Title: TBD

Speakers: Barbara Soda

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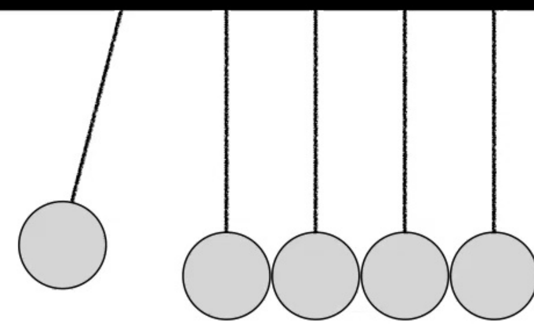


Newton Cradle Spectra

Barbara Šoda, Achim Kempf

Young Researchers Conference, Perimeter Institute, 20th June 2022

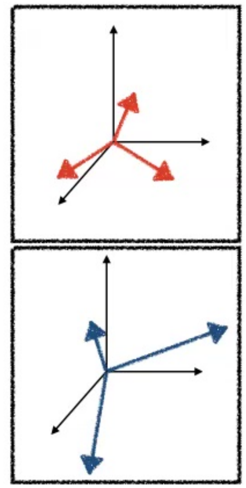
What happens when we add two Hamiltonians?

- Suppose we know the eigenvalues and eigenvectors of a Hamiltonian H_0.

$$H_0 = \sum_i e_i \left|e_i\right\rangle \left\langle e_i\right|$$

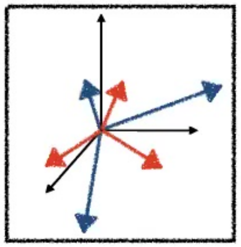
- We know the eigenvectors and eigenvalues of H'.

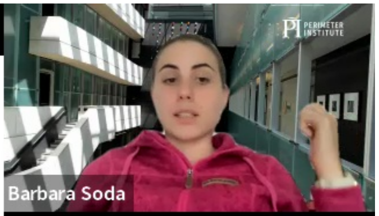
$$H' = \sum_i f_i \left|f_i\right\rangle \left\langle f_i\right|$$



What happens when we add two Hamiltonians?

- What are the eigenvalues and eigenvectors of $H_0 + H'$?
- Typically, we approach the problem perturbatively.
- Nonperturbative results:
  - Wigner-von Neumann result on level repulsion
  - Cauchy interlacing
    - Too few nonperturbative results!





New nonperturbative result: Newton's cradle spectra

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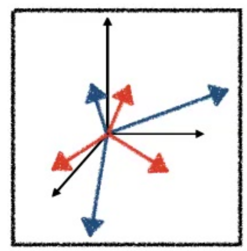

?

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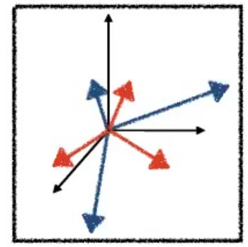
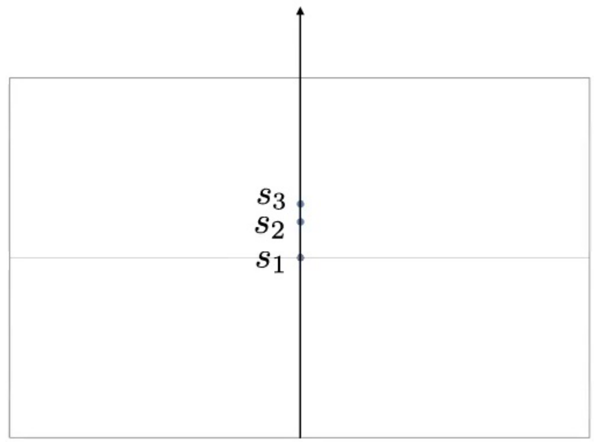


New nonperturbative result: Newton's cradle spectra

Newton's cradle of the spectra

- We start with a self-adjoint matrix S_0.
- We know its eigen-values and -vectors:

$$S_0 = \sum_i s_i \left| s_i \right\rangle \left\langle s_i \right|$$



- e.g. if it acts on a 3-dim Hilbert space $\mathcal{H}$,

  3 eigenvalues s_1, s_2, s_3 .

- What happens when we add a projector onto a 1-dimensional subspace of $\mathcal{H}$?

$$S(\mu) = S_0 + \mu \left| v \right\rangle \left\langle v \right|$$

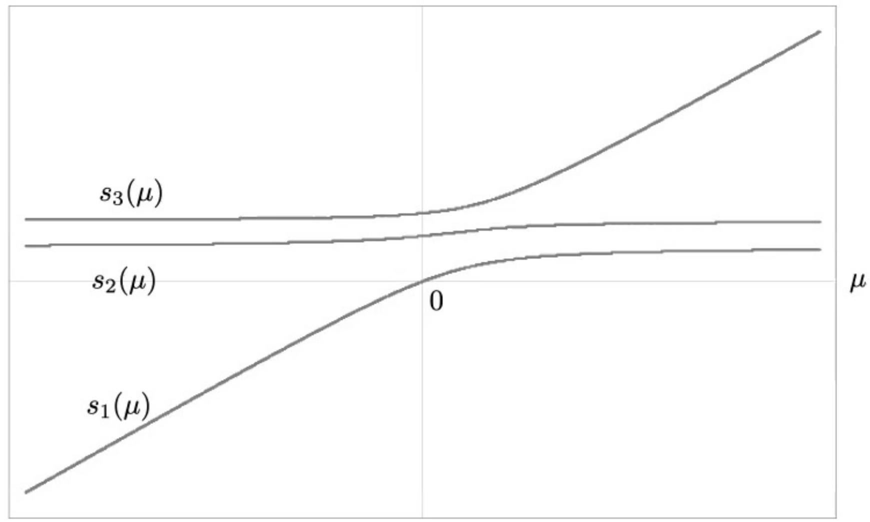
Newton's cradle of the spectra

$$S(\mu) = S_0 + \mu \left| v \right\rangle \left\langle v \right|$$

- Generically $\langle s_i | v \rangle \neq 0, \forall i$,

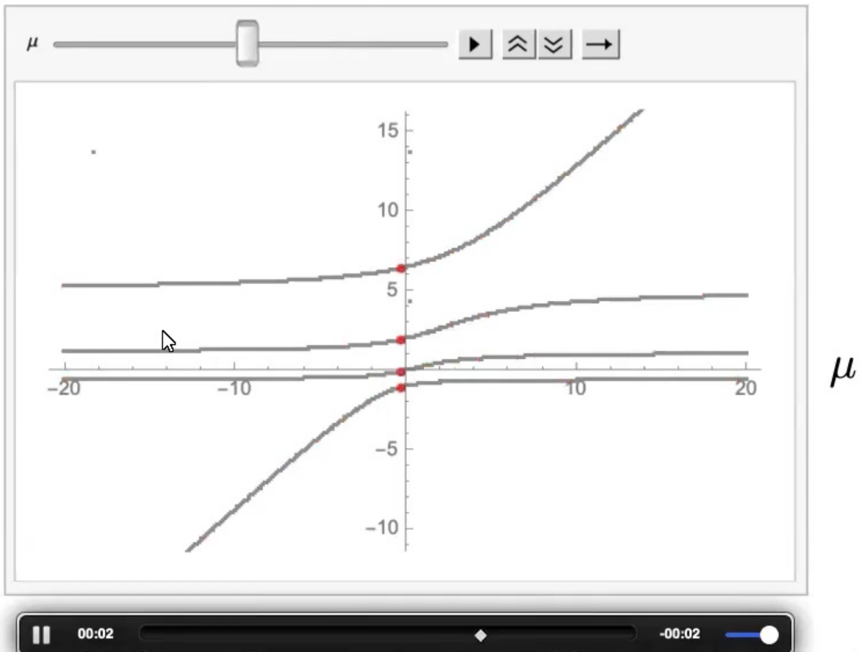
  we get the "Newton cradle".

We call μ the coupling constant.

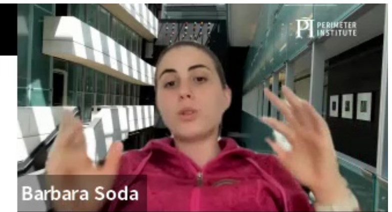


Example with 4 eigenvalues.

Animation of eigenvalues of $S(\mu) = S_0 + \mu \left|v\right\rangle \left\langle v\right|$



Newton cradle-like motion as a function of the coupling constant μ.



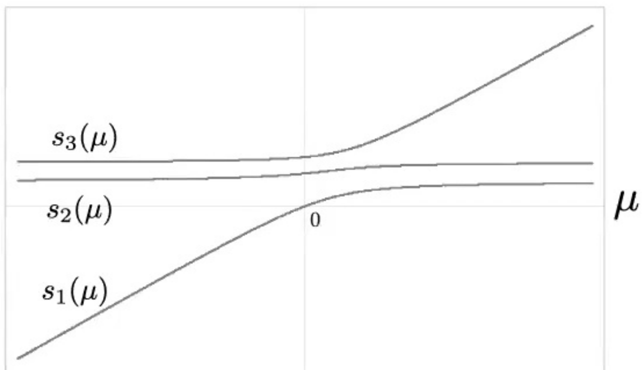
Properties of the Newton cradle of spectra

- The eigenvalues all move in the same direction:

$$\left.\frac{ds_n(\mu)}{d\mu}\right|_{\mu=0} = |v_n|^2 \,, \qquad \frac{d\mu(s)}{ds} = \left(\sum_{m=1}^{N} \frac{|v_m|^2}{s - s_m}\right)^{-2} \sum_{r=1}^{N} \frac{|v_r|^2}{(s - s_r)^2}$$

positive

$$v_n = \langle s_n | v \rangle$$



Properties of the Newton cradle of spectra

- The new eigenvectors $|s\rangle$ expressed in the original eigenbasis (of S_0):

$$\langle s|s_n\rangle = \frac{(-1)^n |v_n|}{s - s_n} \left(\sum_{m=1}^{N} \frac{|v_m|^2}{(s - s_m)^2} \right)^{-1/2} \prod_{r=1}^{N} (-1)^{\theta(s - s_r)}$$

- Where $|s\rangle$ is the eigenvector corresponding to an eigenvalue s :

$$S(\mu)\,|s\rangle = s\,|s\rangle$$

Properties of the Newton cradle of spectra

- The spectra cover the entire real line exactly once :

$$\bigcup_{n=1,\ldots,N} \bigcup_{\mu \in \mathbb{R} \cup \infty} \{s_n(\mu)\} = \mathbb{R} \cup \infty$$

- Therefore, for any $s \in \mathbb{R}$, there is a μ for which s is an eigenvalue of $S(\mu)$:

$$S(\mu)\,|s\rangle = s\,|s\rangle$$

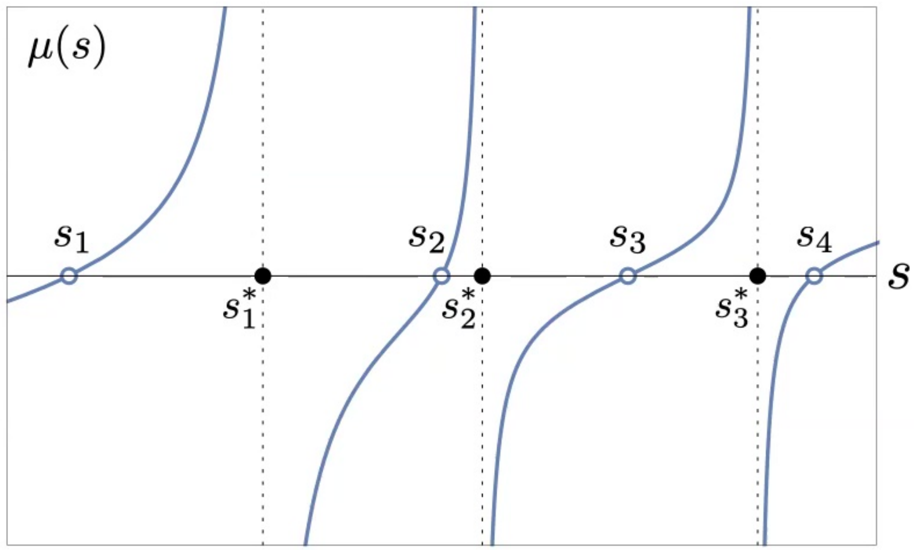
- We can calculate the μ exactly using this formula:

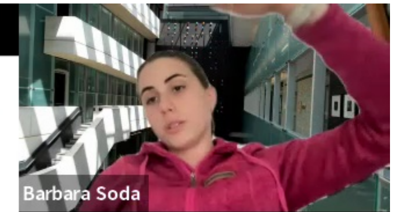
$$\boxed{\mu(s) = \left(\sum_{i=1}^{N} \frac{|v_n|^2}{s - s_n} \right)^{-1}}$$

Coupling constant μ as a function of an eigenvalue s:

$$\mu(s) = \left(\sum_{i=1}^{N} \frac{|v_n|^2}{s - s_n} \right)^{-1}$$

Plot of the dependence:





Remark: Cauchy's interlacing theorem

Theorem paraphrased:

Start with a NxN self-adjoint matrix M, which has eigenvalues $s_1, ..., s_N$.

E.g.

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \text{ has eigenvalues } s_1, s_2, s_3, s_4.$$

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If we cross out a row and a column, the new eigenvalues interlace the old ones.

$$M' = \begin{pmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{pmatrix} \text{ has eigenvalues: } s'_1, s'_2, s'_3$$

$$s_1 \leq s'_1 \leq s_2 \leq s'_2 \leq s_3 \leq s'_3 \leq s_4$$

Cauchy interlacing

We can explain and calculate the new eigenvalues in Cauchy interlacing.

Note the formula: $\mu(s) = \left(\sum_{i=1}^{N} \frac{|v_n|^2}{s - s_n} \right)^{-1}$

What happens when coupling $\mu \to \infty$?

Recall: $S(\mu) = S_0 + \mu \, |v\rangle \langle v|$ Subspace spanned by $|v\rangle$ decouples!

Like deleting a row and a column corresponding to $|v\rangle \langle v|$.

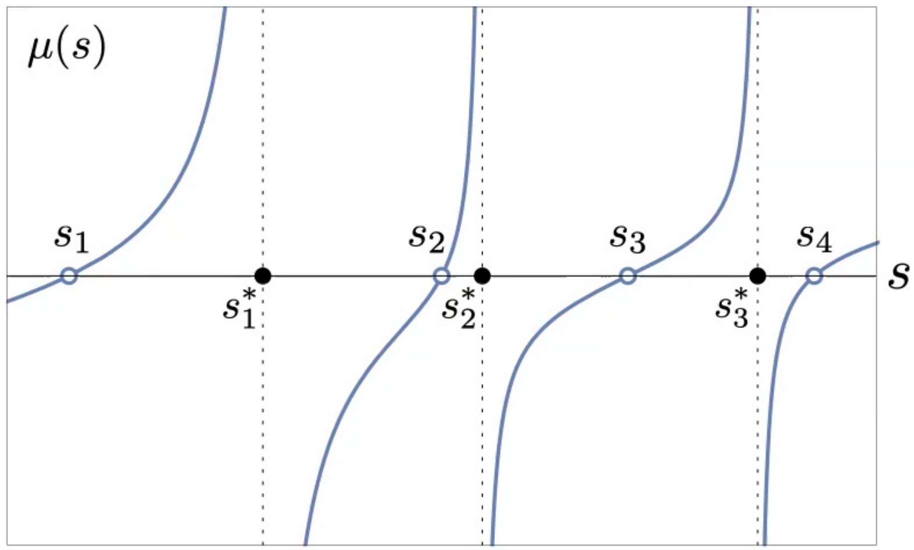
Formula for calculating the interlaced eigenvalues:

$$\sum_{m=1}^{N} \frac{|v_m|^2}{s_n^* - s_m} = 0$$

Coupling constant μ as a function of an eigenvalue s:

$$\mu(s) = \left(\sum_{i=1}^{N} \frac{|v_n|^2}{s - s_n} \right)^{-1}$$

Plot of the dependence:



Note the Cauchy interlacing!

Newton's cradle of unitary operators

- We start with a unitary matrix U_0.
- We know its eigenvalues and vectors:

$$U_0 = \sum_i u_i \left|u_i\right\rangle \left\langle u_i\right|$$

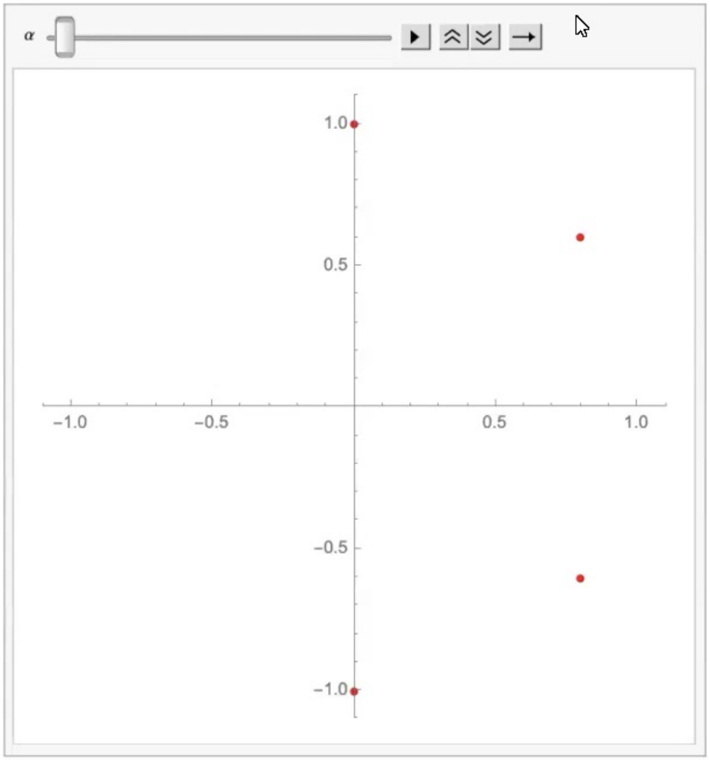
- e.g. if it acts on a 3-dim Hilbert space $\mathcal{H}$,

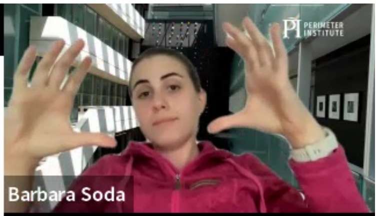
  3 eigenvalues u_1, u_2, u_3.

- We get a behaviour analogous to Newton's cradle when we act with a U(1) operator family:

$$U(\alpha) := \left(\mathbb{1} + \left(e^{i\alpha} - 1\right) \left|w\right\rangle \left\langle w\right|\right) U_0 \quad , \alpha \in [0, 2\pi)$$

Eigenvalues of $U(\alpha) = (\mathbb{1} + (e^{i\alpha} - 1)\,|w\rangle\langle w|)\,U_0$ as a function of α.





The connection between two Newton cradles

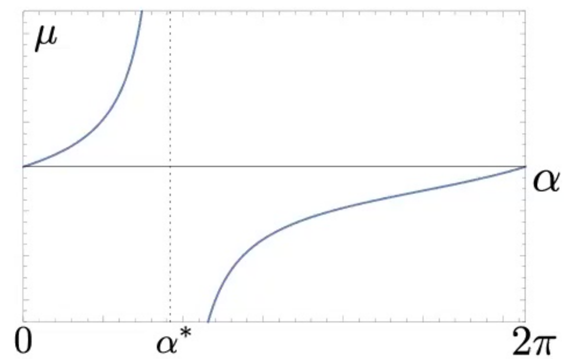
- The self-adjoint and the unitary Newton cradles are related by Cayley transforms:

$$U(\alpha) = (S(\mu) - i\mathbb{1})(S(\mu) + i\mathbb{1})^{-1}\,, \quad S(\mu) = -i(U(\alpha) + \mathbb{1})(U(\alpha) - \mathbb{1})^{-1}$$

- Moebius transform for the eigenvalues: $u_j = \dfrac{s_j - i}{s_j + i}$

- Relationship between μ and α :

$$\mu(\alpha) = \left(\sum_{m=1}^{N} \frac{|v_m|^2}{s_m^2 + 1} \cot\left(\frac{\alpha}{2}\right) - \sum_{k=1}^{N} \frac{|v_k|^2 s_k}{s_k^2 + 1}\right)^{-1}$$



Take home message:

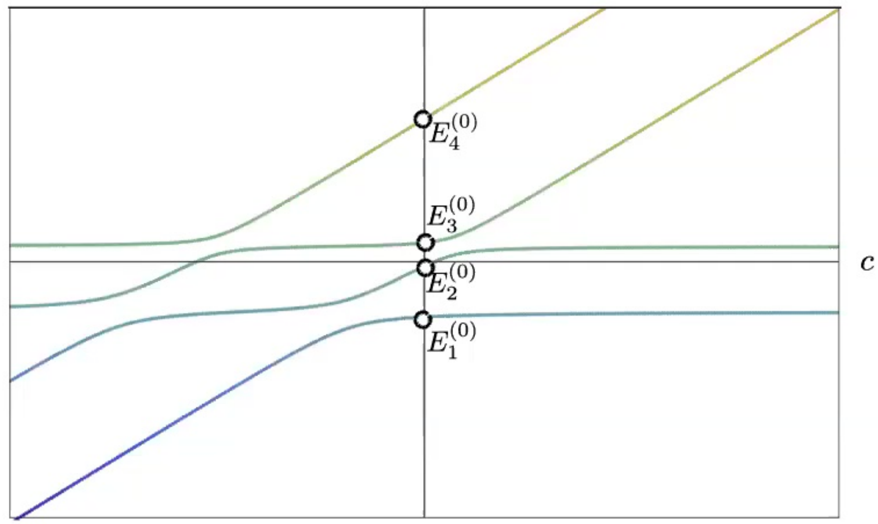
- Addition of self-adjoint operators translates nicely to multiplication of unitaries, but not the exponentiated ones, instead the Cayley transformed unitaries.

~~$H \to e^{iHt}$~~

$$H \to (H - i\mathbb{1})(H + i\mathbb{1})^{-1}$$

Level repulsion

Eigenvalues of $A + cB$, where c is the coupling strength, as a function of c:



Plot of eigenvalues of $A + cB$ as a function of c.

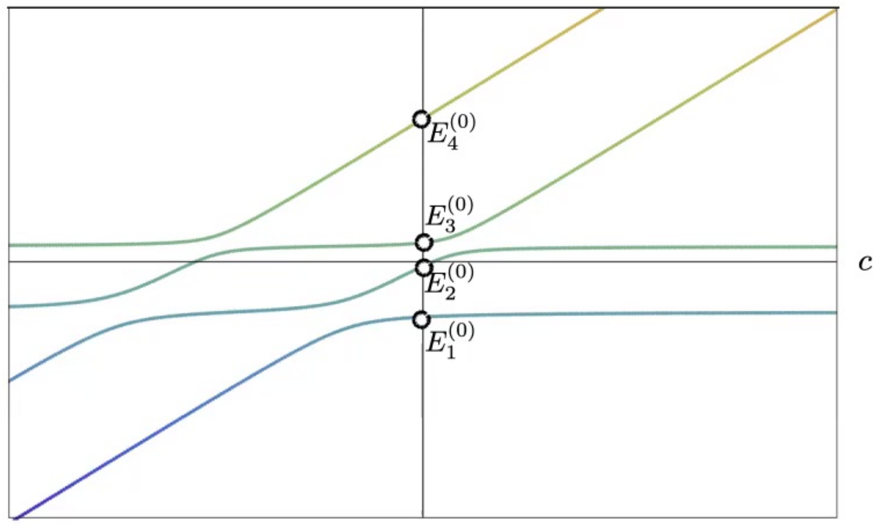
Level repulsion

Eigenvalues of $A + cB$, where c is the coupling strength, as a function of c:

Decompose B:

$$B = \sum_i b_i \left| v_i \right\rangle \left\langle v_i \right|$$

Add projectors $b_i \left| v_i \right\rangle \left\langle v_i \right|$ one by one.



Plot of eigenvalues of $A + cB$ as a function of c.

Level repulsion

Eigenvalues of $A + cB$, where c is the coupling strength, as a function of c:

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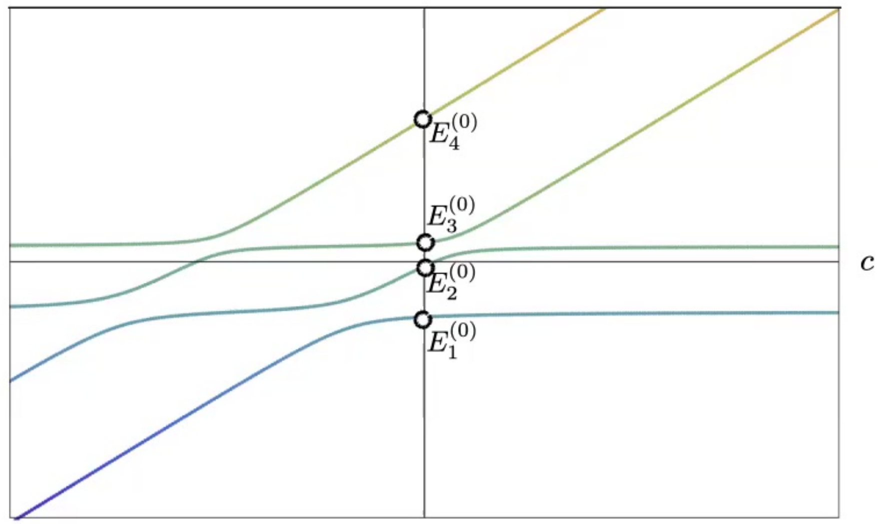
Add projectors $b_i \left|v_i\right\rangle \left\langle v_i\right|$ one by one.

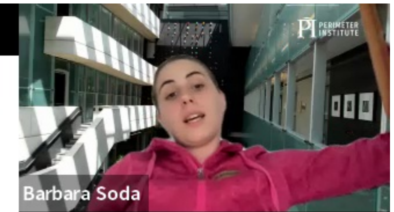
As long as the overlap:

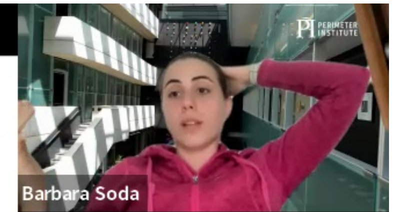
$$\langle a_i(b)|v_i\rangle \neq 0$$

-> no level crossing!



Plot of eigenvalues of $A + cB$ as a function of c.





Summary

- Learned how eigenvalues and eigenvectors change as we add a 1-dim. projector to a Hamiltonian.
    - Eigenvalues move like Newton's cradle.
- We can decompose any Hamiltonian into a sum of 1-dim. projectors ->
    - -> new strategy for understanding addition of Hamiltonians.
- New understanding of Cauchy interlacing, level repulsion…

Applications

- Adiabatic quantum computing:
  - First results: new insights into how algorithmic complexity translates into gap narrowing and therefore the slowdown of computation.
- Shannon sampling: shows equivalence of discrete and continuous representations of information
  - Result: generalized sampling theory allowing finite number of samples and varying information density
- Others in progress: BBT transition in quantum chaos, Casimir-like forces, spectral geometry… wherever there is a spectrum.

Thank you for your attention!

Paper will be on arXiv on Tuesday (tomorrow).