Title: Minimax surfaces and the covariant holographic entropy cone

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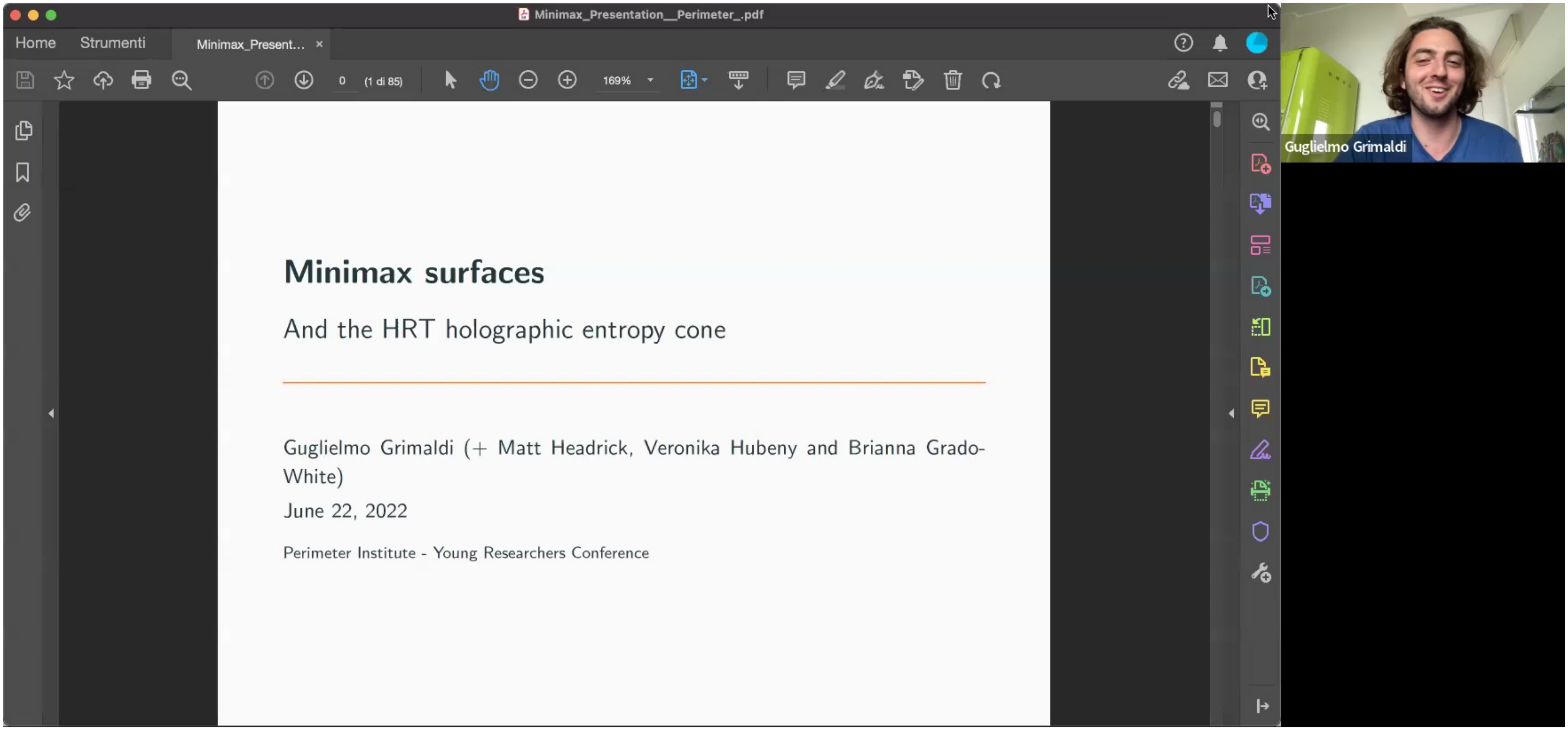
Abstract: I will discuss work-in-progress for defining a new proposal for the covariant holographic entanglement entropy. The proposal instructs us to find maximal spacelike codimension-2 surfaces on timelike hypersurfaces in the bulk, followed by a minimization among all possible hypersurfaces in the right homology class. We describe and prove various properties of such minimax surfaces, and argue for their equivalence with the more familiar HRT and maximin proposals. Finally, we give compelling reasons to be interested in yet another entanglement entropy proposal: minimax surfaces allow us to prove all higher entropy cone inequalities, showing that the RT and HRT holographic entropy cones are indeed equivalent.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Minimax surfaces
And the HRT holographic entropy cone
Guglielmo Grimaldi (+ Matt Headrick, Veronika Hubeny and Brianna Grado-White)
June 22, 2022
Perimeter Institute - Young Researchers Conference
Guglielmo Grimaldi

Plan of the talk

1. Introduction
2. HRT and Maximin
3. Minimax

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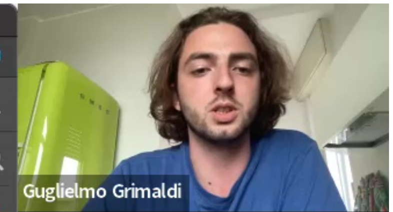


Big picture

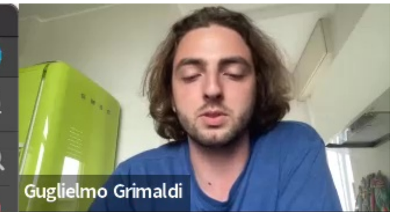
In the classical limit of AdS/CFT, we know that quantum states are dual to classical, smooth, spacetime geometries.

Can we characterize such states better? What are general requirements for a state to have a classical geometric dual?

Studying entanglement entropy (EE) can teach us a lot about these questions.

In QFT, the EE of subregions is required to obey certain inequalities

- positivity $S(A) \geq 0$,
- subadditivity $S(A) + S(B) \geq S(AB)$,
- strong subadditivity $S(AB) + S(BC) \geq S(B) + S(ABC)$.



Holographic EE

For static states in large N strongly coupled CFTs, the entanglement entropy of spatial subregions is geometrically realized in the bulk as the area of certain minimal surfaces.

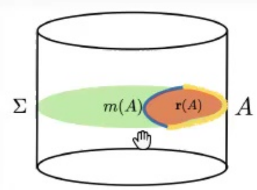
Ryu-Takayanagi (2006)

Pick a boundary time slice that contains A, extend the time slice into a bulk slice Σ. Then

$$S(A) = \text{area}(m(A)) \tag{1}$$

where $m(A) \subset \Sigma$ is the globally minimal surface homologous to A.

The **homology condition**: $\exists\, r \subset \Sigma$ such that $\partial r = A \cup m(A)$



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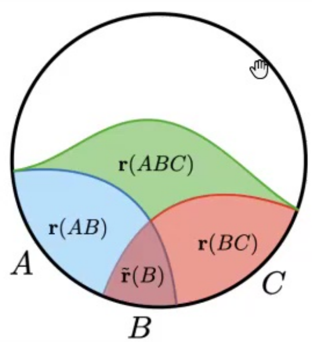
Holographic EE

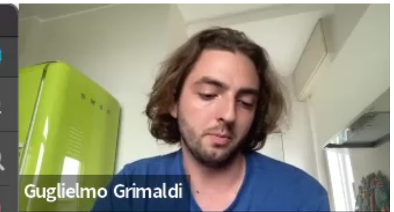
RT obeys positivity, SA and SSA → consistency check

Proofs use an inclusion/exclusion argument **[Headrick-Takayanagi '07]**



$$\tilde{\mathbf{r}}(ABC) = \mathbf{r}(AB) \cup \mathbf{r}(BC)$$
$$\tilde{\mathbf{r}}(B) = \mathbf{r}(AB) \cap \mathbf{r}(BC)$$

$$S(AB) + S(BC) = \tilde{S}(B) + \tilde{S}(ABC)$$
$$\geq S(B) + S(ABC)$$



But RT knows more...

Holographic static states (RT states) are found to obey more inequalities than just the standard ones **[Hayden-Headrick-Maloney '11]**

First inequality found: **monogamy of mutual information** (MMI)

$$S(AB) + S(BC) + S(AC) \geq S(A) + S(B) + S(C) + S(ABC) \qquad (2)$$

Not satisfied by arbitrary quantum states!

First characterization of holographic quantum states.

Soon more inequalities were found **[Bao et al '11]**

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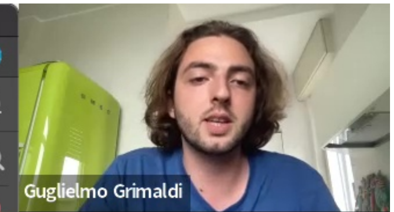

The holographic entropy cone

When one considers more boundary subregions, static states obey new inequalities.

Let subregions be $\{A_1, A_2, \ldots, A_n, O\}$. There are $2^n - 1$ possible combinations of such subregions that make up the set $\mathcal{S}_n$ of possible entropies for n regions.

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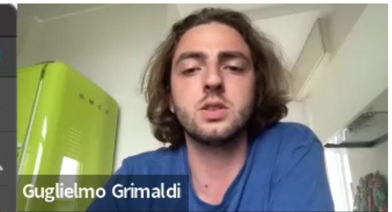

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These entropies define a basis for an entropy space that lives in $\mathbb{R}^{2^n-1}$.

One can prove all of these inequalities for RT using similar proof method as before.

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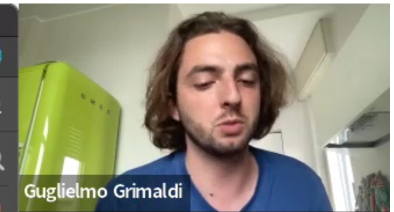

Open questions

What can we say about general holographic states with time dependence?

They were found to obey SSA and MMI, but a general proof for higher parties remains elusive **[Wall '12, Rota-Weinberg '17]**

Recent progress in this direction: for $d = 2$ time dependent states obey same inequalities as static states **[Czech, Dong '19]**

Punchline: we confirm the result in general dimensions by proposing a new holographic EE prescription.

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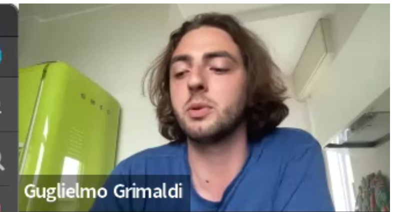

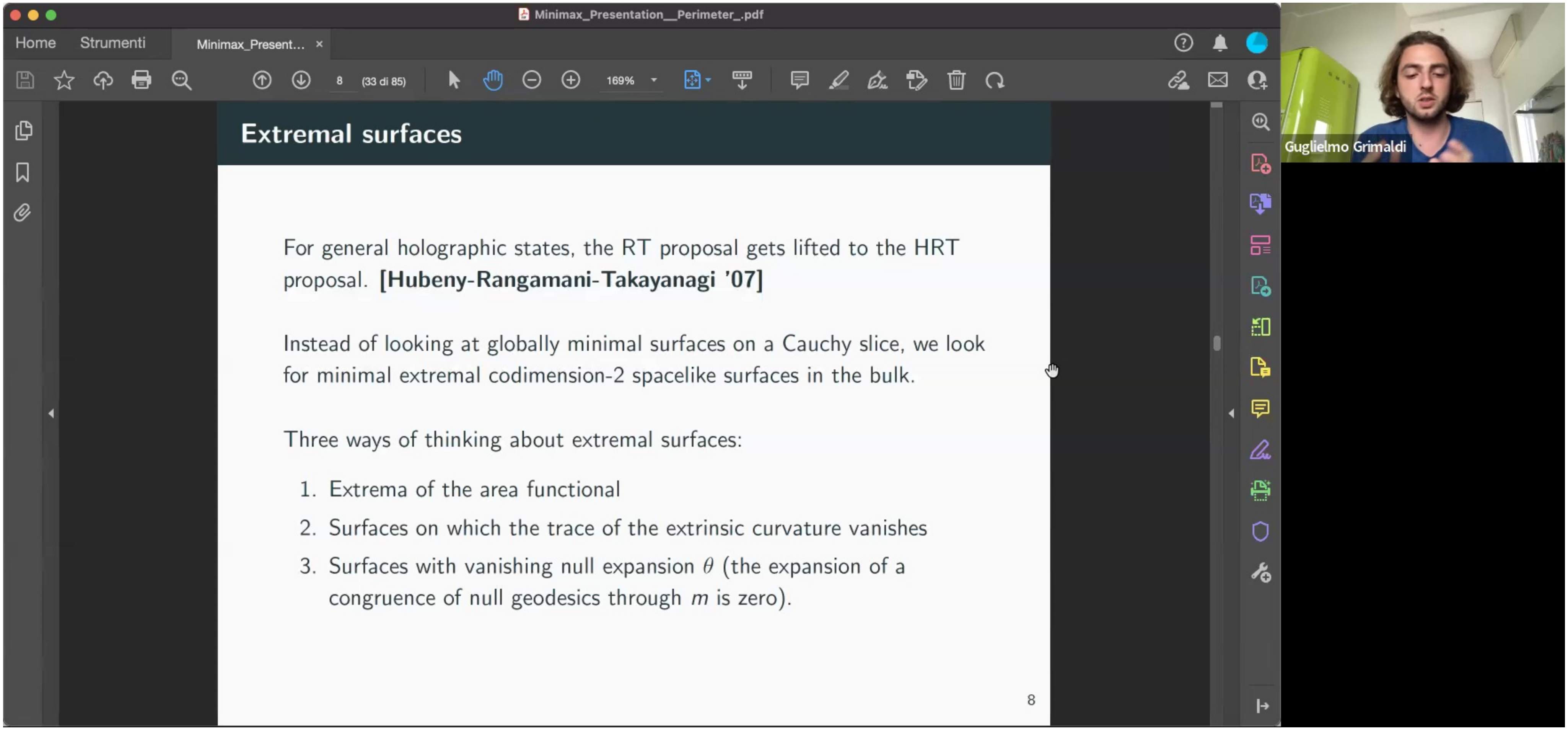
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Extremal surfaces
For general holographic states, the RT proposal gets lifted to the HRT proposal. [Hubeny-Rangamani-Takayanagi '07]
Instead of looking at globally minimal surfaces on a Cauchy slice, we look for minimal extremal codimension-2 spacelike surfaces in the bulk.
Three ways of thinking about extremal surfaces:
1. Extrema of the area functional
2. Surfaces on which the trace of the extrinsic curvature vanishes
3. Surfaces with vanishing null expansion θ (the expansion of a congruence of null geodesics through m is zero).
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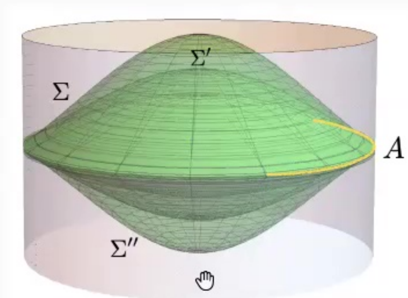
Maximin

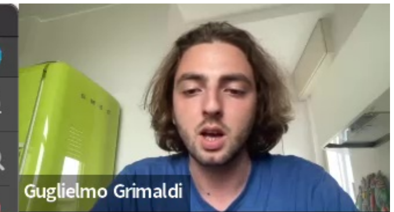
This perspective inspired the so called *maximin* proposal of HRT. **[Wall '12]**

Maximin (Wall, 2012)

Find minimal area surface $m \subset \Sigma$ homologous to A on a Cauchy slice Σ. Then pick the maximal among these minimal surfaces by varying Σ

$$S(A) = \sup_{\Sigma} \inf_{m \sim A} \text{area}(m) \qquad (3)$$





The highs and lows of maximin

Why great: reduces the complicated problem of computing extremal surfaces to finding minimal surfaces on Riemannian manifolds (the Cauchy slices)

Most importantly, maximin allows us to prove SSA and MMI for general holographic states (ask!)

The idea is to somehow "project" all the surfaces onto the same Cauchy slice, and from there use standard static proof.

Any attempt for all higher entropy inequalities fails miserably: for a given inequality LHS $\geq$ RHS, regions in the RHS cross and cannot share the same Cauchy slice.

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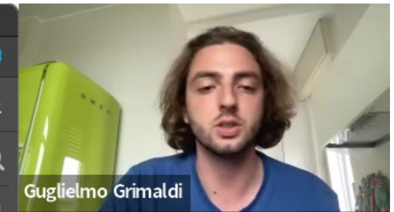

From maximin to minimax

Maximin failed because we relied too much on working on Cauchy slices. Too restrictive.

Main idea: lift up this requirement and work over the entire bulk spacetime. How?

Get rid of Cauchy slices and replaces them with timelike counterparts, which we call **time sheets**

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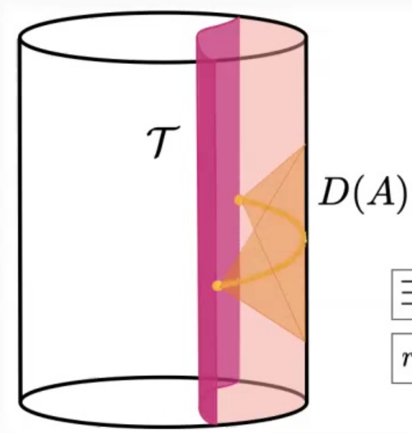
Time sheets

Time sheet

A time sheet $\mathcal{T}$ is a timelike or null hypersurface homologous to $D(A)$ (relative to $J^+(\partial A) \cup J^-(\partial A)$).



$$\exists R \subset \mathcal{M}, \quad \partial R = \mathcal{T} \cup r$$

$$r \cap D(A^c) = \emptyset, \quad D(A) \subset r$$

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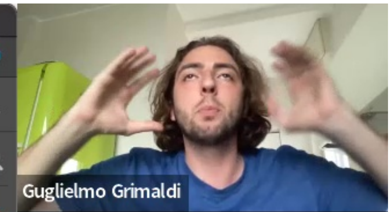

Minimax

Having replaced Cauchy slices with time sheets, we need to understand how to rephrase the problem

On a timelike hypersurface, we can have maximal slices (just like we had minimal slices on Cauchy surfaces). Therefore, we can have the following definition

Minimax

Let A be a boundary region. Pick a time-sheet $\mathcal{T}$, and find the maximal codimension-2 achronal surface γ on $\mathcal{T}$. The minimax surface $m(A)$ is found by picking the minimal of these maximal surfaces among all variations of $\mathcal{T}$

$$S(A) = \inf_{\mathcal{T}} \sup_{\gamma \in \mathcal{T}} \text{area}(\gamma) \tag{4}$$

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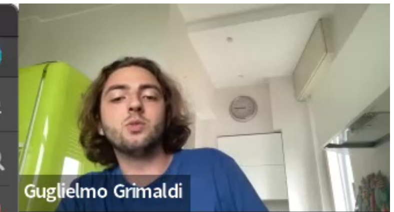

Equivalence with HRT

This is all well and good, but is minimax = HRT?

For AdS spacetimes obeying the null energy condition, yes!

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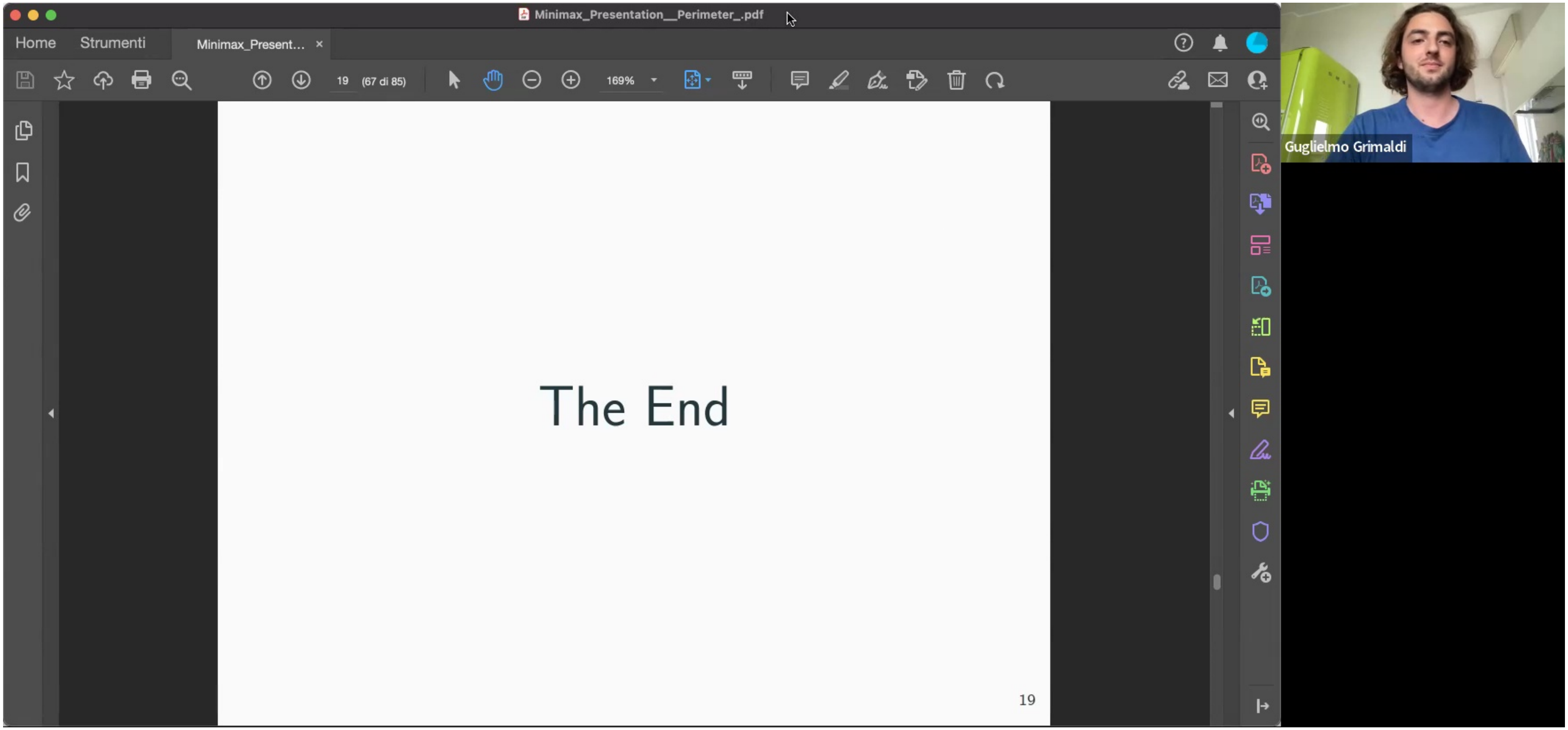
Proof of EE inequalities

So, how does minimax come to the rescue for the higher entropy inequalities?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The End
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