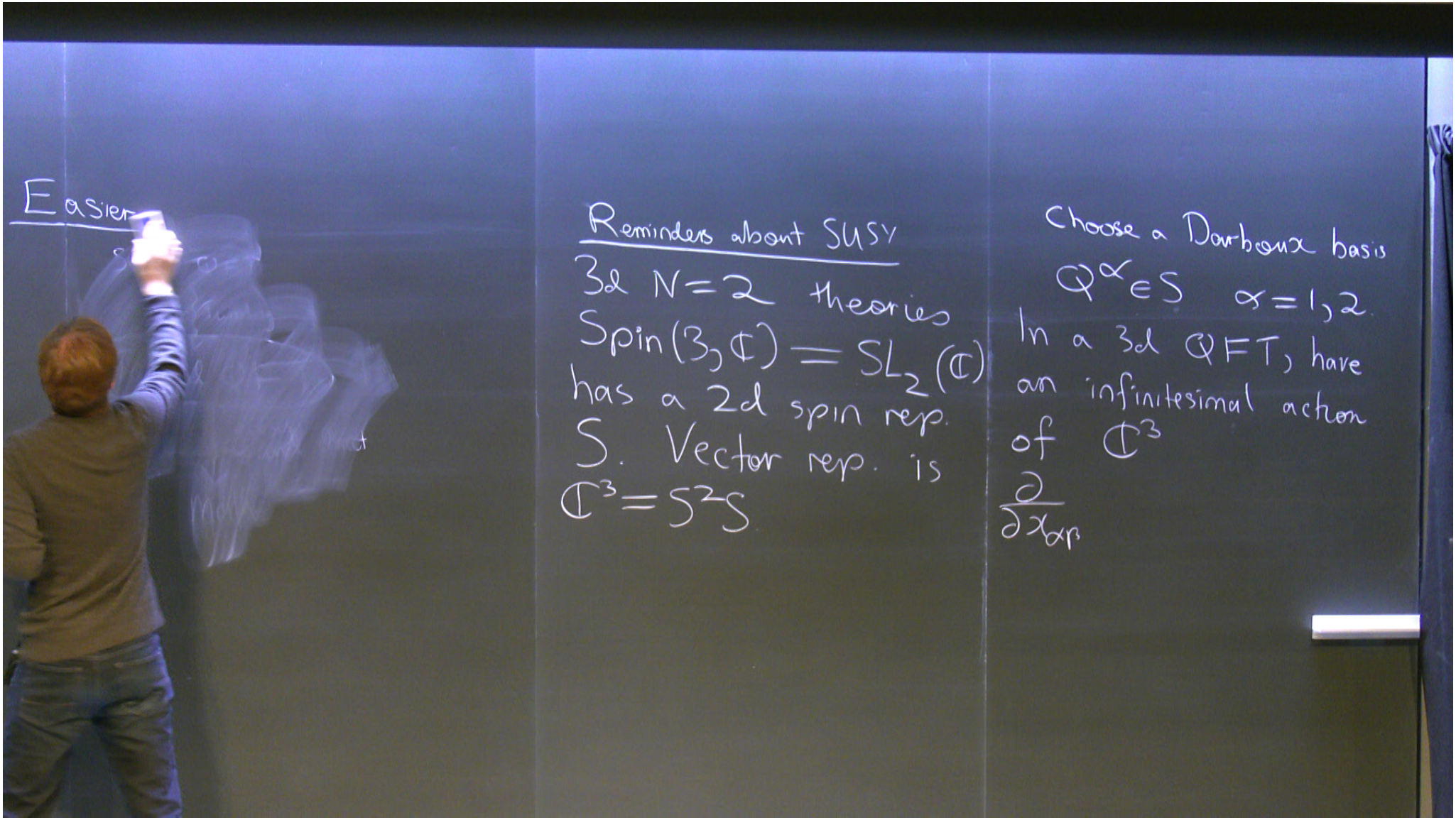
Reminders about SUSY

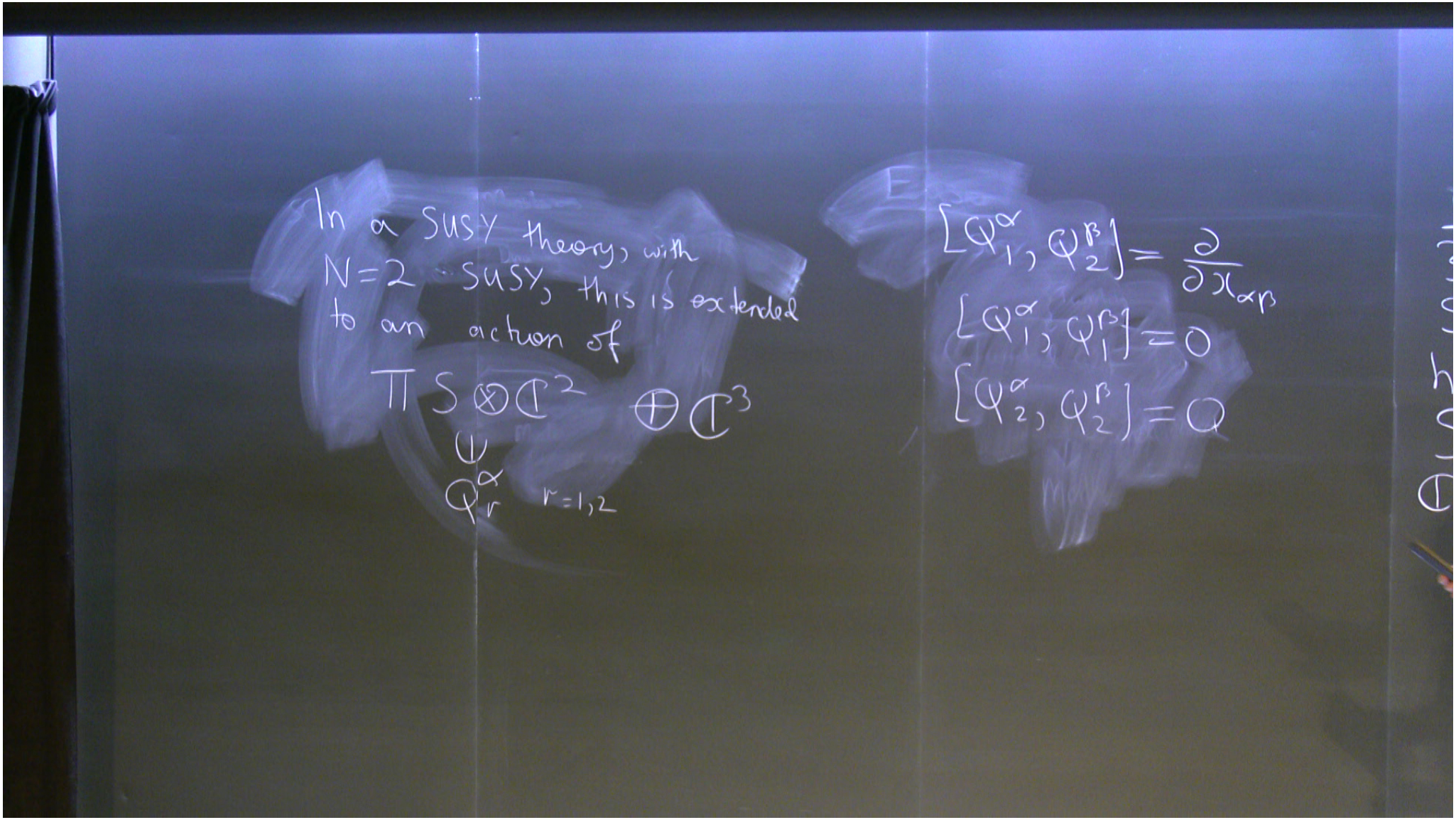
3d $N=2$ theories

$Spin(3,\mathbb{C}) = SL_2(\mathbb{C})$ has a 2d spin rep. S. Vector rep. is $\mathbb{C}^3 = S^2 S$

Choose a Darboux basis

$Q^\alpha \in S \quad \alpha = 1,2$


Easier
Reminders about SUSY
3d N=2 theories
Spin(3,ℂ) = SL2(ℂ)
has a 2d spin rep.
S. Vector rep. is
ℂ³ = S²S
Choose a Darboux basis
Q^α ∈ S α = 1,2.
In a 3d QFT, have
an infinitesimal action
of ℂ³
∂/∂x_αβ


In a Susy theory, with N=2 Susy, this is extended to an action of
$\Pi S \otimes \mathbb{C}^2 \oplus \mathbb{C}^3$
$Q^\alpha_r \quad r=1,2$
$[Q^\alpha_1, Q^\beta_2] = \frac{\partial}{\partial x_{\alpha\beta}}$
$[Q^\alpha_1, Q^\beta_1] = 0$
$[Q^\alpha_2, Q^\beta_2] = 0$

$$[Q_1^\alpha, Q_2^\beta] = \frac{\partial}{\partial x_{\alpha\beta}}$$

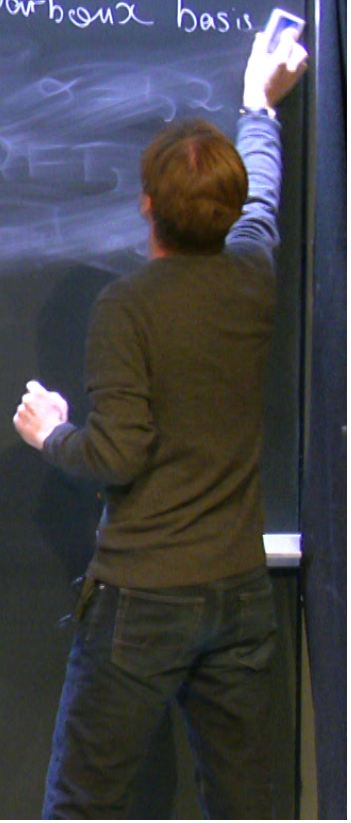
$$[Q_1^\alpha, Q_1^\beta] = 0$$

$$[Q_2^\alpha, Q_2^\beta] = 0$$

Take $Q = Q_1^1$ in the algebra. $Q^2 = \frac{1}{2}[Q, Q] = 0$

Twisting: Replaces everything in the QFT by its Q-cohomology

Choose a Darboux basis



$$[Q_1^\alpha, Q_2^\beta] = \frac{\partial}{\partial x_{\alpha\beta}}$$

$$[Q_1^\alpha, Q_1^\beta] = 0$$

$$[Q_2^\alpha, Q_2^\beta] = 0$$

Take $Q = Q_1^1 + Q_1^2$ in the algebra. $Q^2 = \frac{1}{2}[Q, Q] = 0$

Twisting: Replaces everything in the QFT by its Q-cohomology

$\mathcal{H}$ = Hilbert space of QFT,

$H^*(\mathcal{H}, Q)$ is Hilbert space of twisted theory

Similarly, for Obs, local operators.

Take $Q = \text{some } Q^{SUSY}_i$ in the algebra. $Q^2 = \frac{1}{2}[Q,Q] = 0$

Twisting: Replaces everything in the QFT by its Q-cohomology.

$\mathcal{H}$ = Hilbert space of basis QFT,

$H^*(\mathcal{H}, Q)$ is Hilbert space of twisted theory.

Similarly, for Obs, local operators.

How do local ops. in twisted theory behave?

As Q is a symmetry, correlation functions of Q-closed operators descend to Q-cohomology.

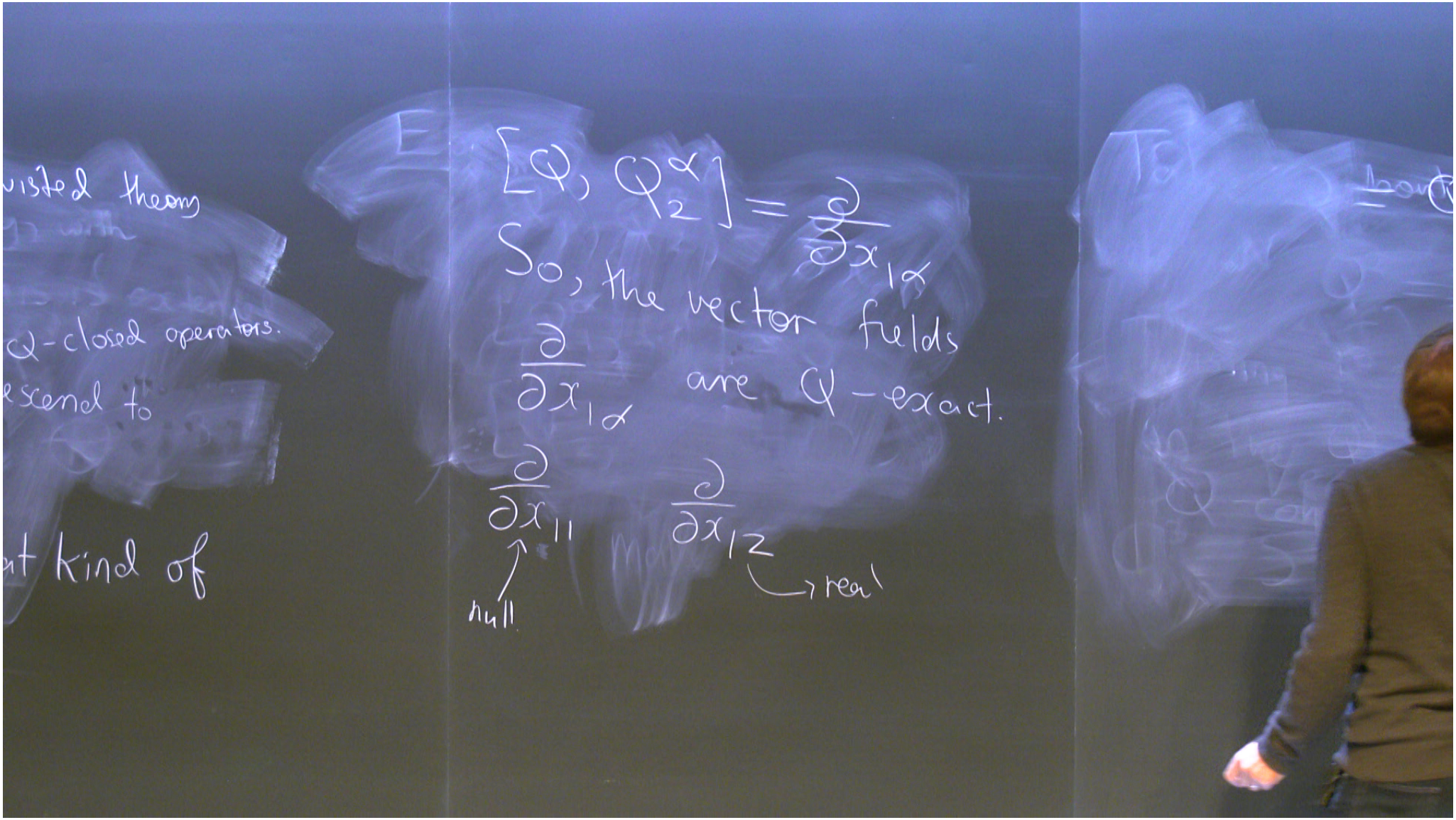
$\mathcal{O}_1 \dots \mathcal{O}_n \in H^i(\mathcal{O}_{PS}, Q)$

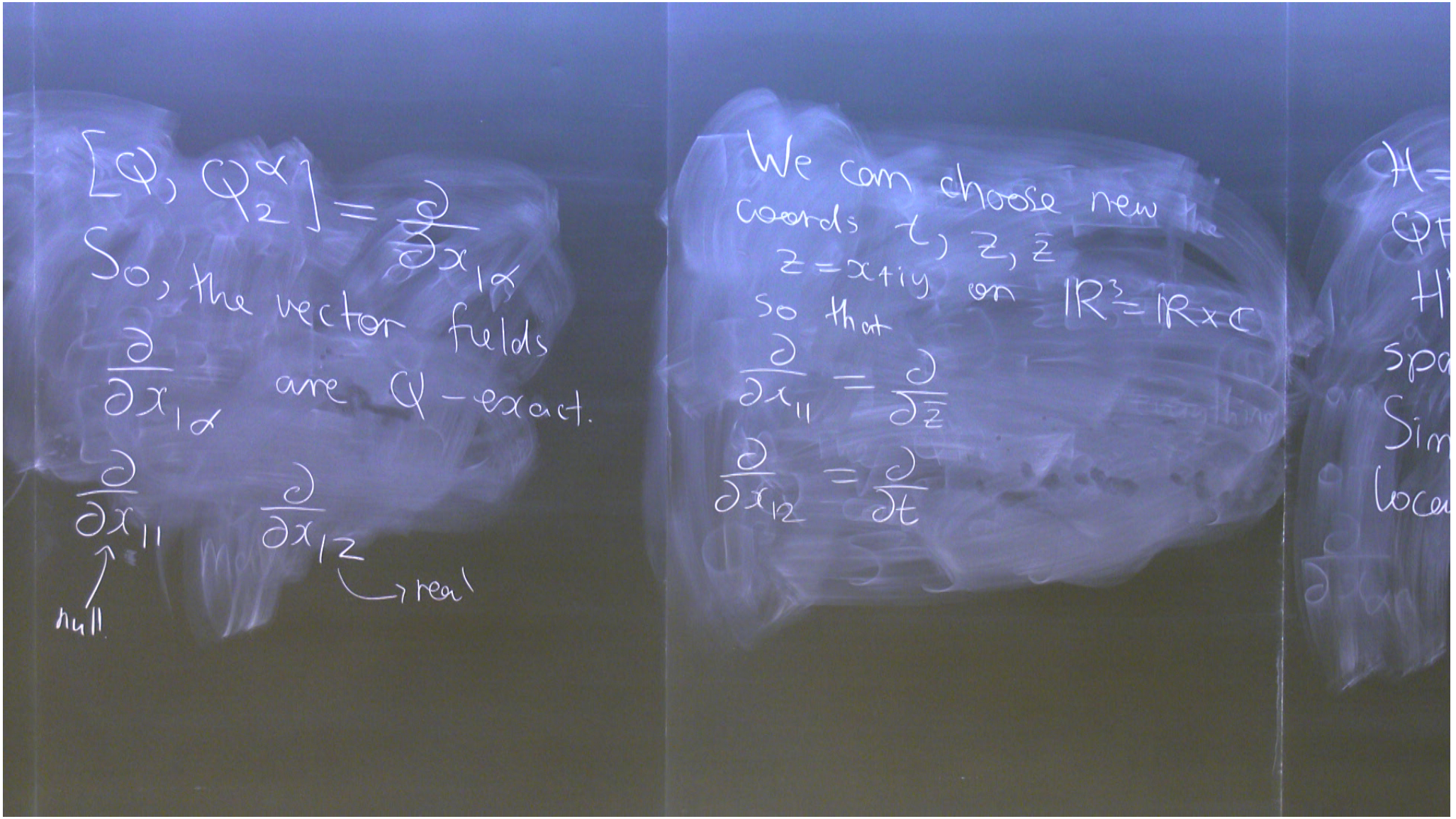
$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle$ what kind of functions are these?

$[Q_1^\alpha, Q_2^\beta] =$

$[Q_1^\alpha, Q_1^\beta] =$

$[Q_2^\alpha, Q_2^\beta] =$


visted theory
Q-closed operators.
escend to
t kind of
$[Q, Q_2^\alpha] = \frac{\partial}{\partial x_{1\alpha}}$
So, the vector fields
$\frac{\partial}{\partial x_{1\alpha}}$ are Q-exact.
$\frac{\partial}{\partial x_{11}}$
$\frac{\partial}{\partial x_{12}}$
null
real


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So, the vector fields
$\frac{\partial}{\partial x_{1\alpha}}$ are Q-exact.
$\frac{\partial}{\partial x_{11}}$
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null
real
We can choose new coords $t, z, \bar{z}$
$z = x + iy$ on $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$
so that
$\frac{\partial}{\partial x_{11}} = \frac{\partial}{\partial \bar{z}}$
$\frac{\partial}{\partial x_{12}} = \frac{\partial}{\partial t}$

We can choose new coords $t, z, \bar{z}$

$z = x_1 + iy$ on $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$

so that

$$\frac{\partial}{\partial x_{11}} = \frac{\partial}{\partial \bar{z}}$$

$$\frac{\partial}{\partial x_{12}} = \frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial t_i} \langle \mathcal{O}_1(t_1, z_1) \cdots \mathcal{O}_n(t_n, z_n) \rangle = 0$$

$$\frac{\partial}{\partial z_i} (\quad) = 0$$

as,

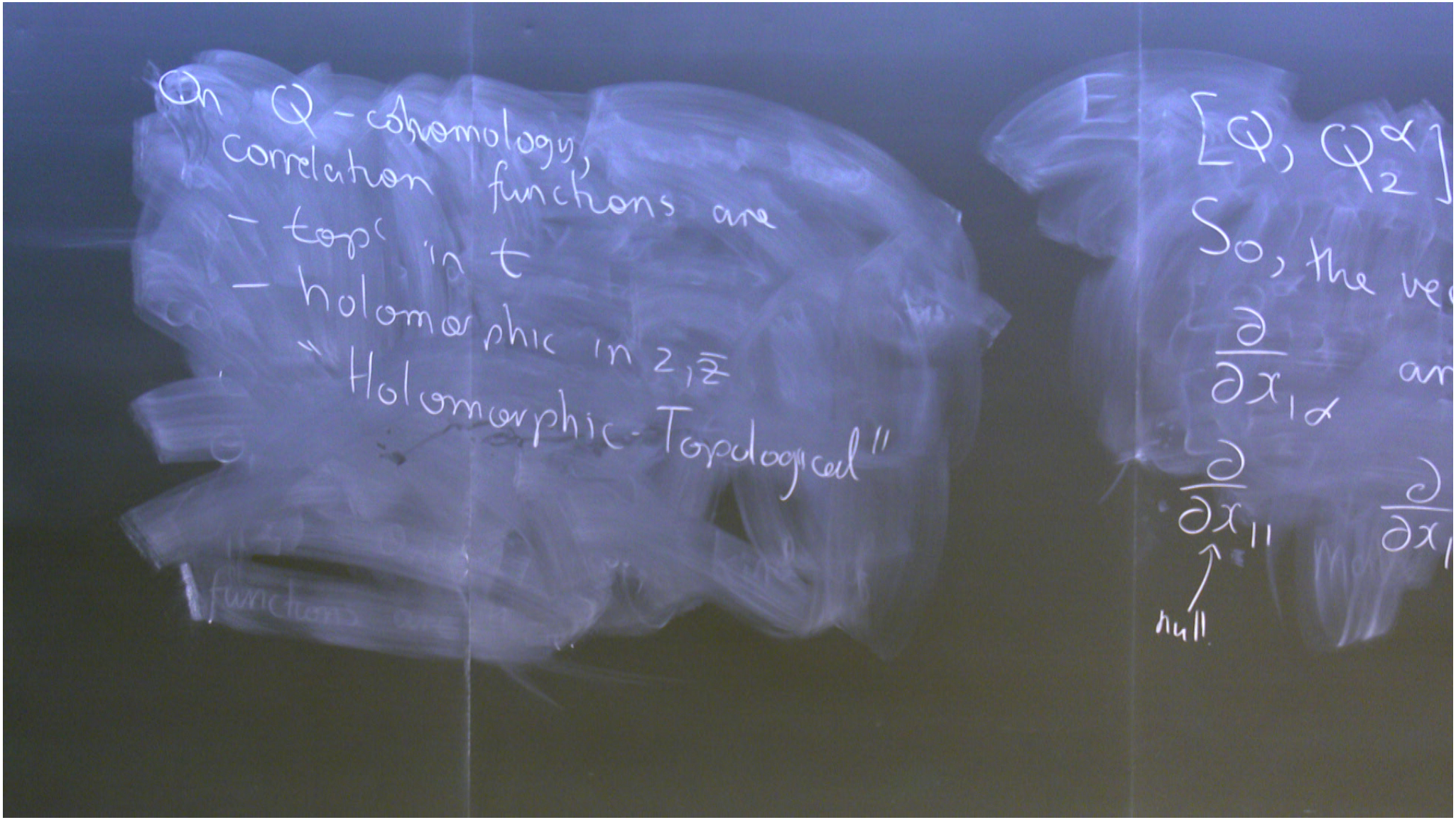
$$\left\langle \frac{\partial}{\partial t_i} \mathcal{O}_1 \cdots \right\rangle$$

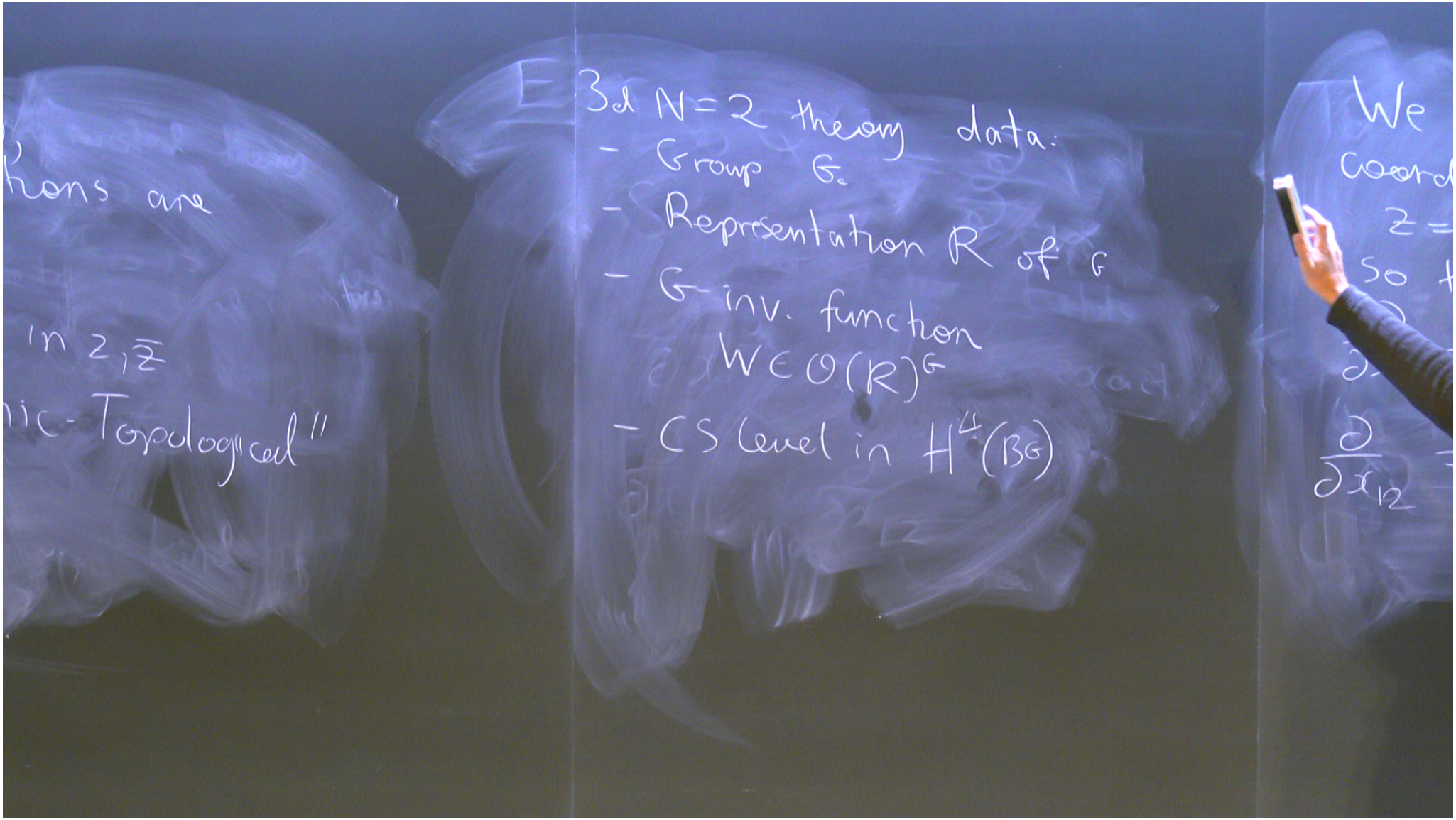
$$= \langle [Q, Q_2^2] \mathcal{O}_1 \cdots \rangle$$

$$= \langle Q Q_2^2 \mathcal{O}_1 \cdots \rangle$$

$$= \langle Q_2^2 \mathcal{O}_1, Q \mathcal{O}_2, \cdots \rangle + \cdots$$

$$= 0$$


On Q-cohomology,
correlation functions are
- topo in t
- holomorphic in z, z̄
" Holomorphic-Topological "
[Q, Q_2^α]
So, the ve
∂/∂x_{1α}
∂/∂x_{11}
∂/∂x_1
null


hons are
in z, z̄
ic-Topological"
3d N=2 theory data:
- Group G.
- Representation R of G
- G-inv. function
W ∈ O(R)^G
- CS level in H^4(BG)
We
Coord
z =
so t
∂/∂x12

3d $N=2$ theory data:

- Group G.
- Representation R of G
- G-inv. function $W \in \mathcal{O}(R)^G$
- CS level in $H^4(BG)$

We will write a Lagrangian for twisted-theory based on this.

To start, $R=0$ (pure gauge)

Fields of the twisted theory are

$$A \in \left(\Omega^1(\mathbb{R}\times\mathbb{C})/dz\right)\otimes \mathfrak{g}$$

$$A = A_t dt + A_{\bar z} d\bar z$$

We will write a Lagrangian for twisted-theory based on this.

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$B \in dz\,\Omega^0(\mathbb{R}\times\mathbb{C}) \otimes \mathfrak{g}$

Lagrangian is

$\int BF(A) + k_{cs} \int A\partial A$

not consider this so much, $k=0$

theory data:
G
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function
$\mathcal{O}(R)^G$
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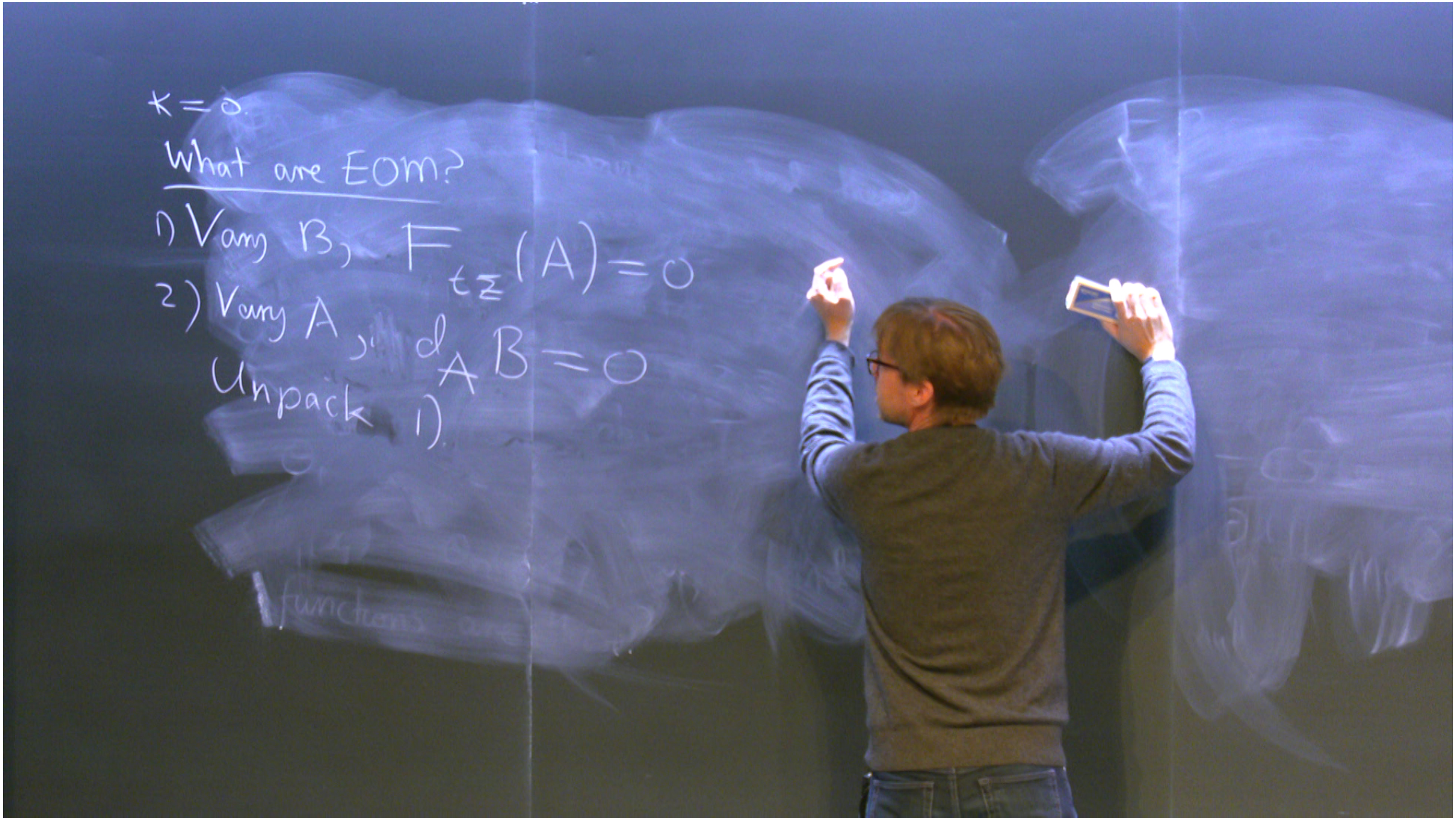
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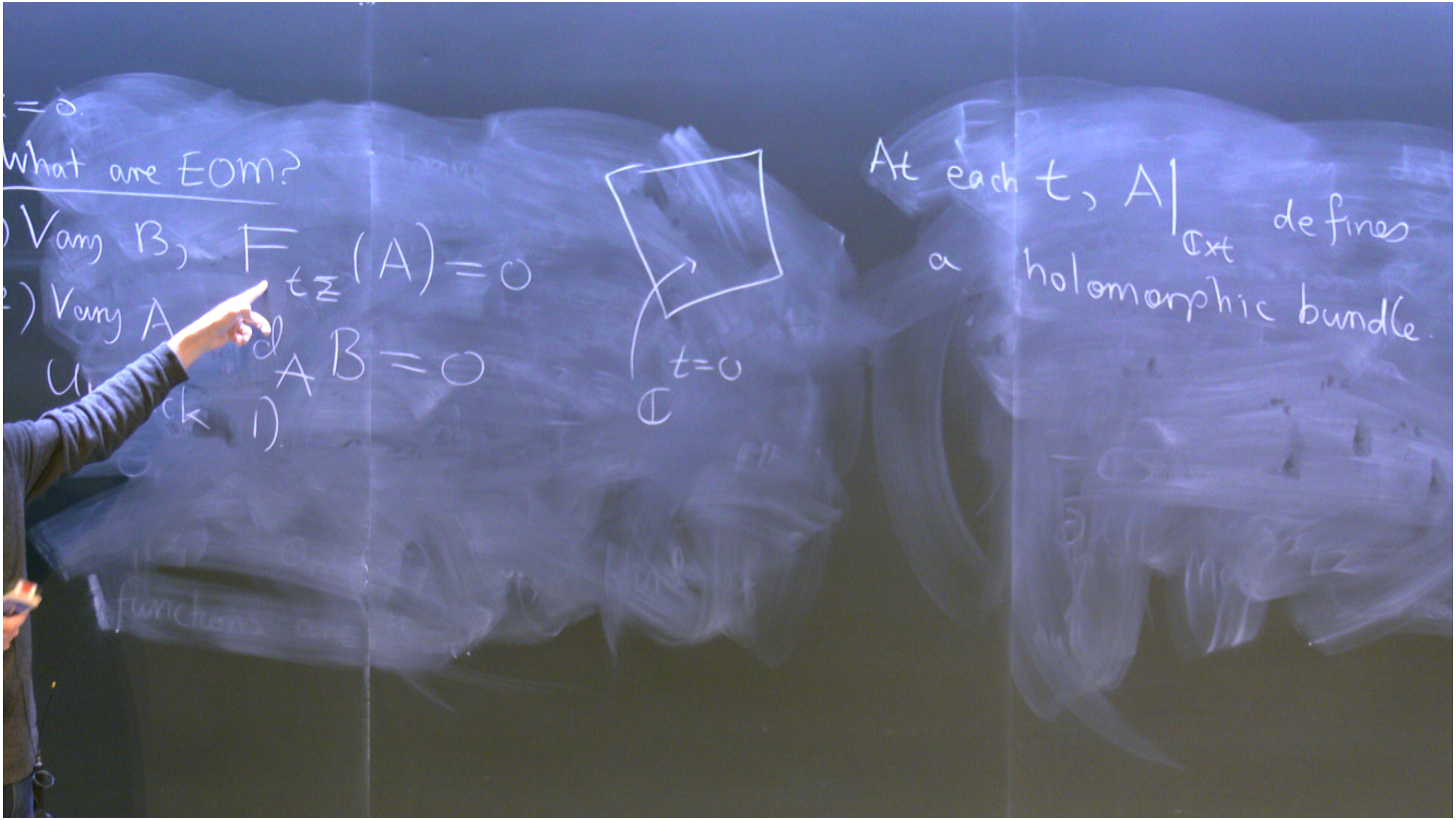
$B \in dz\,\Omega^0(\mathbb{R}\times\mathbb{C}) \otimes \mathfrak{g}^*$

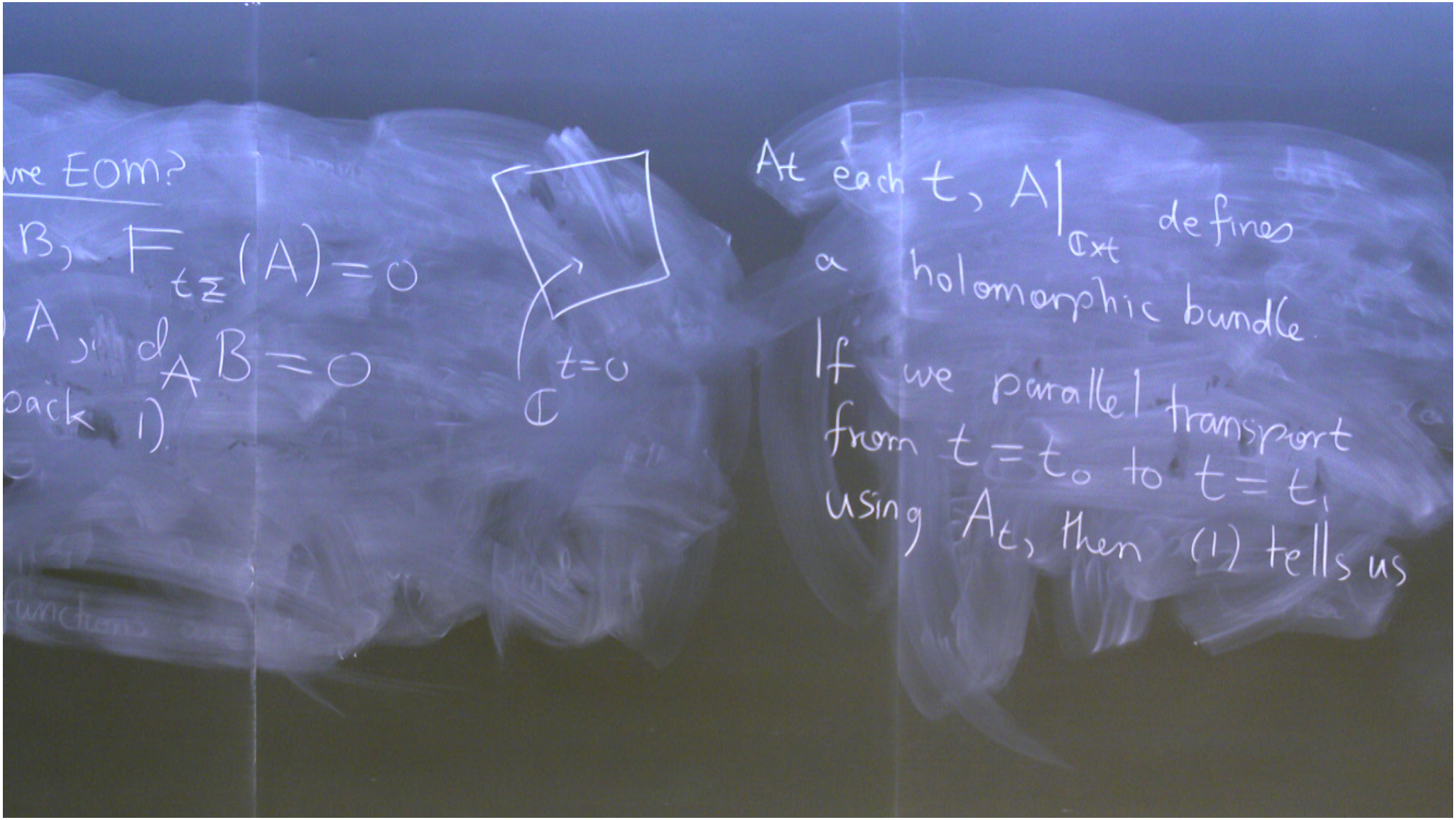
Lagrangian is

$$\int BF(A) + \underbrace{k_{cs}\int A\partial A}$$

not consider this so much, $k=0$


k = 0.
What are EOM?
1) Vary B, $F_{tz}(A) = 0$
2) Vary A, $d_A B = 0$
Unpack 1)


=0.
What are EOM?
) Vary B, $F_{tz}(A)=0$
) Vary A, $d_A B=0$
$t=0$
$\mathbb{C}$
At each t, $A|_{\mathbb{C}xt}$ defines
a holomorphic bundle.


re EOM?
B, $F_{tz}(A) = 0$
A, $d_A B = 0$
back 1)
$t=0$
$\mathbb{C}$
At each t, $A|_{\mathbb{C}_t}$ defines
a holomorphic bundle.
If we parallel transport
from $t=t_0$ to $t=t_1$
using A_t, then (1) tells us

defines

phic bundle

llel transport

o to $t = t_1$

hen (1) tells us

this is an iso. of holomorphic bundles

$G = GL_n$, $\psi \in \mathbb{C}^n$,

$(\partial_{\bar{z}} + A_{\bar{z}})\,\psi(z, t_0) = 0$

Define $\psi(z, t)$ by parallel transport.

$(\partial_{\bar{z}} + A_{\bar{z}})\,\psi(z, t) = 0$

as, $(\partial_t + A_t)(\partial_z + A_{\bar{z}})\,\psi(z, t)$

$B \in d_z \Omega^0(\mathbb{R}$

Lagrangian is

$\int BF(A)$

this is an iso. of holomorphic bundles.

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Lagrangian is

$$\int BF(A) + \underbrace{k_{cs}\int A\partial A}$$

not consider this so much, $K = 0$

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What happens with matter? $(W = 0)$

Fields of the twisted theory are

$\varphi \in \Omega^0(\mathbb{R}\times\mathbb{C}, R)$

$\eta \in (\Omega^1(\mathbb{R}\times\mathbb{C})/dz) \otimes R^*$

this is an iso. of holomorphic bundles

$G = GL_n$, $\psi \in \mathbb{C}^n$,

$(\partial_{\bar z} + A_{\bar z})\,\psi(z, t_0) = 0$

Define $\psi(z,t)$ by parallel transport.

$(\partial_{\bar z} + A_{\bar z})\,\psi(z,t) = 0$

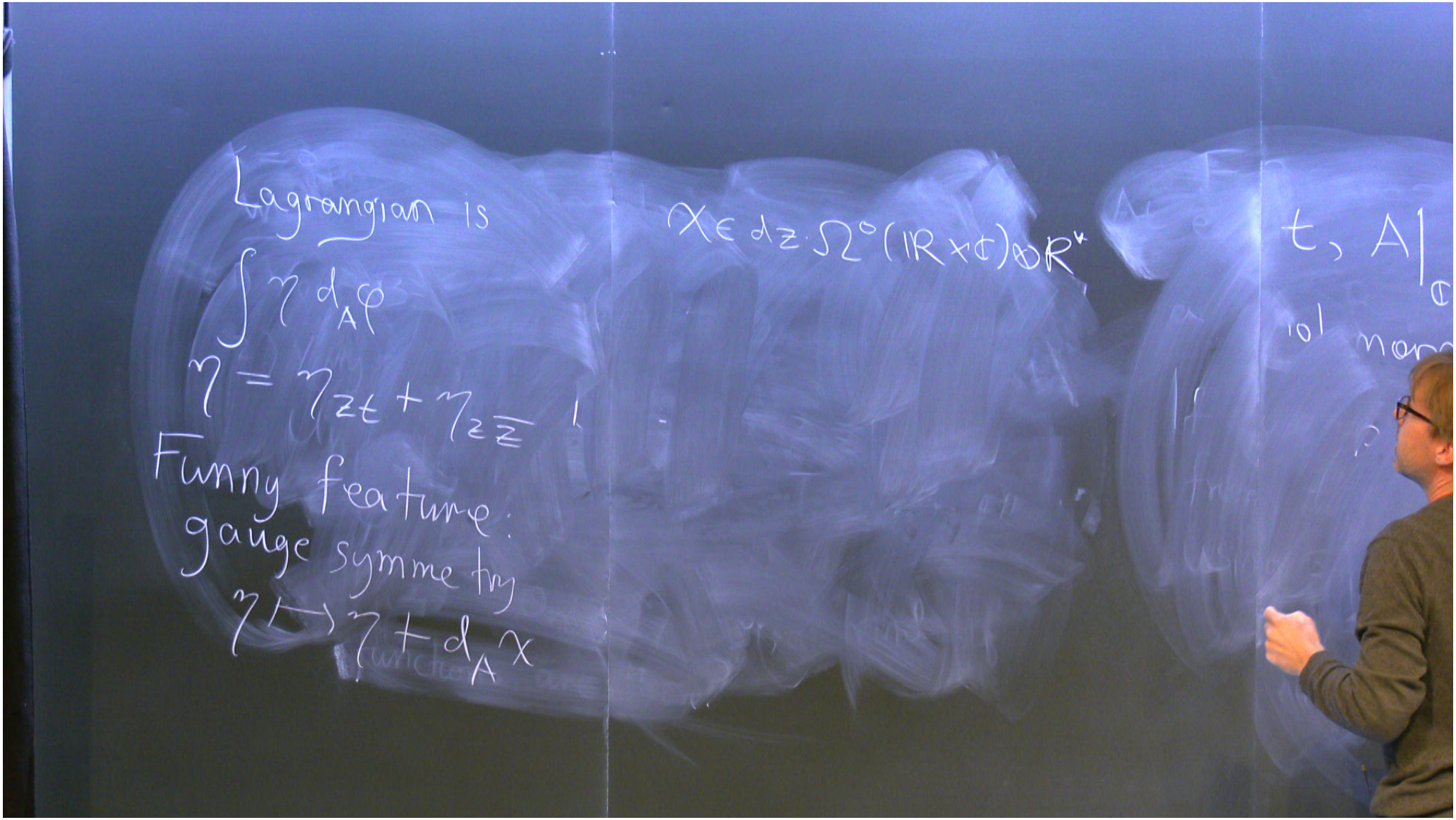
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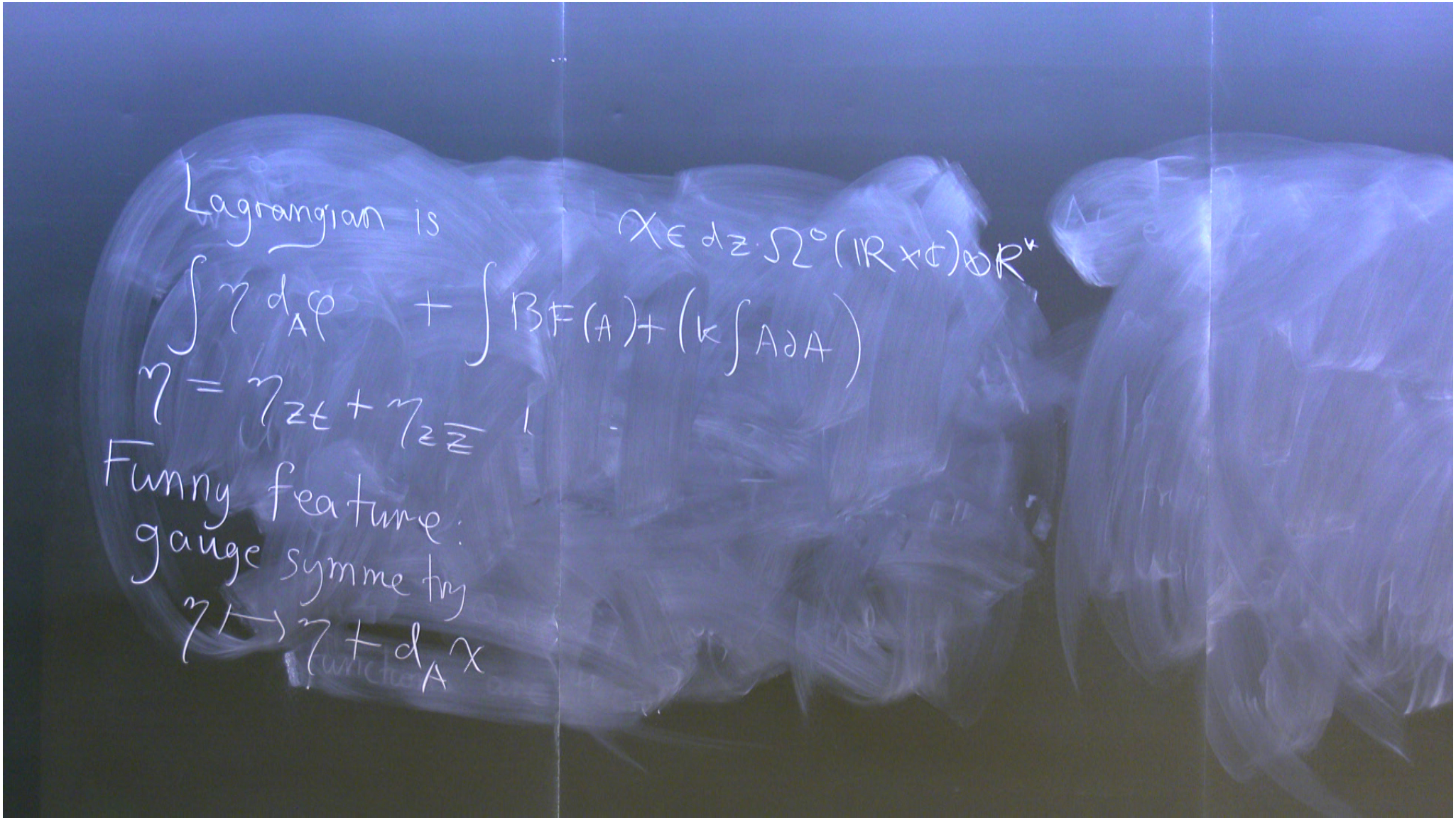
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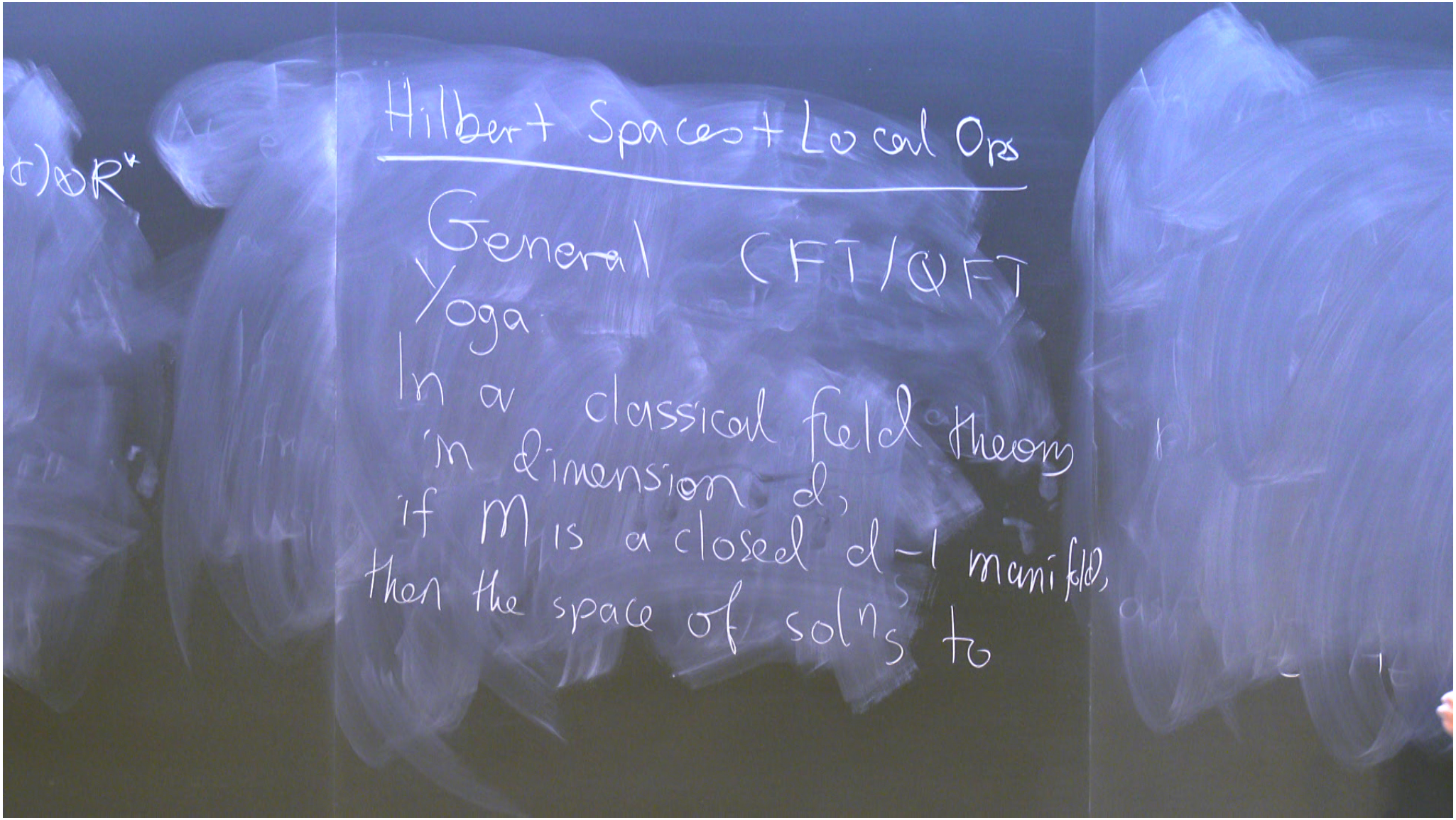
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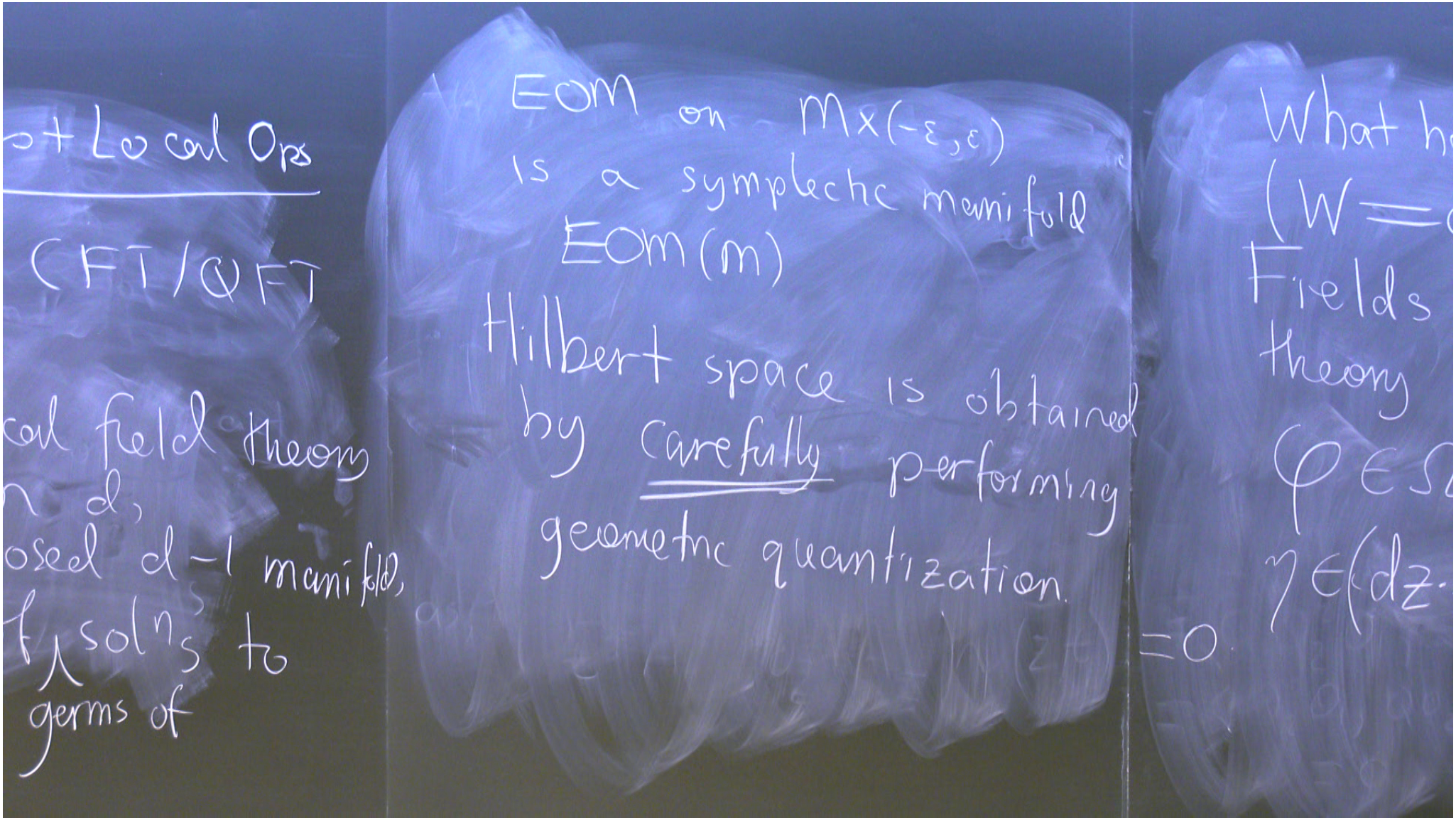
$\varphi \in \Omega^0(\mathbb{R}\times\mathbb{C}, R)$

$\eta \in (dz \cdot \Omega^1(\mathbb{R}\times\mathbb{C}, \ldots)) \otimes R^*$


Lagrangian is
$\int \eta \, d_A \varphi$
$\eta = \eta_{zt} + \eta_{z\bar{z}}$
Funny feature:
gauge symmetry
$\eta \mapsto \eta + d_A \chi$
$\chi \in dz \cdot \Omega^0(\mathbb{R} \times \mathbb{C}) \oplus \mathbb{R}^k$
$t, A|_{\mathbb{C}}$


Lagrangian is
$\chi \in dz \cdot \Omega^0(\mathbb{R}+\mathbb{C}) \oplus R^*$
$\int \eta\, d_A \varphi + \int BF(A) + \left(k \int A \partial A\right)$
$\eta = \eta_{z\bar{z}} + \eta_{z\bar{z}}$
Funny feature:
gauge symmetry
$\eta \mapsto \eta + d_A \chi$


Hilbert Spaces + Local Ops
General QFT/QFT
Yoga
In a classical field theory
in dimension d,
if M is a closed d-1 manifold,
then the space of soln's to


+ Local Ops
CFT/QFT
al field theory
n d,
osed d-1 manifold,
f λ soln's to
germs of
EOM on M x (-ε,ε)
is a symplectic manifold
EOM(M)
Hilbert space is obtained
by carefully performing
geometric quantization.
=0
What h
(W=
Fields
theory
φ ∈ S
γ ∈ (dz

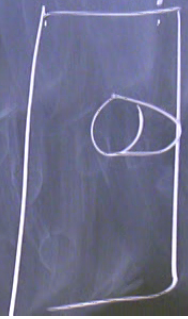
EOM on $M \times (-\varepsilon, \varepsilon)$
is a symplectic manifold
$EOM(M)$

Hilbert space is obtained
by carefully performing
geometric quantization.

$= 0$

Local operators
= Hilbert space on S^2

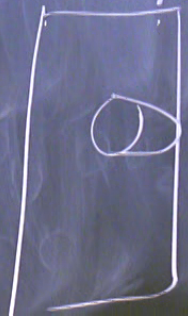
Easier Local operators
on the boundary with
boundary condition
$A = 0$
(no matter for now)
Hilbert space on the disc
ending on
boundary.

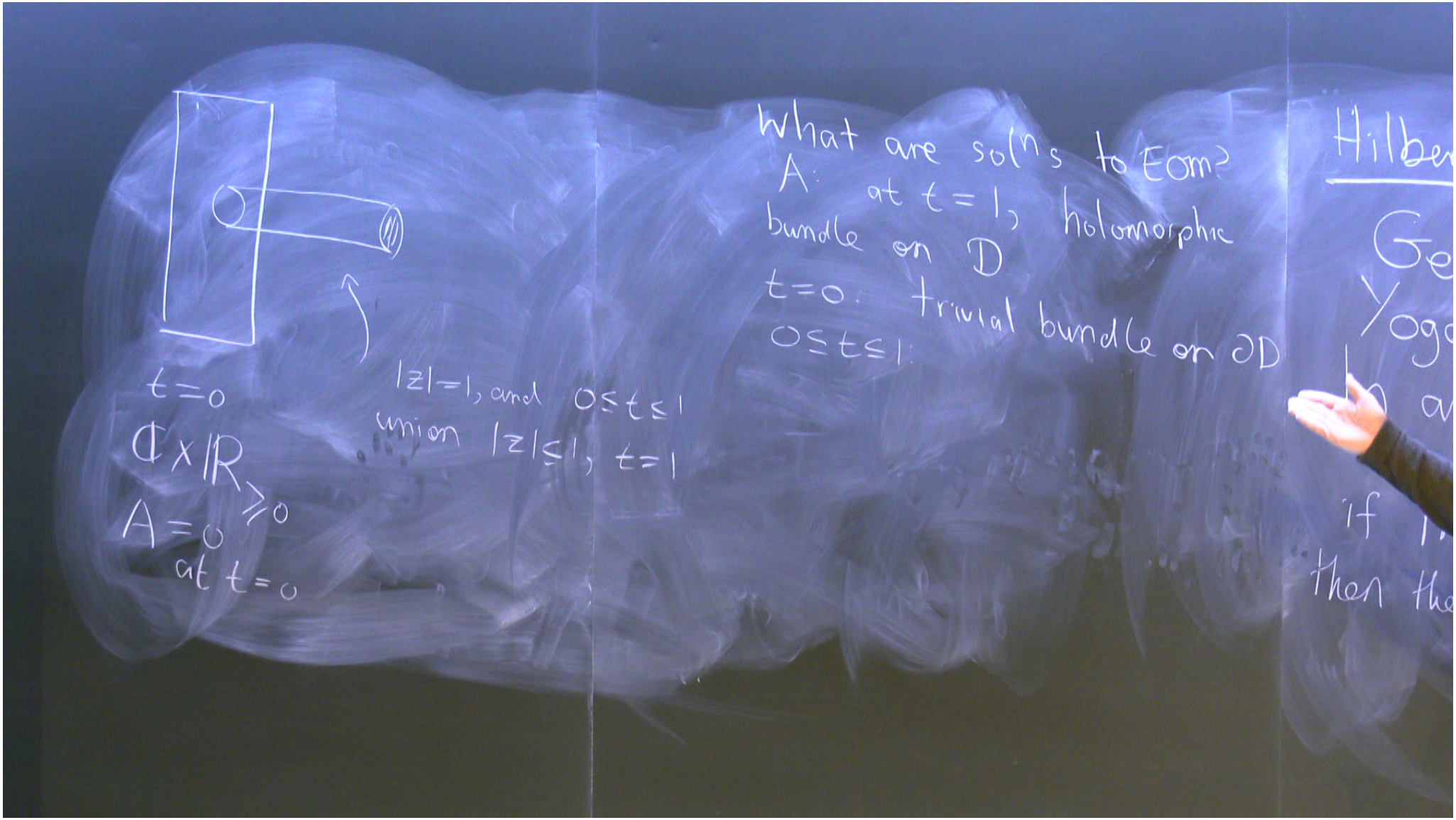


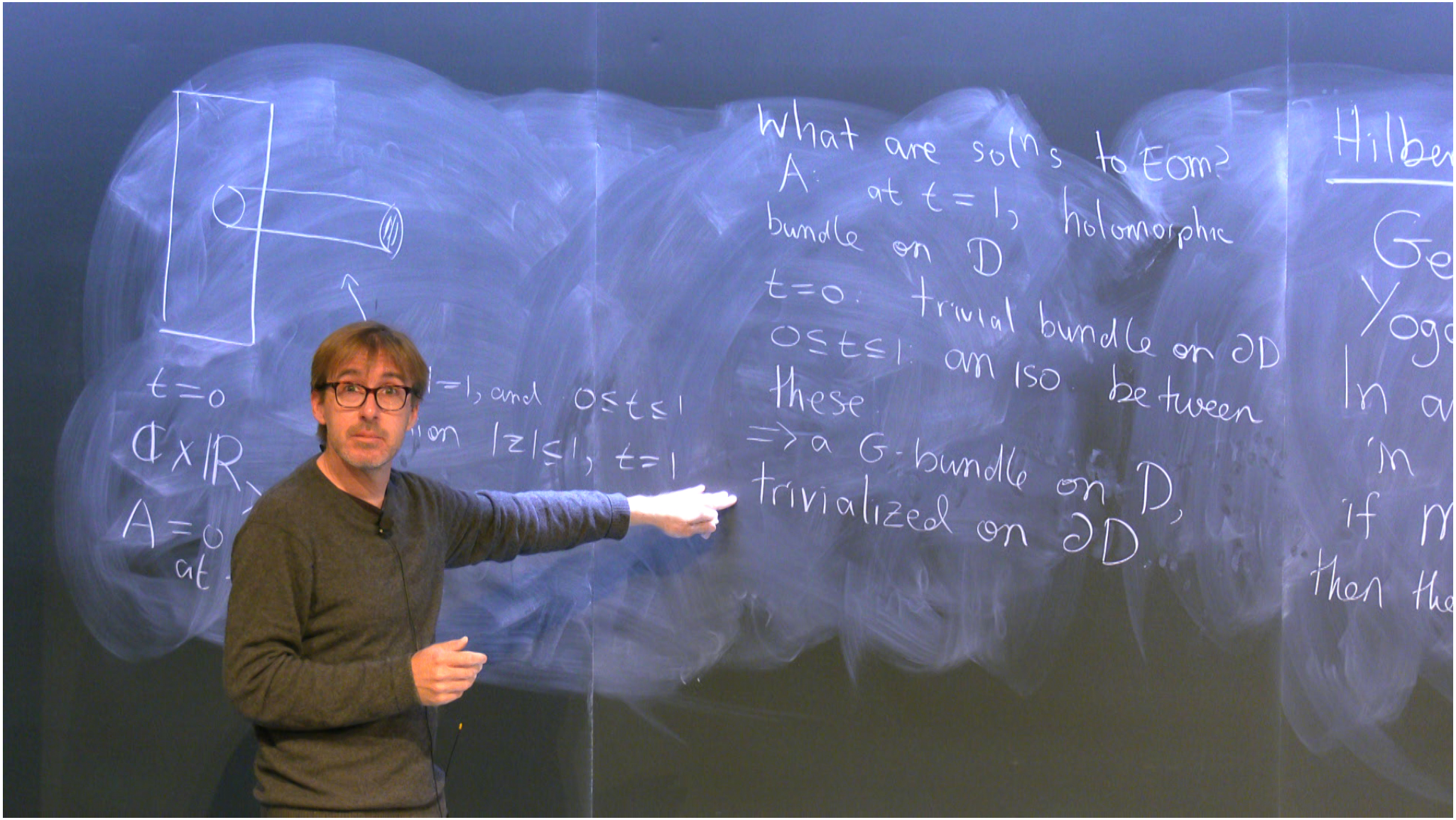
EOM on $M \times (-\varepsilon, \varepsilon)$
is a symplectic manifold
$EOM(M)$ (the phase space)
Hilbert space is obtained
by carefully performing
geometric quantization.

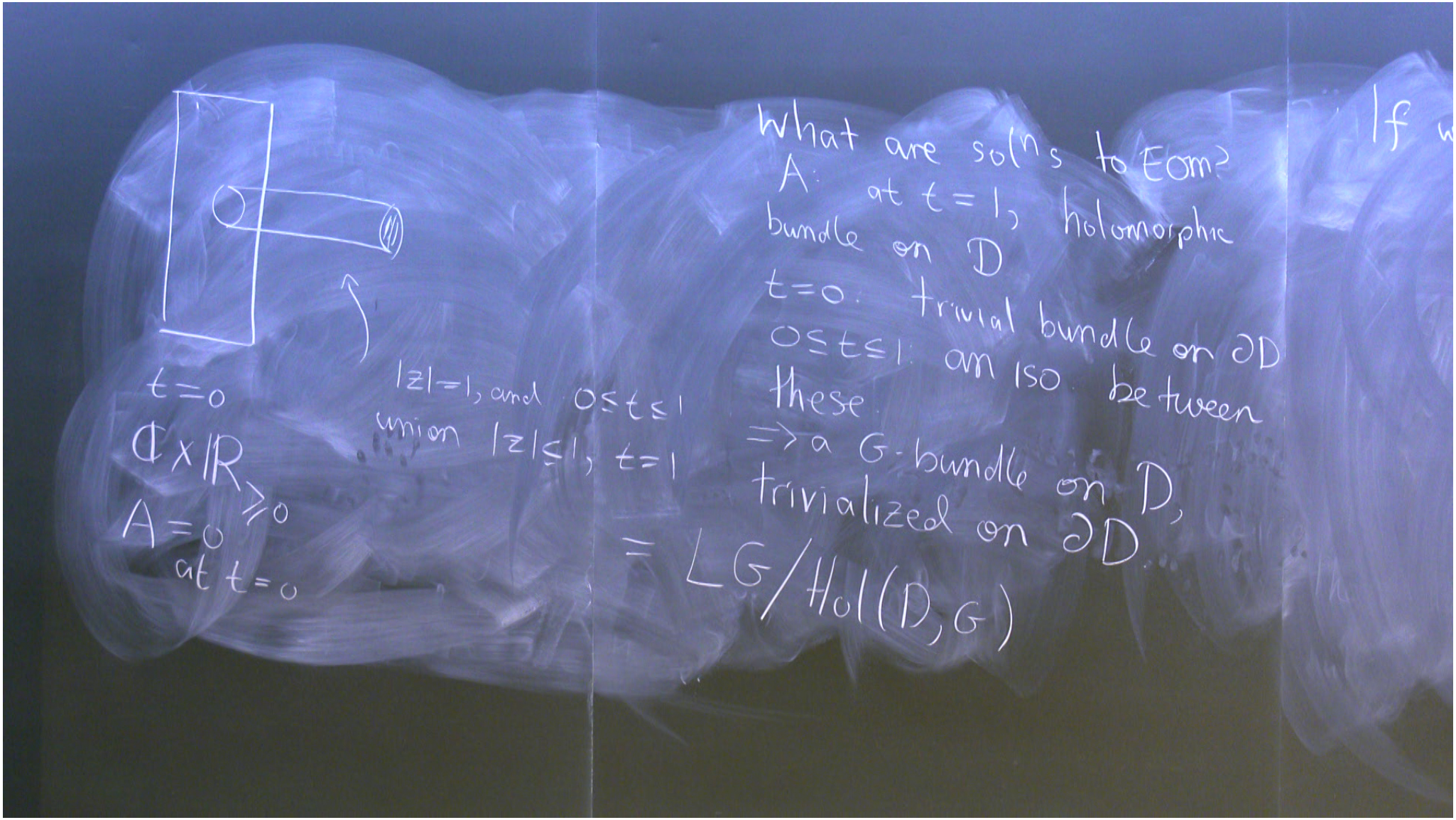
$= 0$

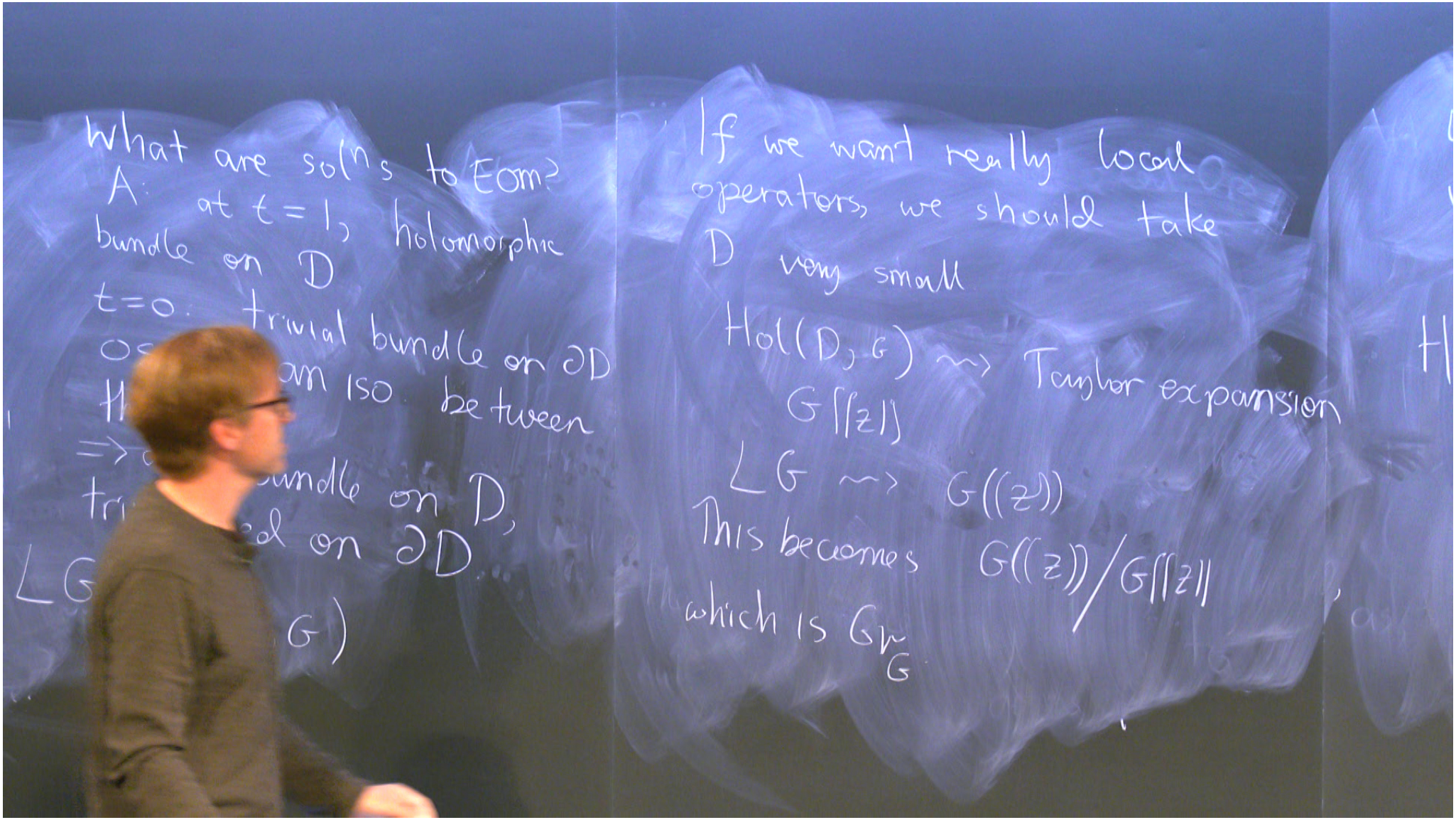
Local operators
= Hilbert space on S^2
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Hilbert space on the disc
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What are soln's to Eom?
A: at t=1, holomorphic bundle on D
t=0: trivial bundle on ∂D
0≤t≤1
Hilbe
Ge
Yog
if
then the
t=0
|z|=1, and 0≤t≤1
union |z|≤1, t=1
ℂ×ℝ
A=0 ≥0
at t=0


What are soln's to EOM?
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⇒ a G-bundle on D, trivialized on ∂D
t=0
A=0
Hilbe


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A: at t=1, holomorphic bundle on D
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0≤t≤1: an iso between these
⇒ a G-bundle on D, trivialized on ∂D
= LG/Hol(D, G)
t=0
|z|=1, and 0≤t≤1
union |z|≤1, t=1
ℂ × ℝ≥0
A=0 at t=0


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an iso between
bundle on D,
on ∂D
G)
If we want really local
operators, we should take
D very small
Hol(D, G) ⇝ Taylor expansion
G[[z]]
LG ⇝ G((z))
This becomes G((z))/G[[z]]
which is G_∨G

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Hol(D, G) ⇝ Taylor expansion
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B no constraints at
t=0.

$B \in dz\,\mathrm{Hol}(D, g^*)$
$= dz\, g^*[[z]]$

(better: section on D
of $K \oplus$ associated bundle
for g^*)

$B \in T^*_A Gr_G$

B no constraints at $t=0$.

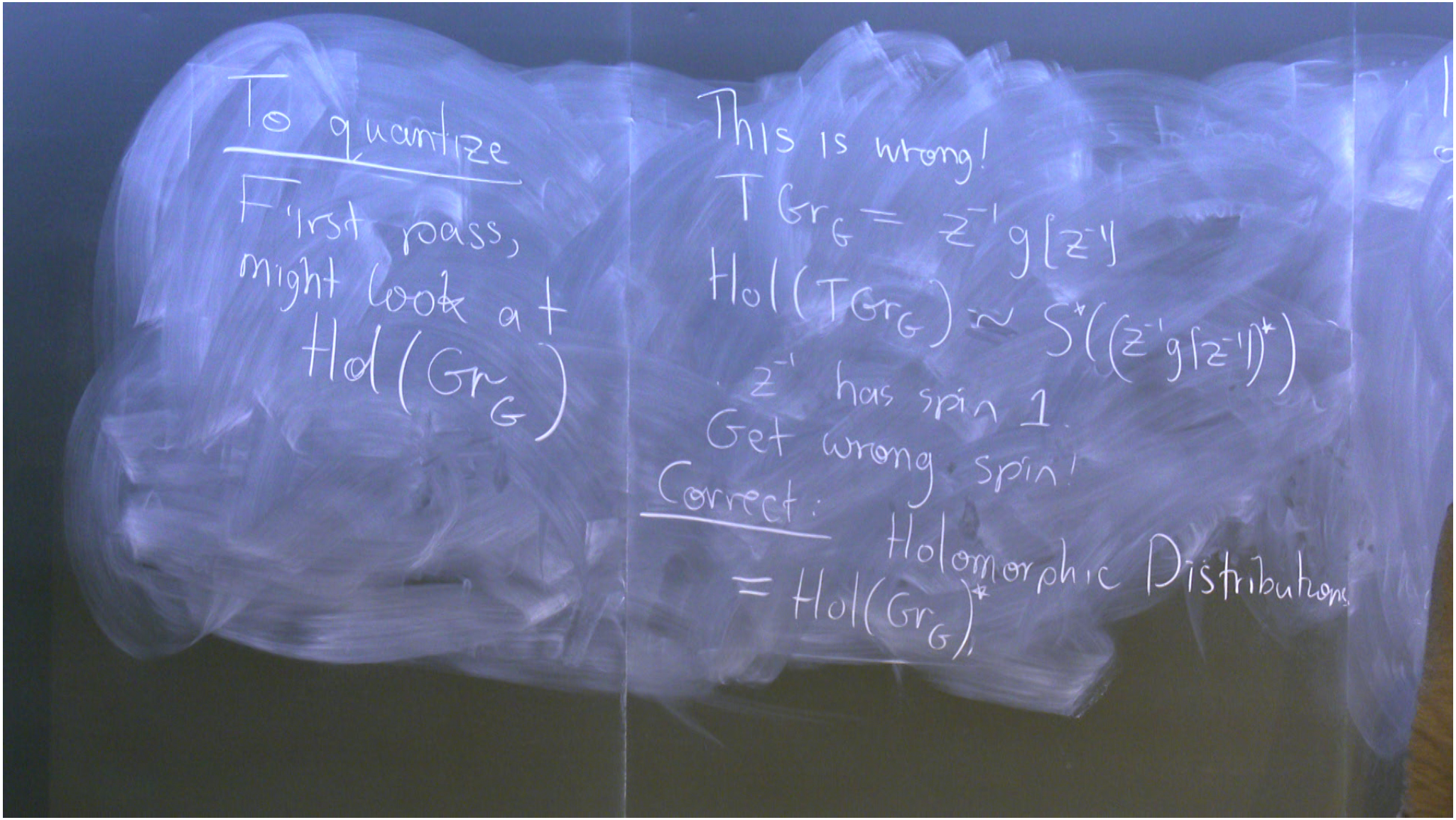
$B \in dz\,\mathrm{Hol}(D, g^*) = dz\, g^*[[z]]$

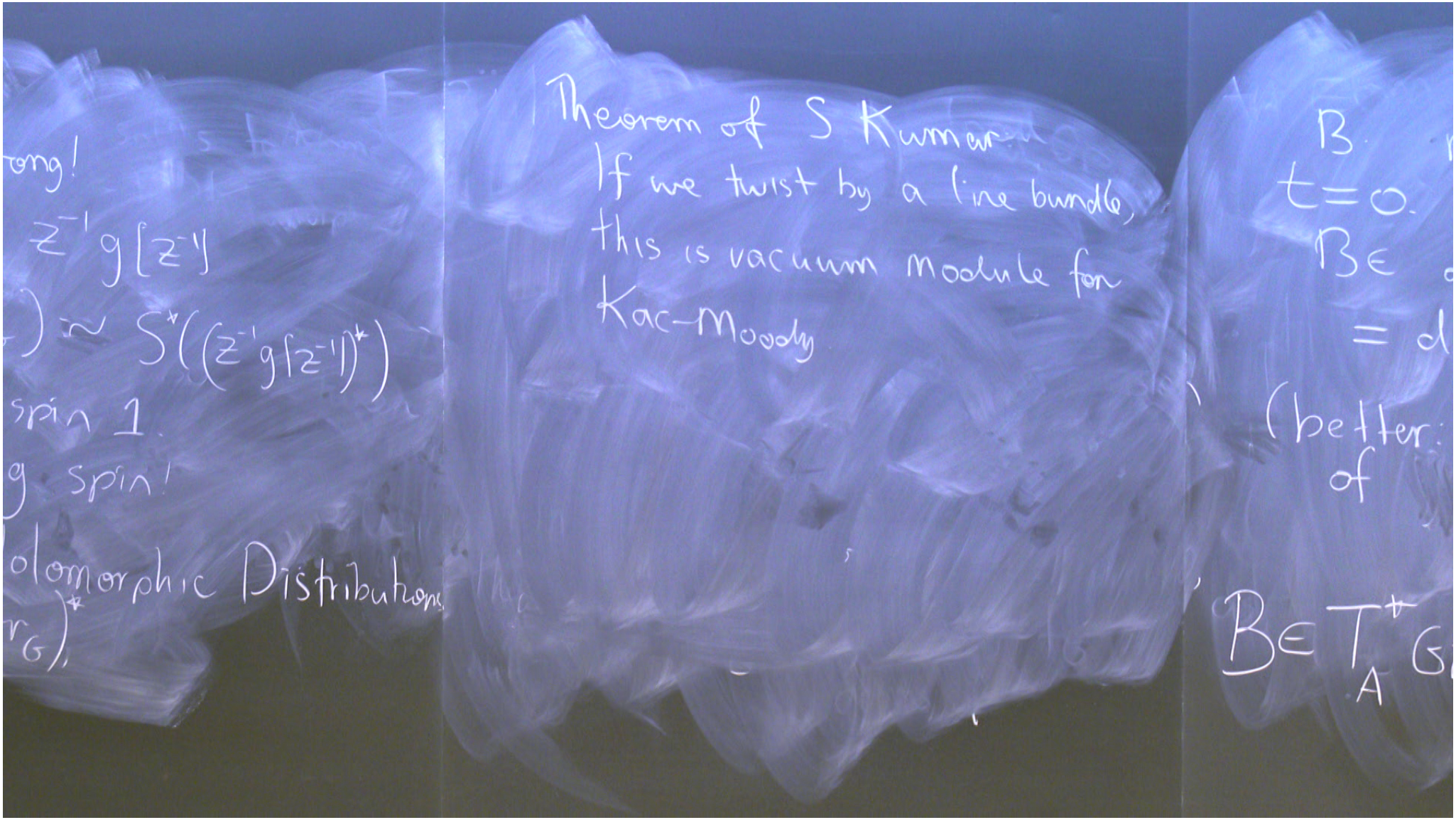
(better: section on D of $K \otimes$ associated bundle for g^*)

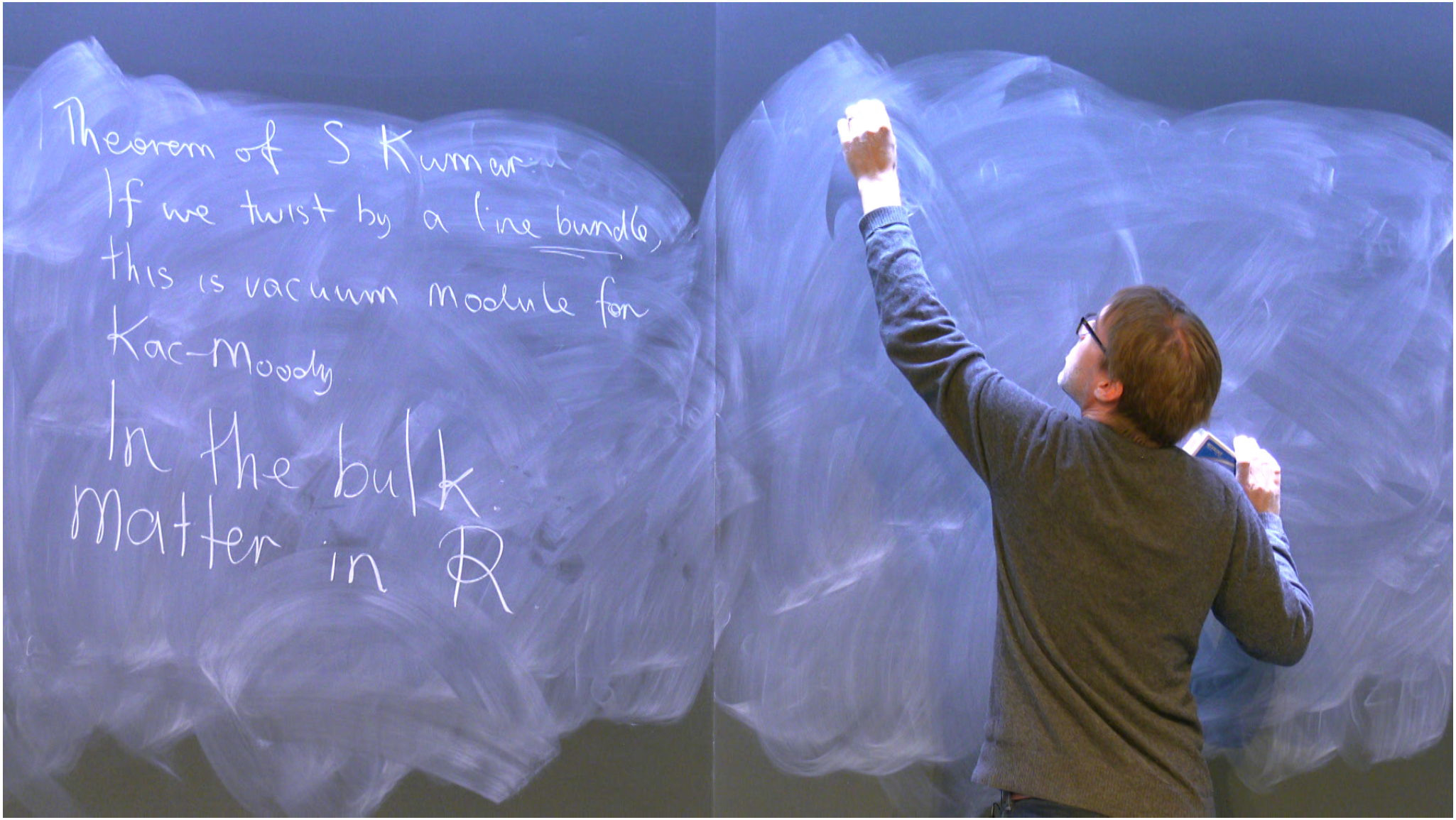
$B \in T^*_A Gr_G$

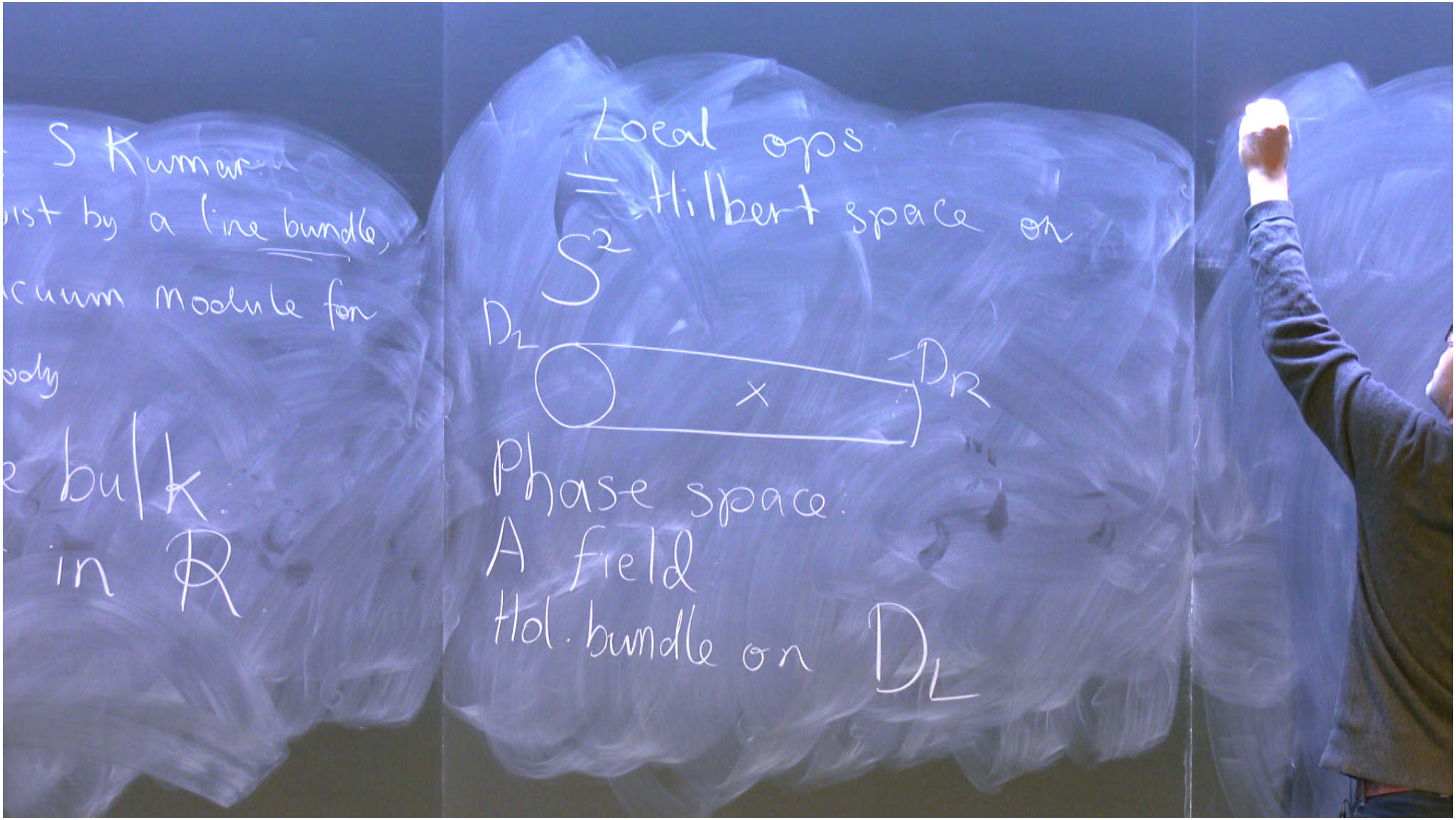
$T_0 Gr_G = z^{-1} g[z^{-1}]$ dual to $dz\, g^*[[z]]$ using residues

Phase Space $= T^* Gr_G$

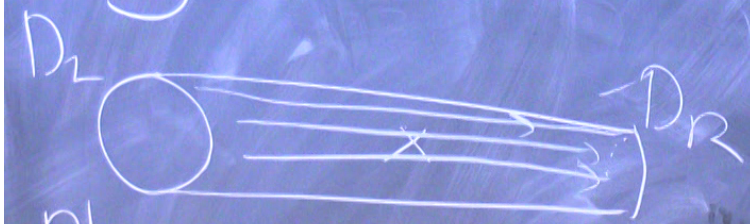

To quantize
First pass, might look at
Hol(Gr_G)
This is wrong!
T Gr_G = z^{-1} g[z^{-1}]
Hol(T Gr_G) ≃ S*((z^{-1} g[z^{-1}])*)
z^{-1} has spin 1.
Get wrong spin!
Correct:
Holomorphic Distributions
= Hol(Gr_G)*


ong!
$z^{-1}g[z^{-1}]$
$\simeq S^*((z^{-1}g[z^{-1}])^*)$
spin 1
g spin!
olomorphic Distributions
$r_G)^*$
Theorem of S Kumar
If we twist by a line bundle,
this is vacuum module for
Kac-Moody
B
t=0
Be
= d
(better:
of
BeT^+_A G


Theorem of S Kumar:
If we twist by a line bundle,
this is vacuum module for
Kac-Moody
In the bulk:
matter in R

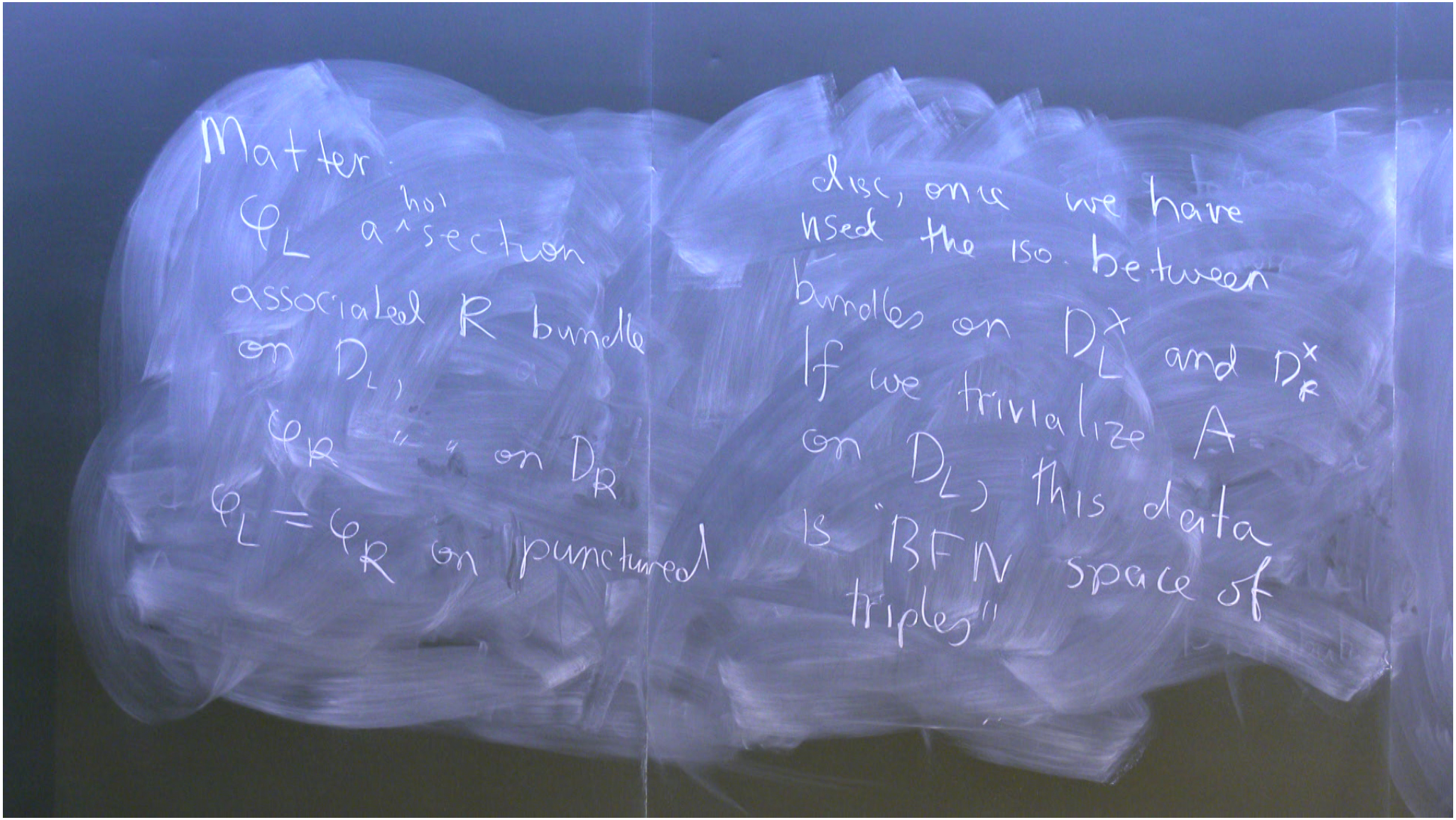

S Kumar
ist by a line bundle,
cuum module for
ody
e bulk
in R
Local ops
= Hilbert space on
S^2
D_L
D_R
Phase space:
A field
Hol. bundle on D_L

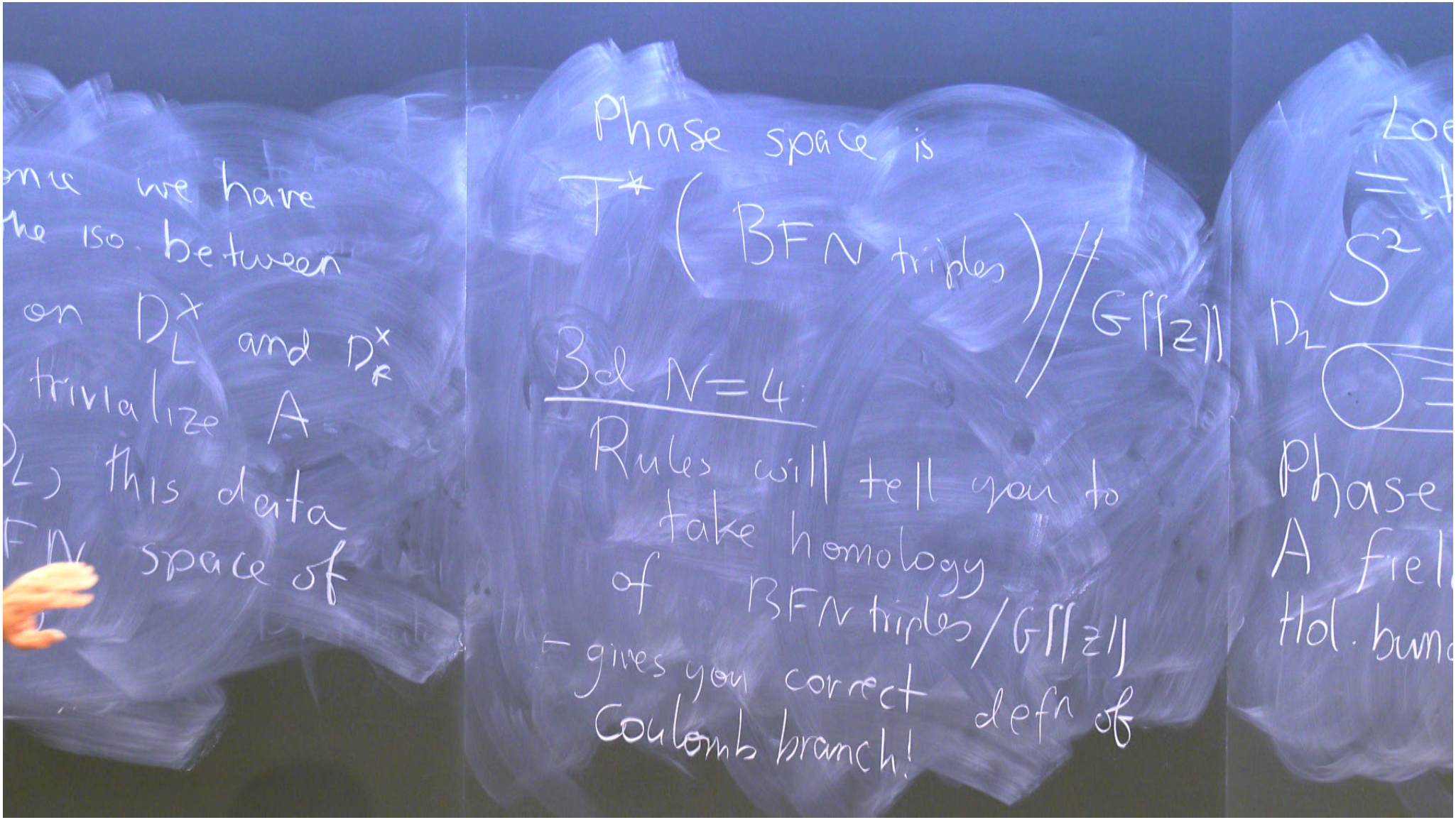
Local ops
$\cong$ Hilbert space on S^2

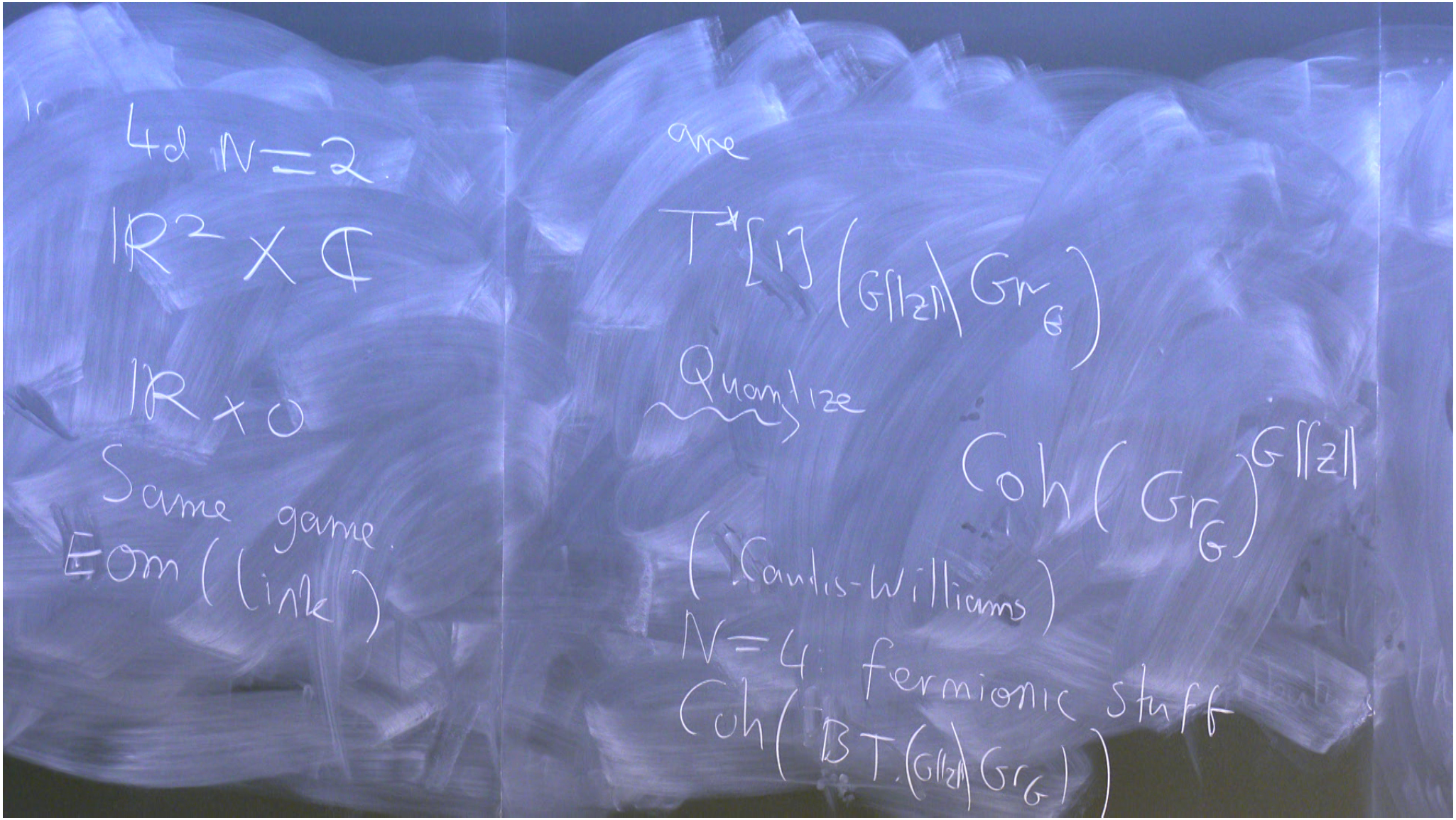


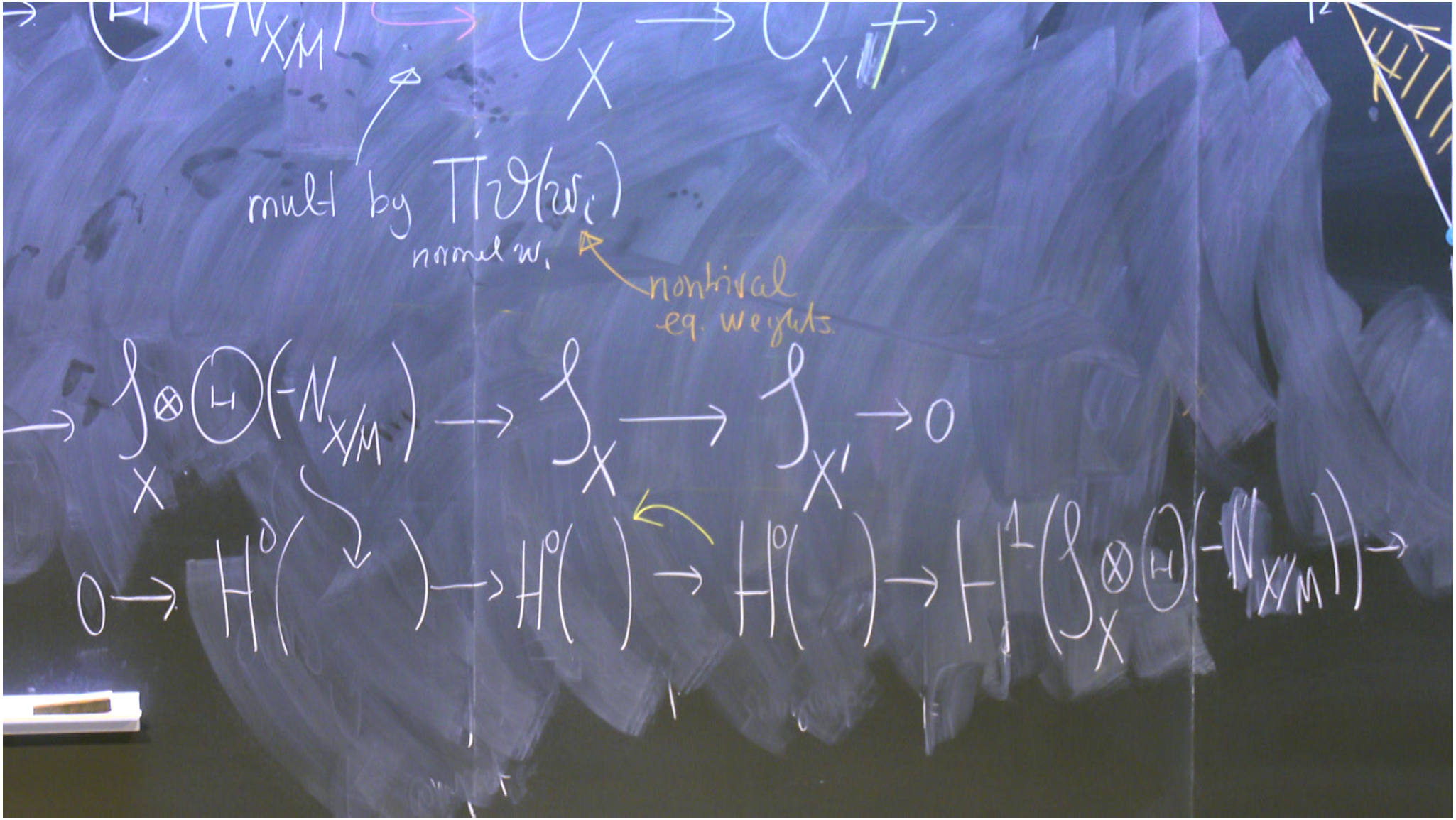
Phase space:
A field
Hol. bundle on D_L

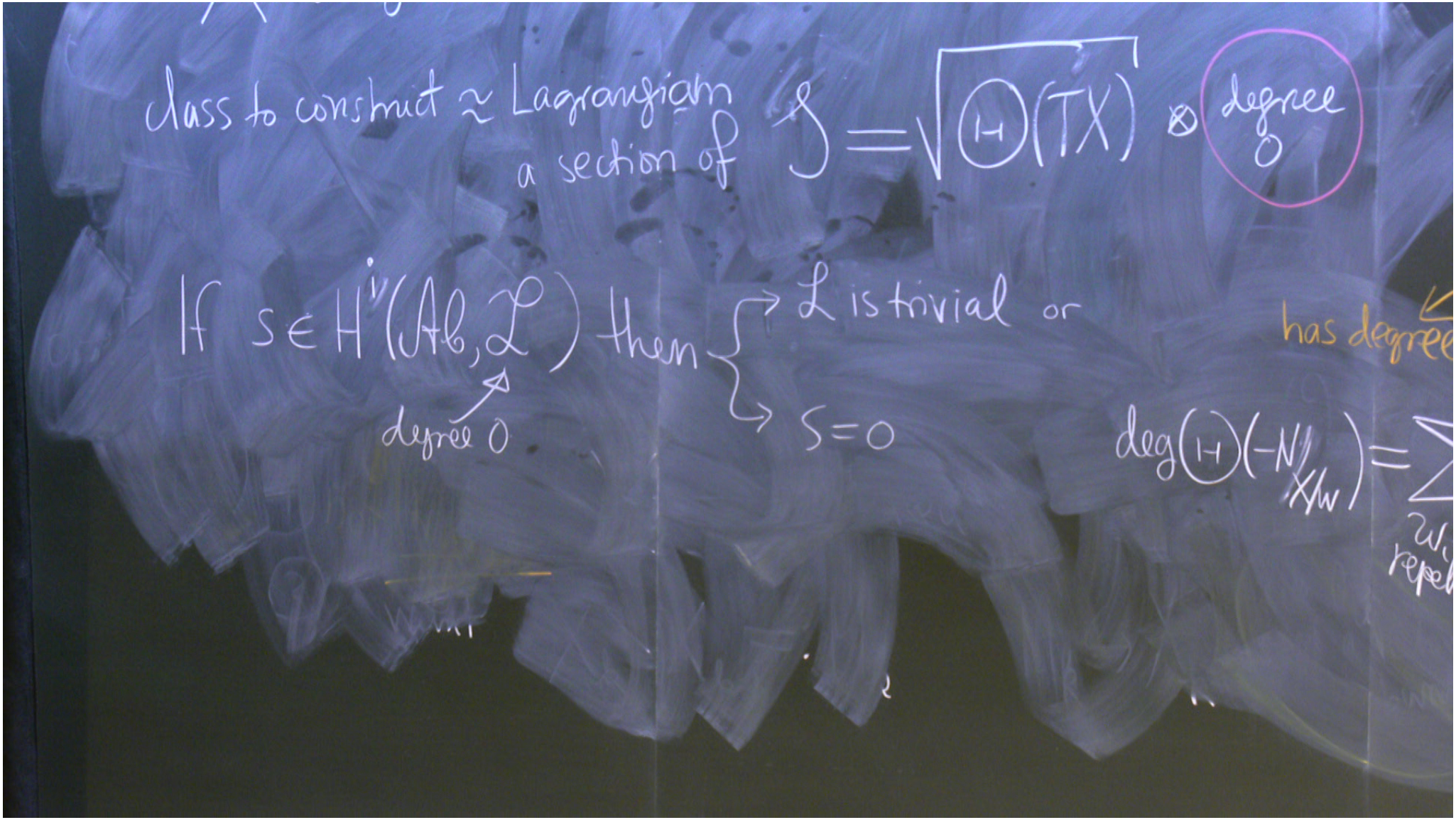
Hol. bundle on D_R
An iso between these on $\partial D_L = \partial D_R$
Assume iso. is on $D_L^\times$, with finite order of poles

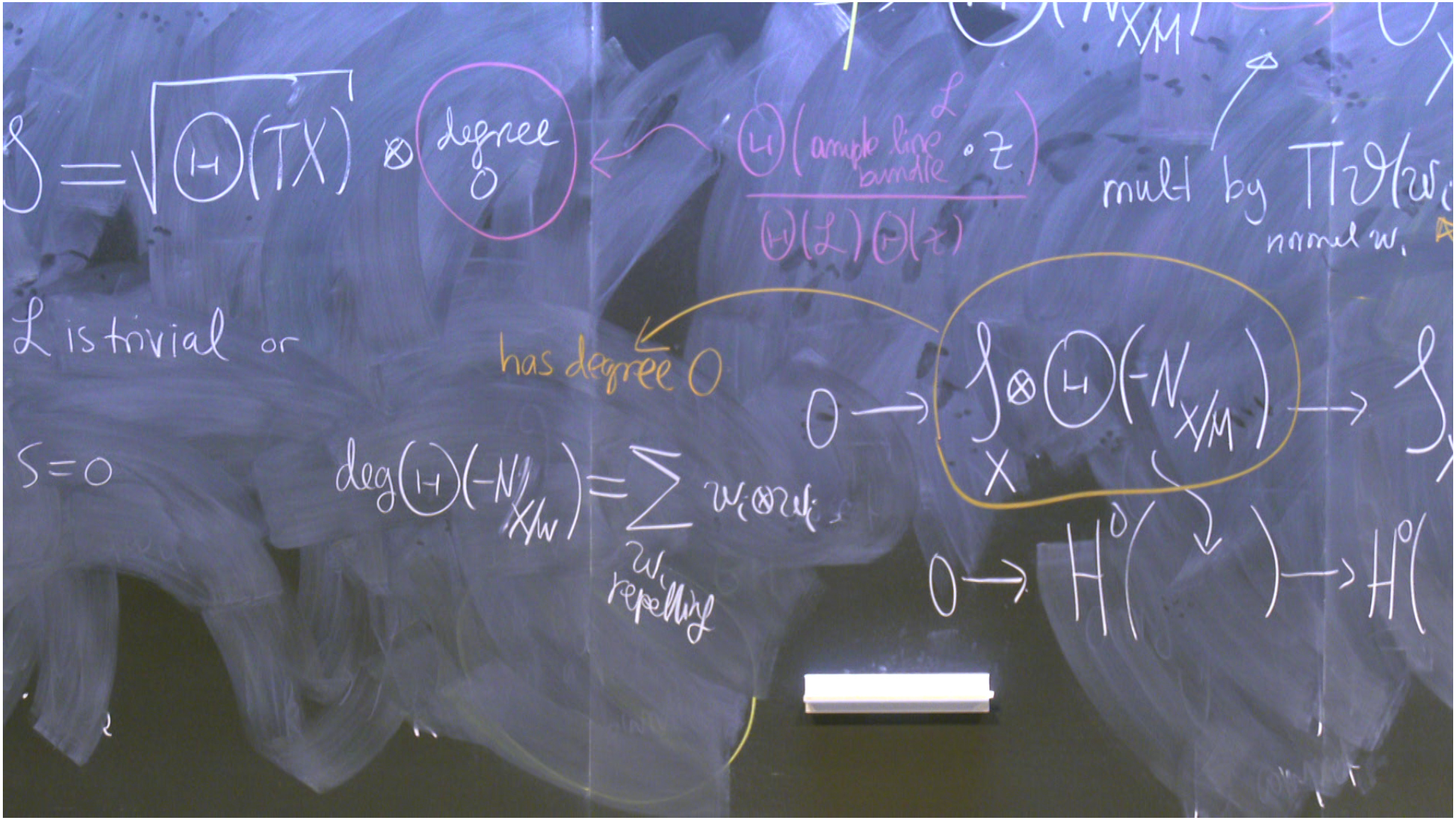

Matter.
φ_L a hol section
associated R bundle
on D_L,
φ_R " " on D_R
$\varphi_L = \varphi_R$ on punctured
disc, once we have
used the iso between
bundles on $D_L^\times$
If we trivialize A
on D_L, this data
is "BFN space of
triples"

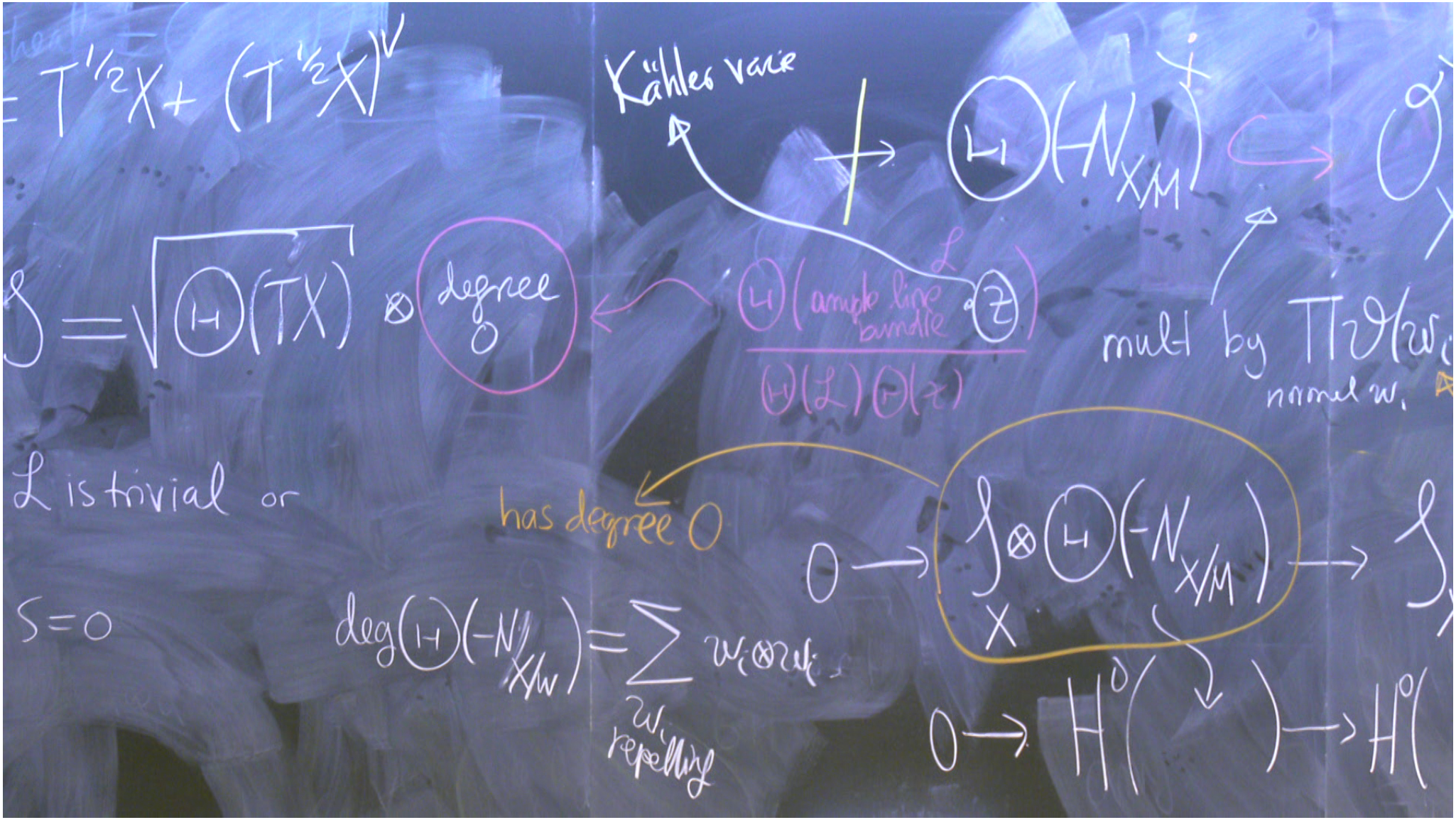

Phase space is
T^* (BFN triples) // G[[z]]
3d N=4:
Rules will tell you to take homology of BFN triples / G[[z]]
- gives you correct defn of Coulomb branch!
S^2
D_2
Phase
A fiel
Hol. bund

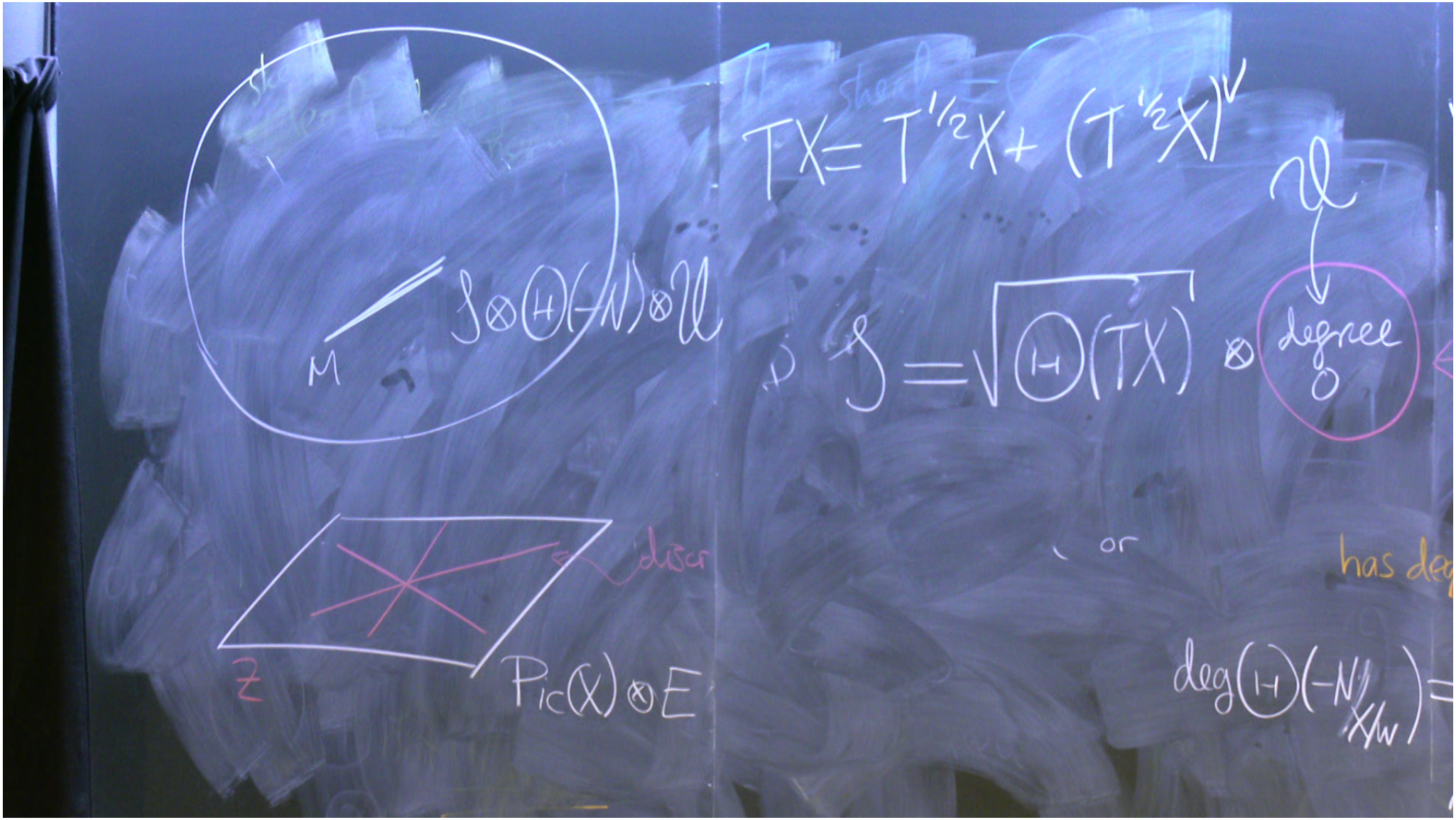

4d N=2
$\mathbb{R}^2 \times \mathbb{C}$
$\mathbb{R}_{>0}$
Same game.
Eom((inf k))
are
$T^*[1](G[[z]]\backslash Gr_G)$
Quantize
$Coh(Gr_G)^{G[[z]]}$
(Gaudis-Williams)
N=4: fermionic stuff
$Coh(BT.(G[[z]]\backslash Gr_G))$

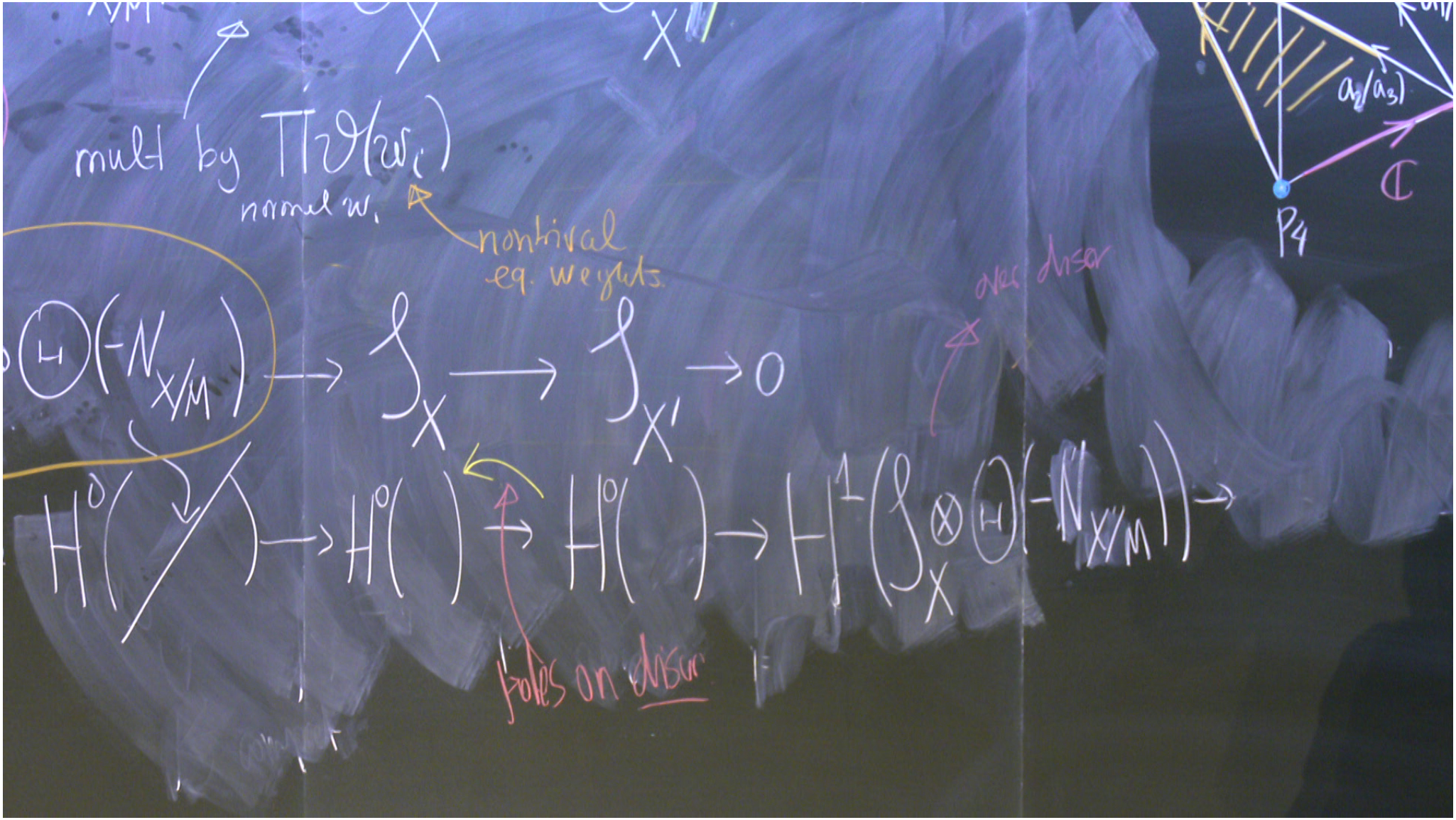

mult by $\prod \vartheta(w_i)$
normal w_i
nontrivial
eq. weights
$\to \mathcal{I}_X \otimes \bigwedge^{\bullet}(-N_{X/M}) \to \mathcal{I}_X \to \mathcal{I}_{X'} \to 0$
$0 \to H^0(\quad) \to H^0(\quad) \to H^0(\quad) \to H^1(\mathcal{I}_X \otimes \bigwedge^{\bullet}(-N_{X/M})) \to$

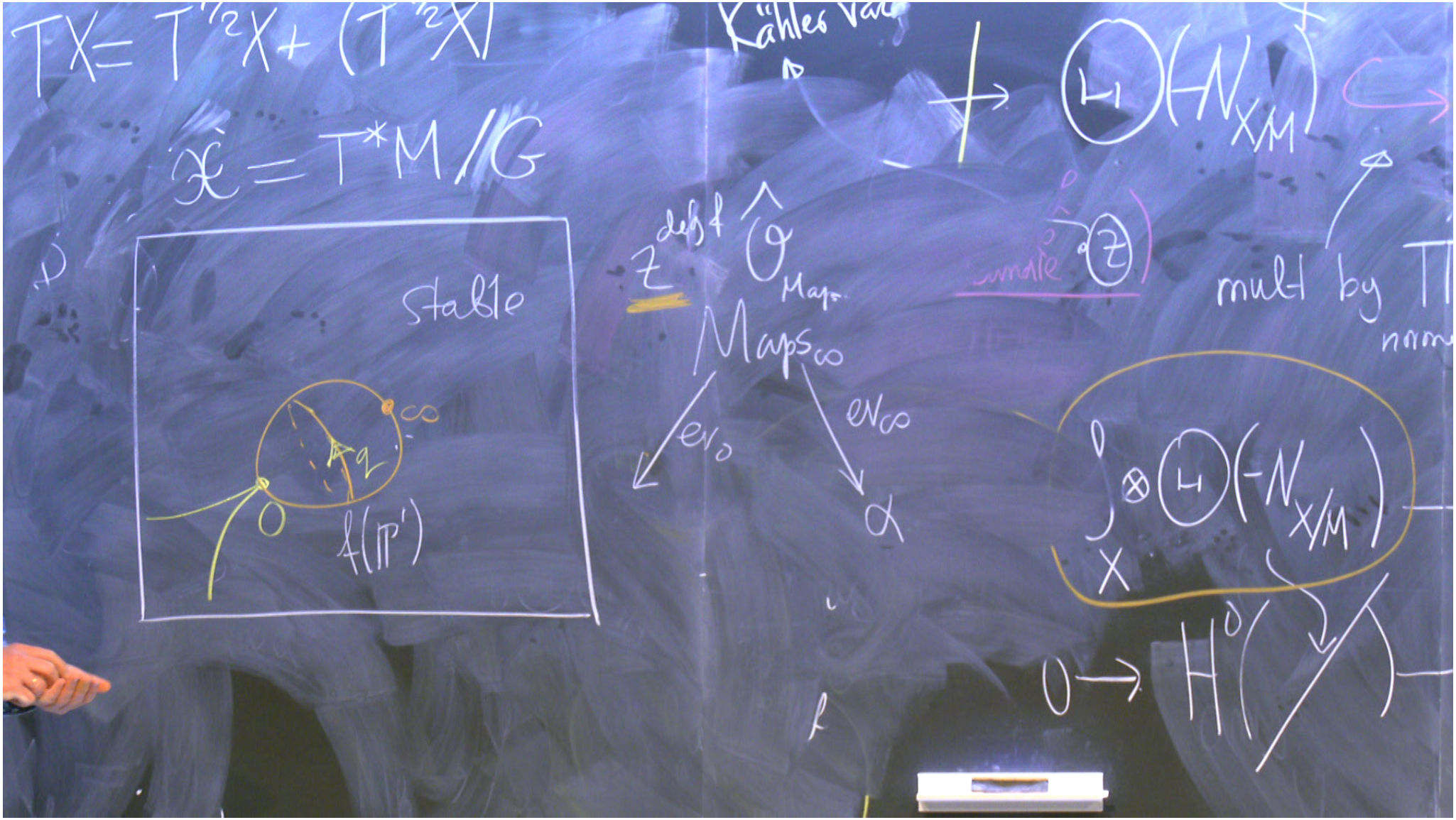

class to construct ≃ Lagrangian
a section of
$\mathcal{S} = \sqrt{\hat{A}(TX)} \otimes$ degree 0
If $s \in H^{i}(Ab, \mathcal{L})$ then
$\mathcal{L}$ is trivial or
$s = 0$
degree 0
has degree
$\deg(\hat{A})(-N_{X/W}) = \sum$


$\mathcal{L}$ is trivial or
has degree 0
$s=0$
$\deg(\wedge)(-N_{X/M}) = \sum_{w_i \text{ repelling}} w_i \otimes w_i$
mult by $\prod_{\text{normal } w_i} (1-w_i)$
$0 \to \int_X \otimes (\wedge)(-N_{X/M}) \to$
$0 \to H^0(\quad) \to H^0($


Kähler case
degree
ample line bundle
mult by
has degree 0
L is trivial or
S=0
repelling


M
$S \otimes \Theta(-N) \otimes \mathcal{U}$
$TX = T^{1/2}X + (T^{1/2}X)^{\vee}$
$\mathcal{U}$
$S = \sqrt{\Theta(TX)} \otimes$ degree 0
or
has deg
Z
disc
$Pic(X) \otimes E$
$\deg(\Theta)(-N_{X/W}) =$


mult by $\prod \vartheta(w_i)$
normal w_i
nontrival eq. weights
$\mathcal{O}(-N_{X/M})$
$\mathcal{I}_X$
$\mathcal{I}_{X'}$
0
$H^0(\quad) \to H^0(\quad) \to H^0(\quad) \to H^1(\mathcal{I}_X \otimes \mathcal{O}(-N_{X'/M})) \to$
poles on divisor
a_2/a_3
$\mathbb{C}$
P_4


TX = T''^2X + (T''^2X)
Kähler
stable
Maps
mult by