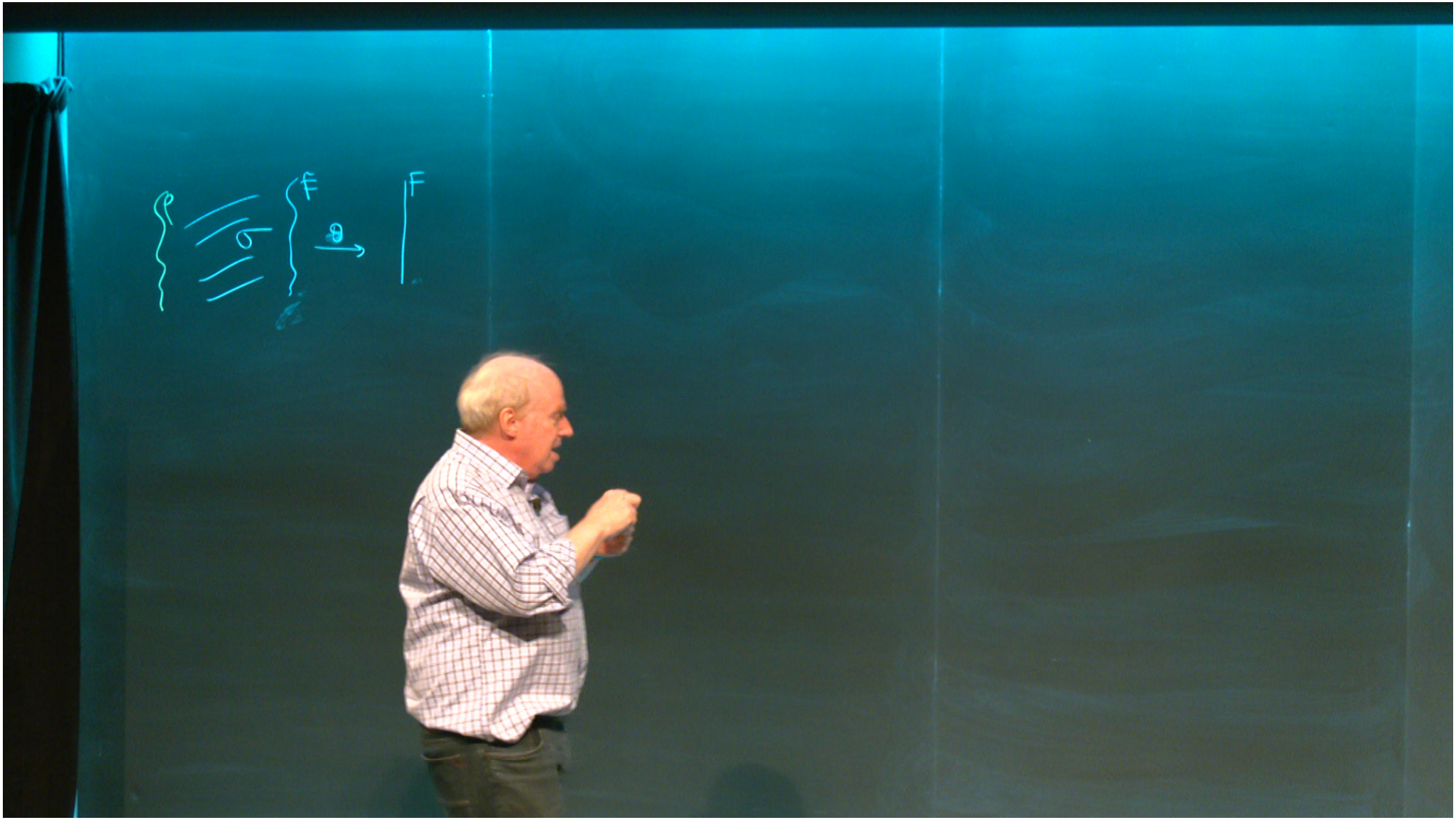
Title: Finite symmetry in QFT

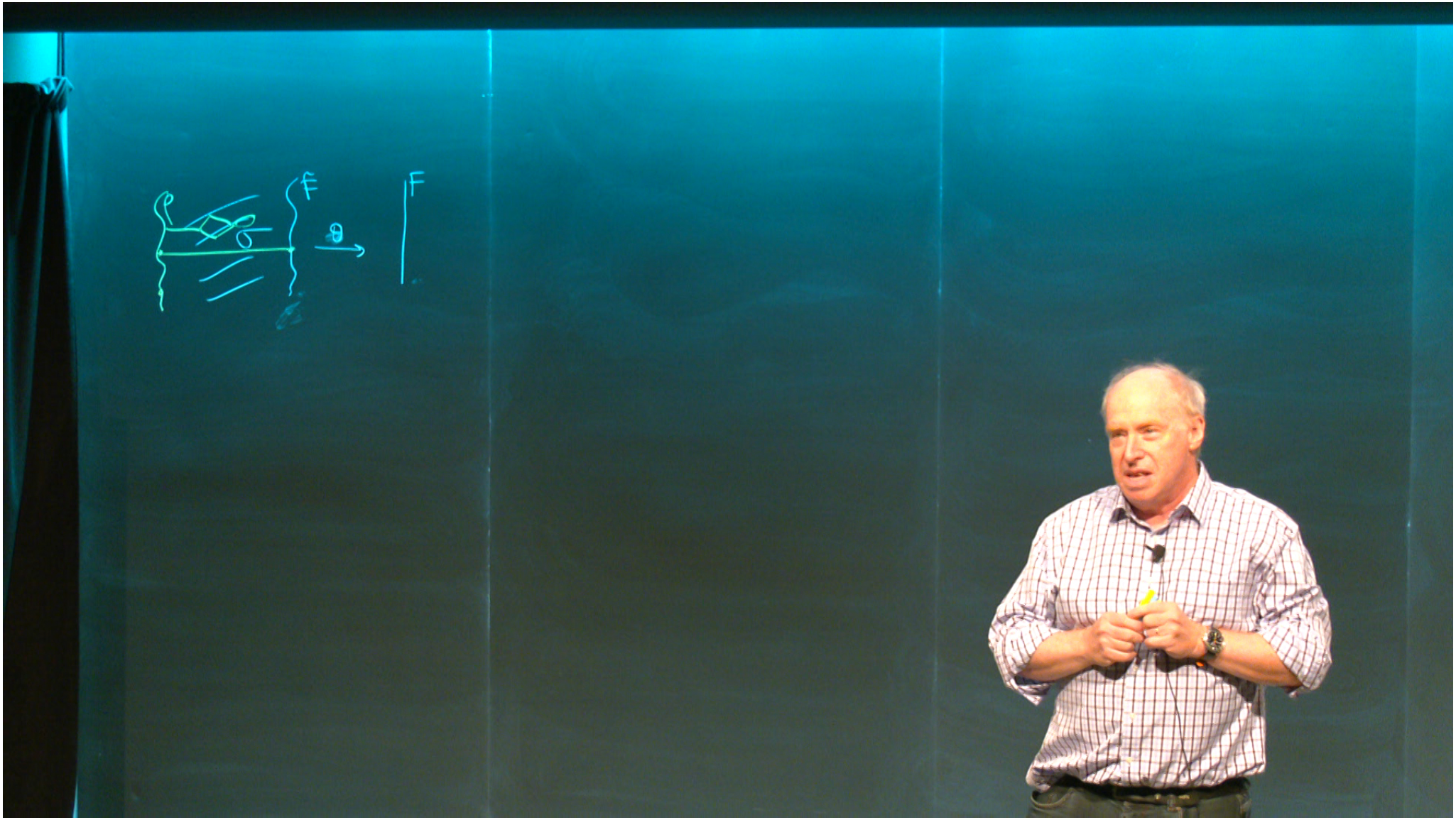
Speakers: Dan Freed

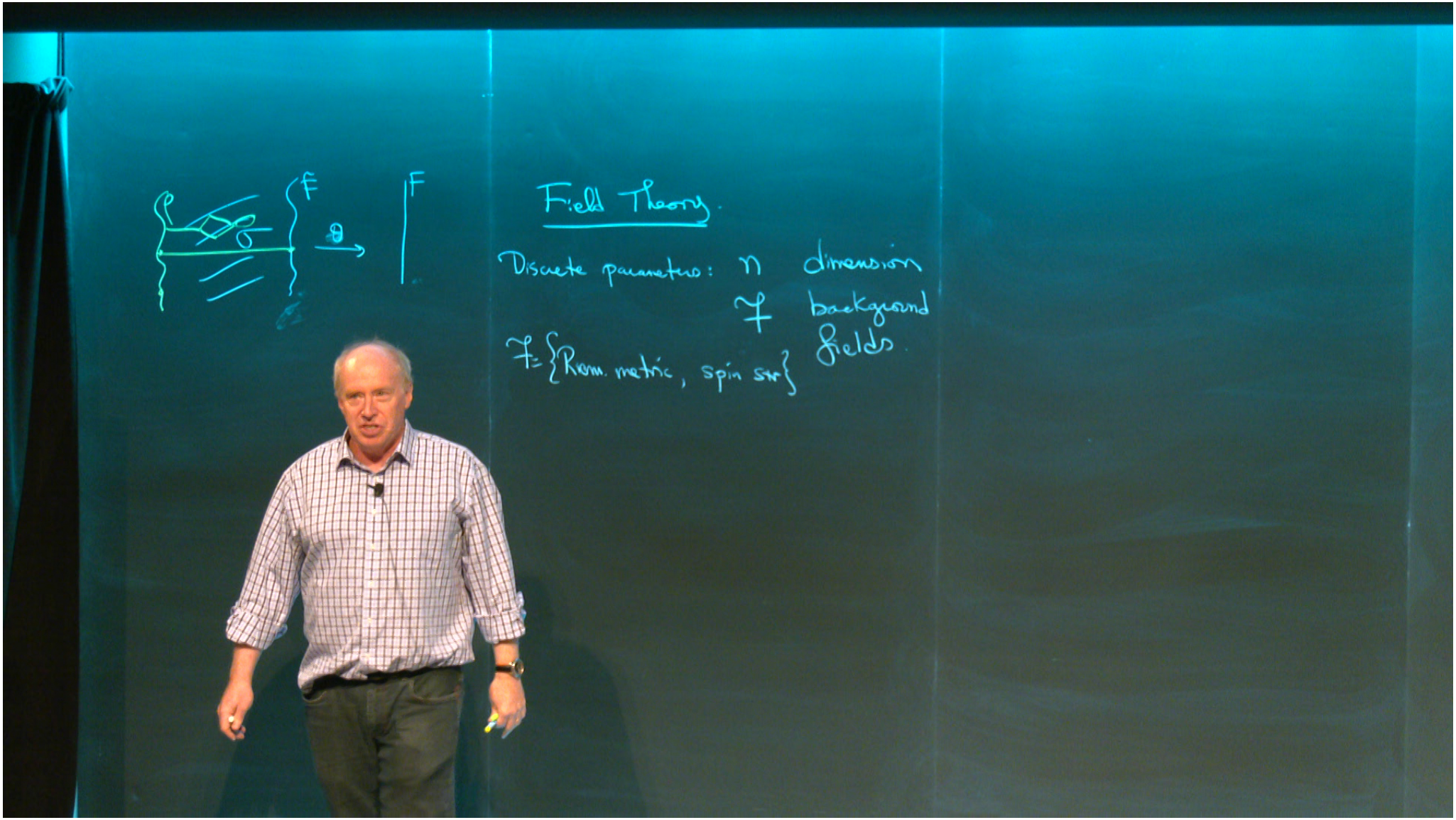
Collection: Global Categorical Symmetries

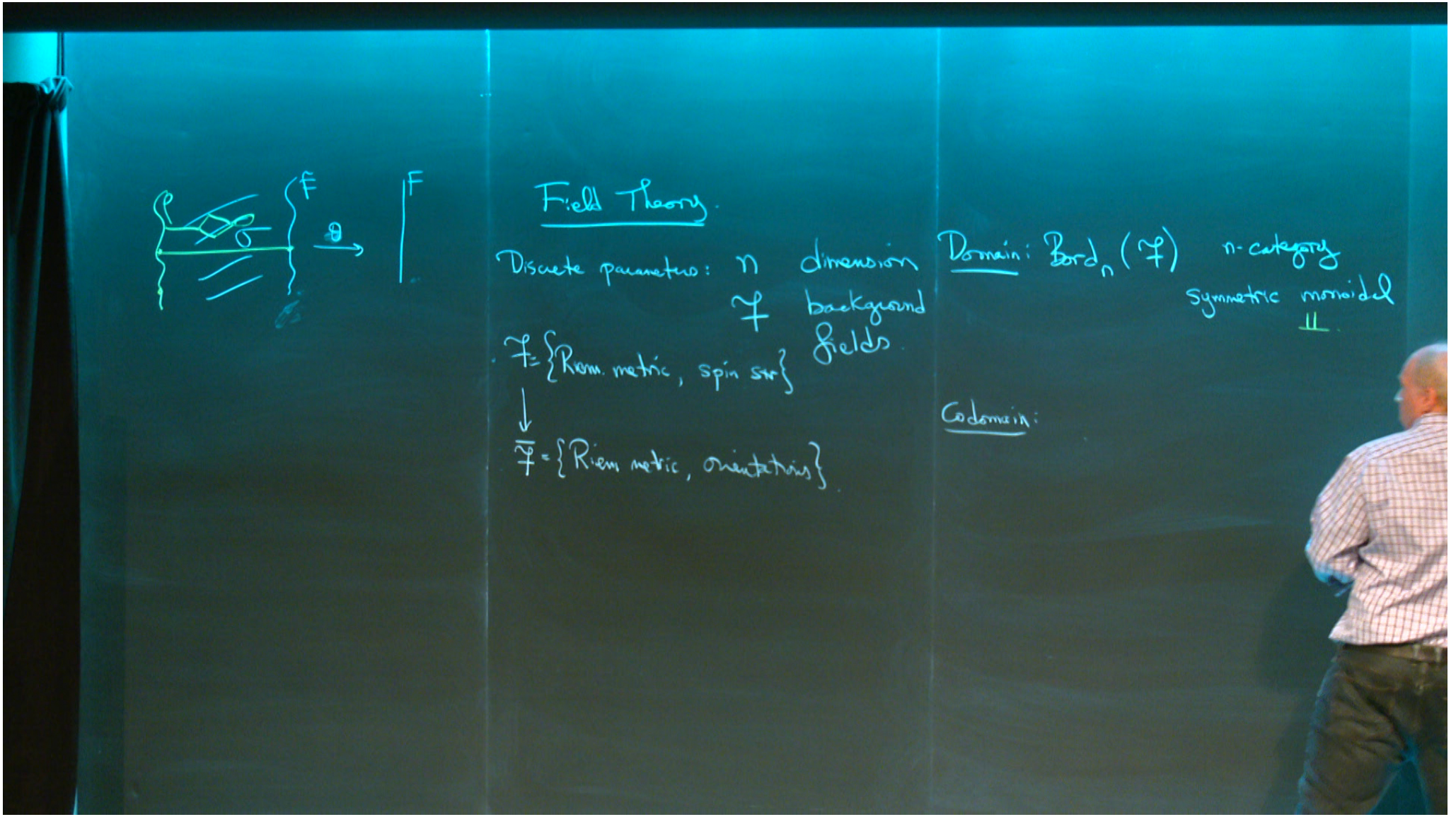
Date: June 14, 2022 - 9:00 AM

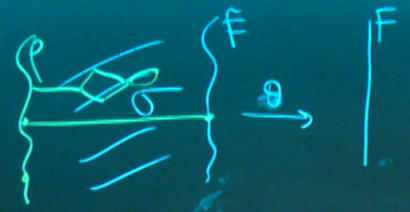
URL: https://pirsa.org/22060030


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Field Theory
Discrete parameters: n dimension
ℱ background fields
ℱ = {Riem. metric, spin str}


F
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Field Theory
Discrete parameters: n dimension
$\mathcal{Y}$ background fields
$\mathcal{Y} = \{$Riem. metric, spin str$\}$
$\overline{\mathcal{Y}} = \{$Riem metric, orientation$\}$
Domain: $\mathrm{Bord}_n(\mathcal{Y})$ n-category
Symmetric monoidal
Codomain:



Field Theory

Discrete parameters: n dimension
$\mathcal{Y}$ background fields

$\mathcal{Y} = \{$Riem. metric, spin str$\}$

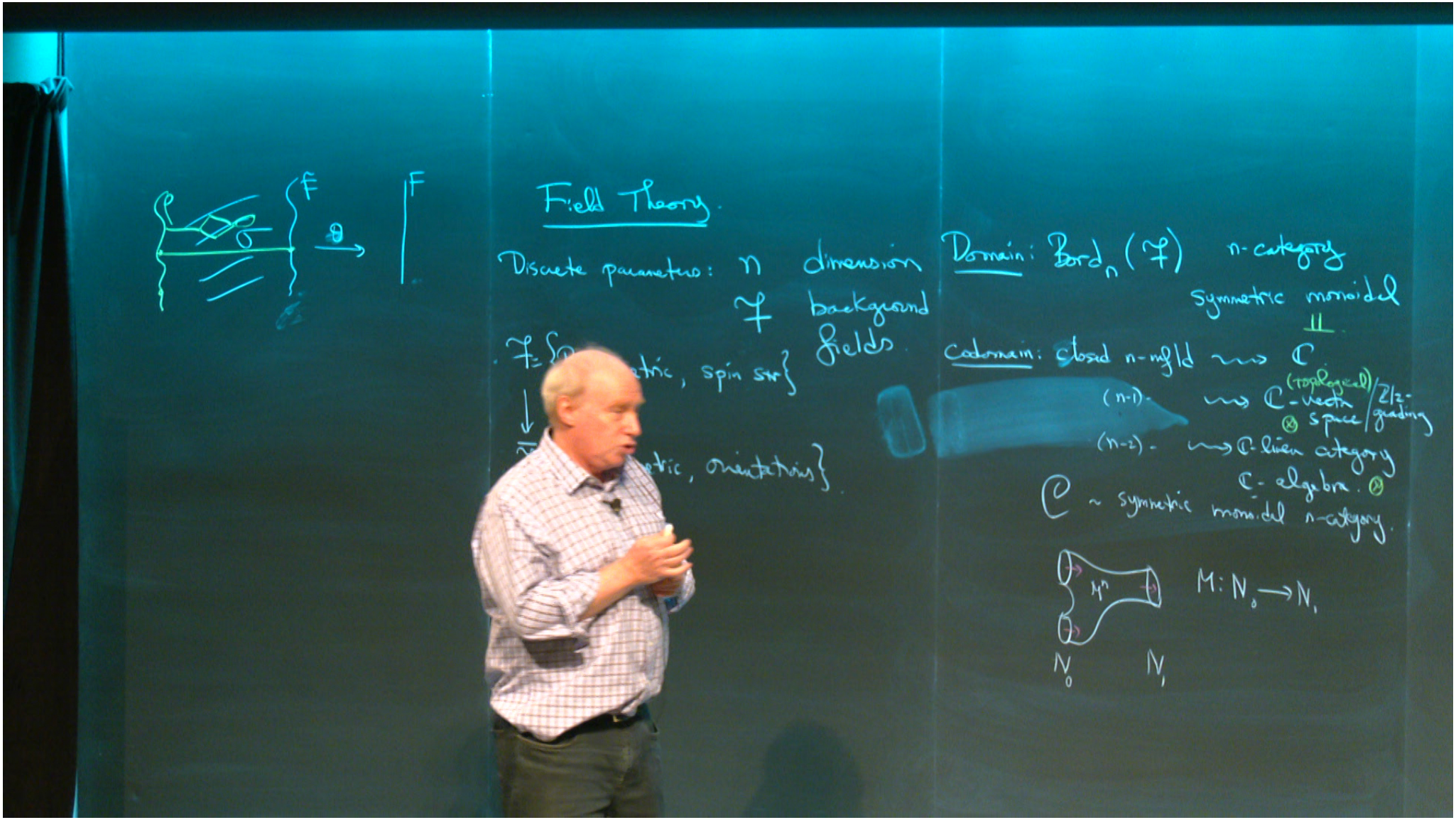
$\downarrow$

$\overline{\mathcal{Y}} = \{$Riem metric, orientation$\}$

Domain: $\mathrm{Bord}_n(\mathcal{Y})$ n-category
Symmetric monoidal

Codomain: closed n-mfld $\rightsquigarrow$ $\mathbb{C}$
$(n-1)$- $\rightsquigarrow$ $\mathbb{C}$-(topological) vect space $\otimes$ / $\mathbb{Z}/2$-grading
$(n-2)$- $\rightsquigarrow$ $\mathbb{C}$-linear category
$\mathbb{C}$-algebra $\otimes$

$\mathcal{C}$ ~ symmetric monoidal n-category.


F
F
Field Theory
Discrete parameters: n dimension
$\mathcal{Y}$ background fields
Domain: $\mathrm{Bord}_n(\mathcal{Y})$ n-category
Symmetric monoidal
Codomain: closed n-mfld
(topological)
(n-1)-
$\mathbb{C}$-vect space
$\mathbb{Z}/2$-grading
(n-2)-
$\mathbb{C}$-linear category
$\mathbb{C}$-algebra
$\mathcal{C}$ ~ symmetric monoidal n-category.
$M: N_0 \to N_1$
N_0
N_1

Domain: $\mathrm{Bord}_n(\mathcal{T})$ n-category

symmetric monoidal

$\amalg$

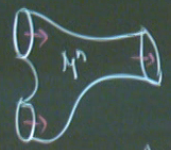
Codomain: closed n-mfld $\rightsquigarrow \mathbb{C}$

$(n-1)$- $\rightsquigarrow \mathbb{C}$ (topological) $\mathbb{C}$-vector space $\otimes$ / $\mathbb{Z}/2$-grading

$(n-2)$- $\rightsquigarrow \mathbb{C}$-linear category

$\mathbb{C}$-algebra $\otimes$

$\mathcal{C}$ ~ symmetric monoidal n-category.



$M: N_0 \to N_1$

Def$_n$: A field thy is a homomorphism

$F: \mathrm{Bord}_n(\mathcal{T}) \to \mathcal{C}$.



M^n closed

$\hat{M}: \amalg S_i \to \phi^{n-1}$ $\quad F(M) \in \mathbb{C}$

$F(\hat{M}): \otimes F(S_i) \to \mathbb{C}$ correlation fⁿ

Domain: $\mathrm{Bord}_n(\mathcal{Y})$ n-category

symmetric monoidal

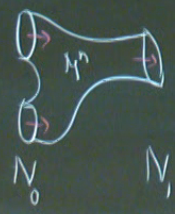
Codomain: closed n-mfld $\rightsquigarrow$ $\mathbb{C}$

$(n-1)$- $\rightsquigarrow$ $\mathbb{C}$-(topological) vect$_{\otimes}$ space / $\mathbb{Z}/2$-grading

$(n-2)$- $\rightsquigarrow$ $\mathbb{C}$-linear category

$\mathbb{C}$-algebra. $\otimes$

$\mathcal{C}$ ~ symmetric monoidal n-category.



$M: N_0 \longrightarrow N_1$

Defn: A field thy is a homomphism

$F: \mathrm{Bord}_n(\mathcal{Y}) \longrightarrow \mathcal{C}$.



M^n closed

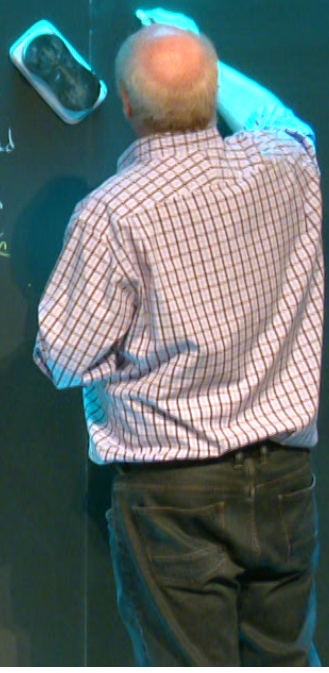
$\hat{M}: \amalg S_i \longrightarrow \phi^{n-1}$

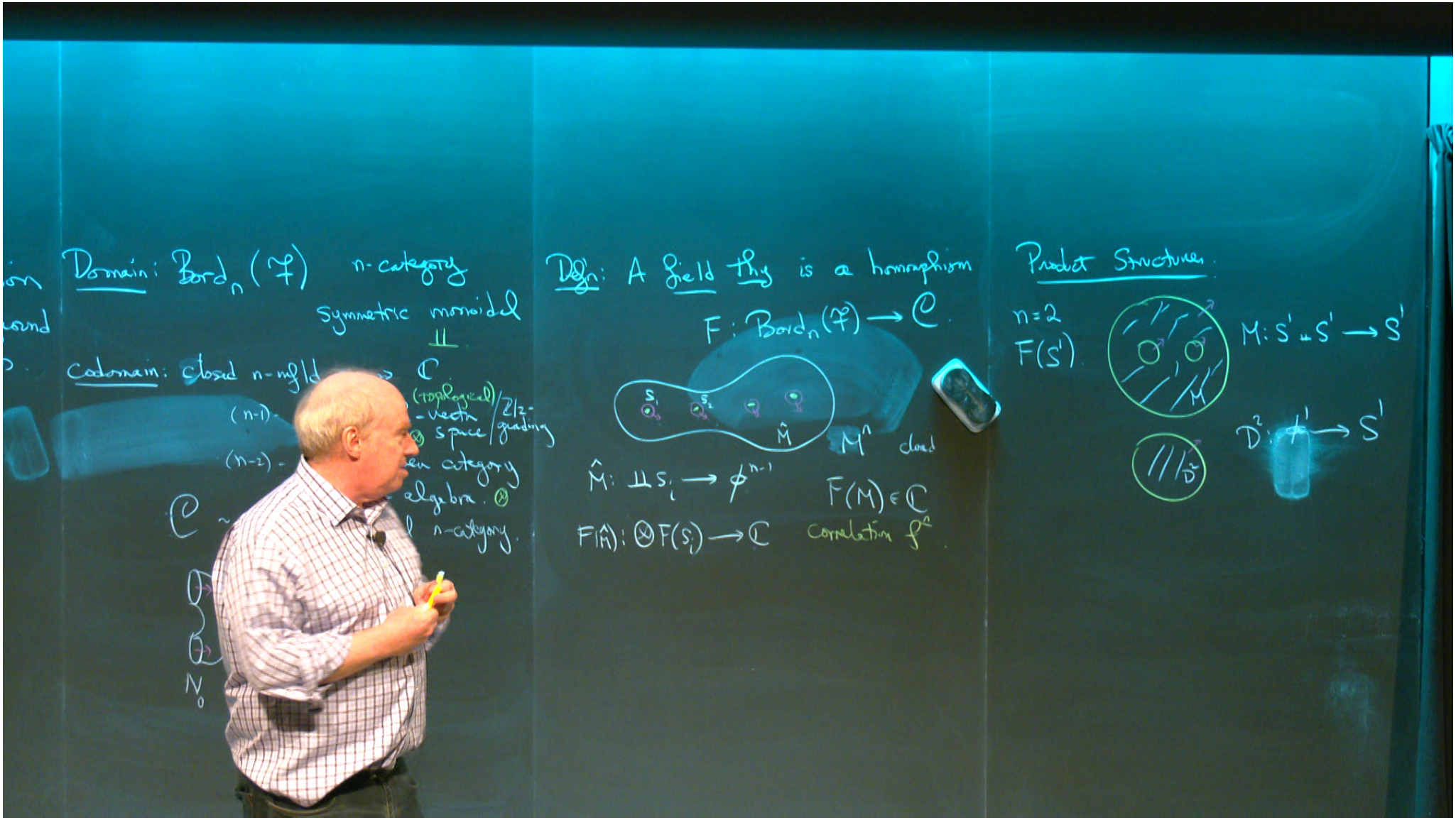
$F(M) \in \mathbb{C}$

$F(\hat{M}): \otimes F(S_i) \longrightarrow \mathbb{C}$ correlation f^n

Product Structure:

$n=2$




Domain: $\mathrm{Bord}_n(\mathcal{T})$ n-category
symmetric monoidal
$\amalg$
Codomain: closed n-mfld
$\mathbb{C}$
(topological) - vector space / $\mathbb{Z}/2$-grading
(n-1)-
(n-2)-
algebra.
n-category.
Defn: A field thy is a homomorphism
$F: \mathrm{Bord}_n(\mathcal{T}) \to \mathbb{C}$.
$\hat{M}: \amalg S_i \to \phi^{n-1}$
$F(\hat{M}): \otimes F(S_i) \to \mathbb{C}$
M^n closed
$F(M) \in \mathbb{C}$
correlation fn.
Product Structure.
n=2
$F(S^1)$
$M: S^1 \amalg S^1 \to S^1$
$D^2: \phi^1 \to S^1$

Domain: $\mathrm{Bord}_n(\mathcal{T})$ n-category

symmetric monoidal

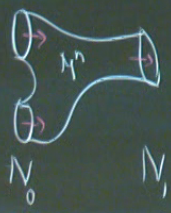
Codomain: closed n-mfld $\rightsquigarrow$ $\mathbb{C}$

$(n-1)$- $\rightsquigarrow$ $\mathbb{C}$ - (topological) vect$_{\mathbb{C}}$ $\otimes$ space / $\mathbb{Z}/2$-grading

$(n-2)$- $\rightsquigarrow$ $\mathbb{C}$-linear category

$\mathbb{C}$ - algebra $\otimes$

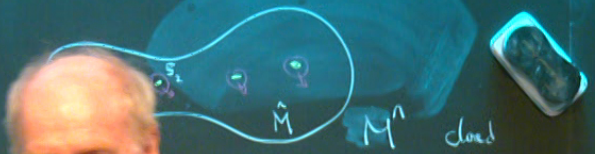
$\mathcal{C}$ ~ symmetric monoidal n-category.



$M: N_0 \longrightarrow N_1$

Defn: A field thy is a homomphism

$F: \mathrm{Bord}_n(\mathcal{T}) \longrightarrow \mathcal{C}$.



M^n closed

$\hat{M} \longrightarrow \phi^{n-1}$

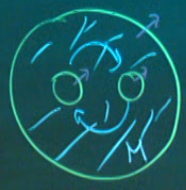
$F(M) \in \mathbb{C}$

correlation f^n.

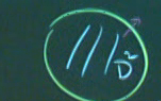
Product Structures

$n=2$

$F(S^1)$



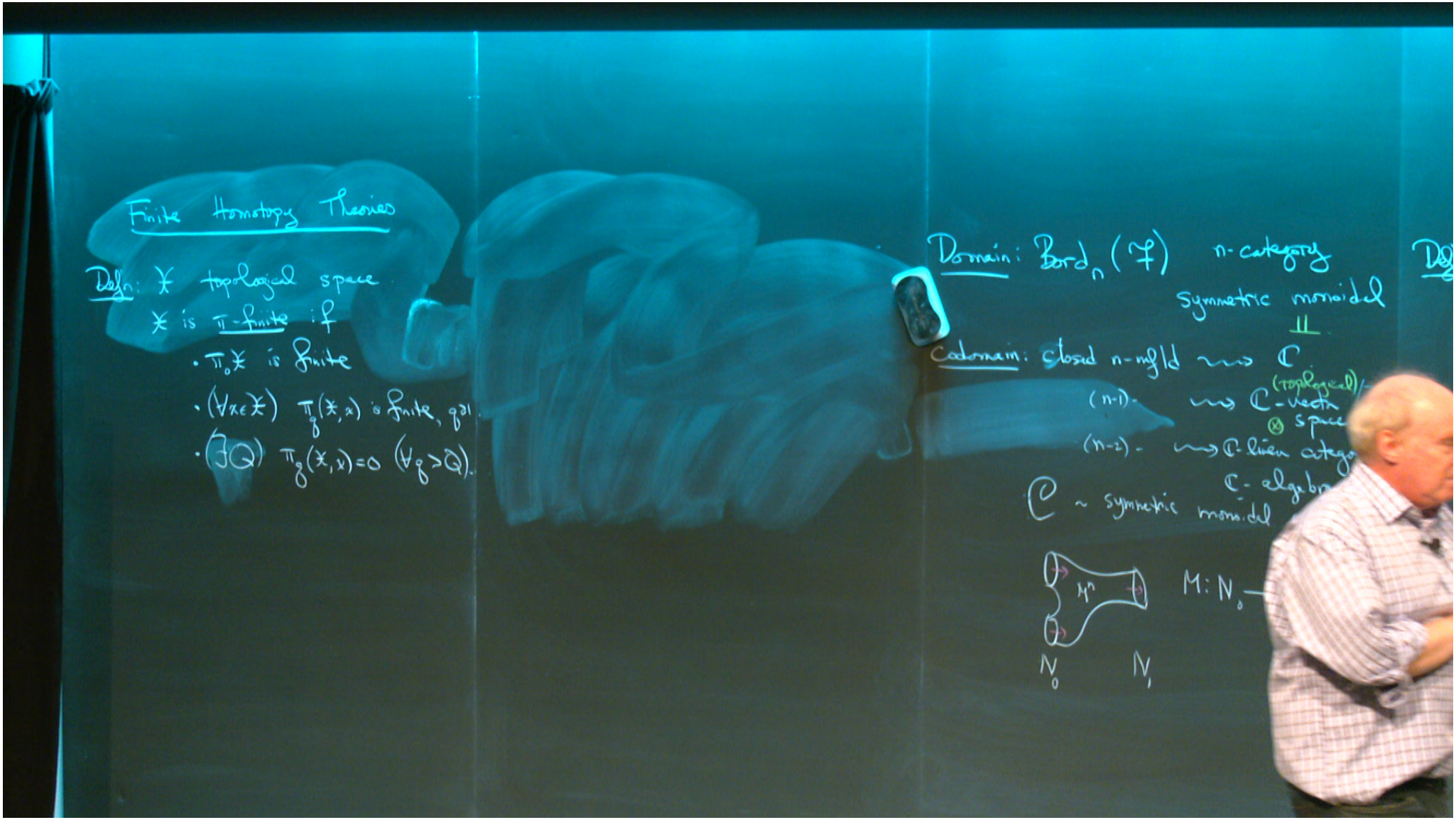
$M: S^1 \amalg S^1 \longrightarrow S^1$

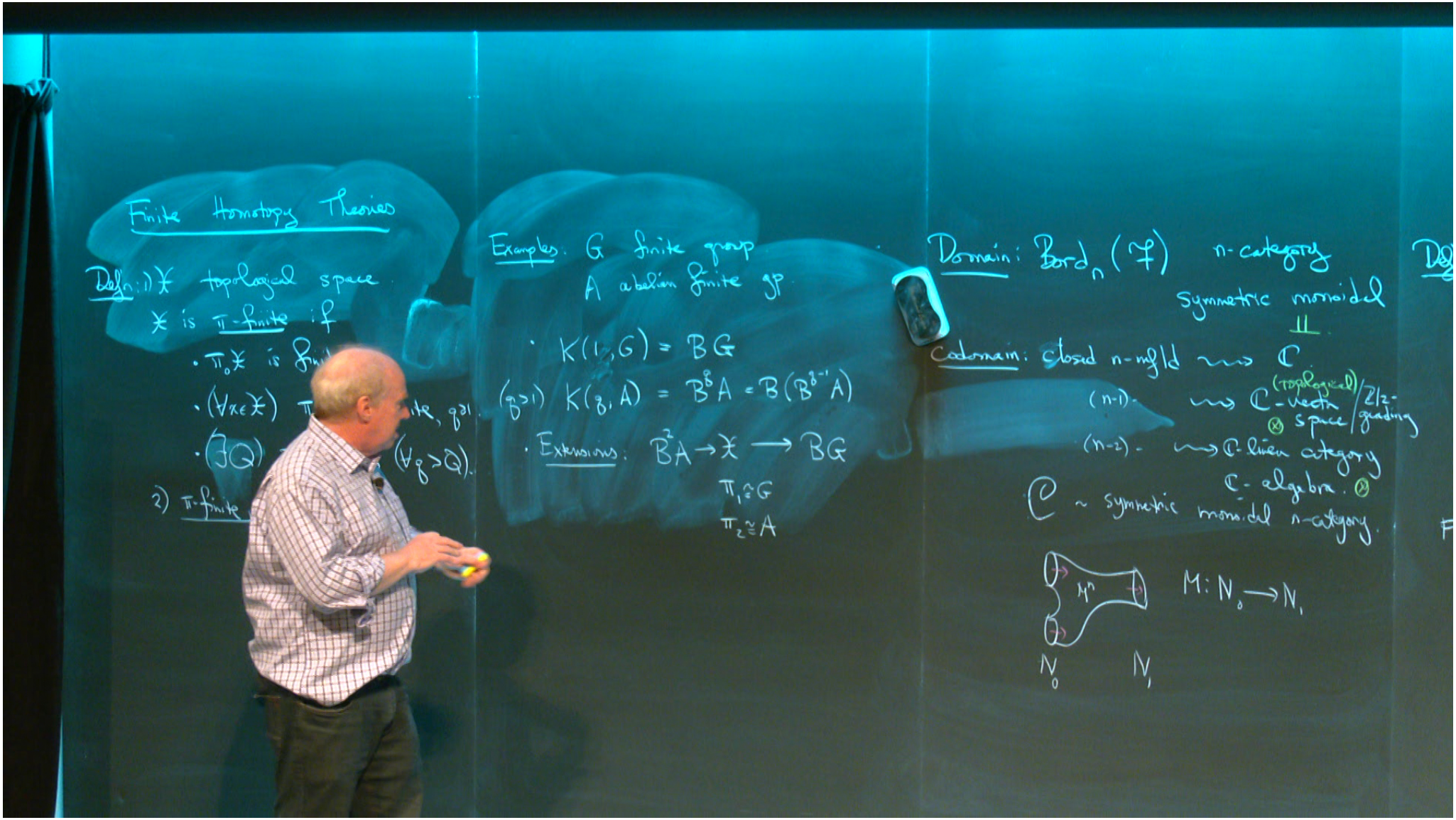


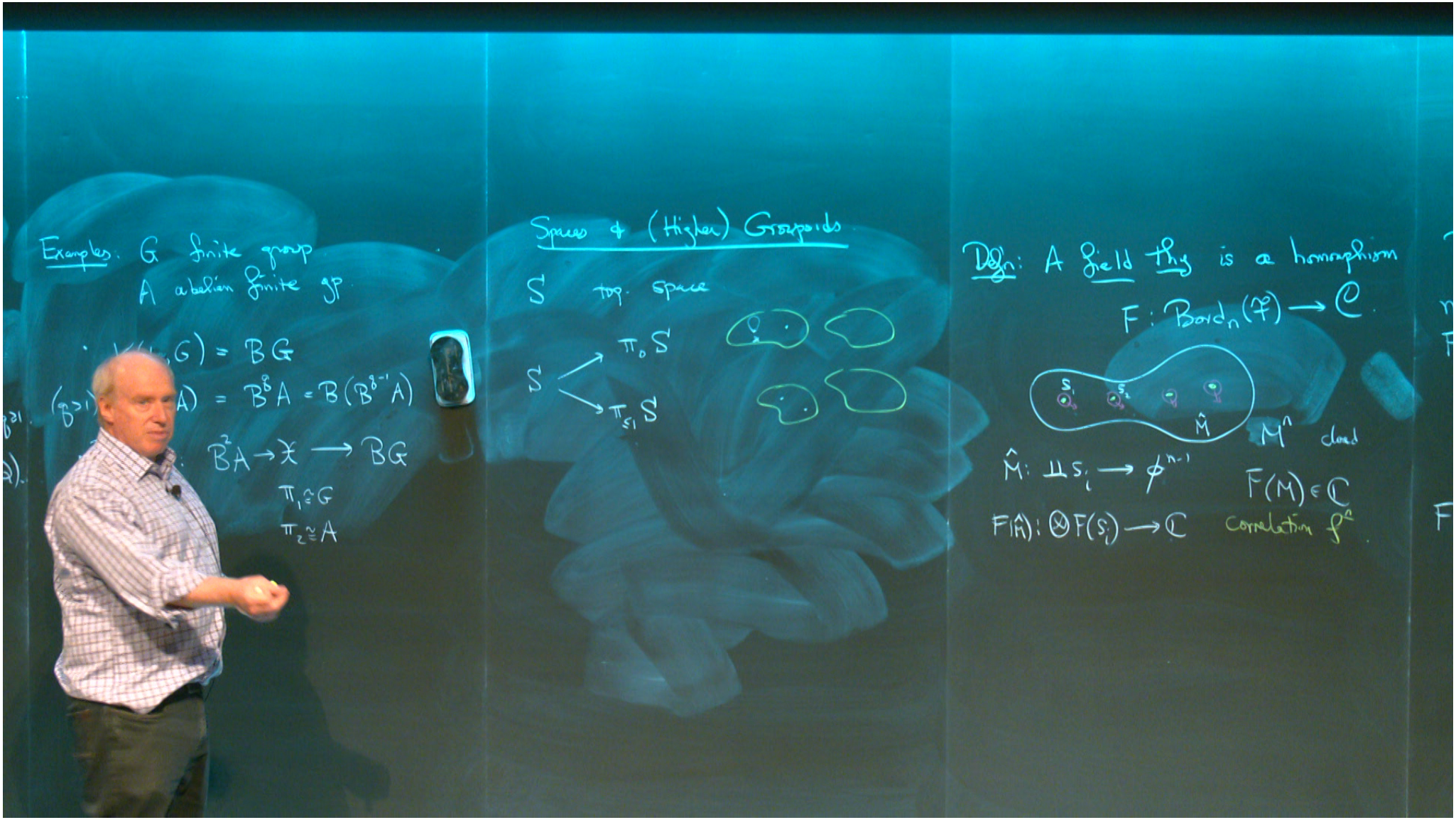
$D^2: I^1 \longrightarrow S^1$

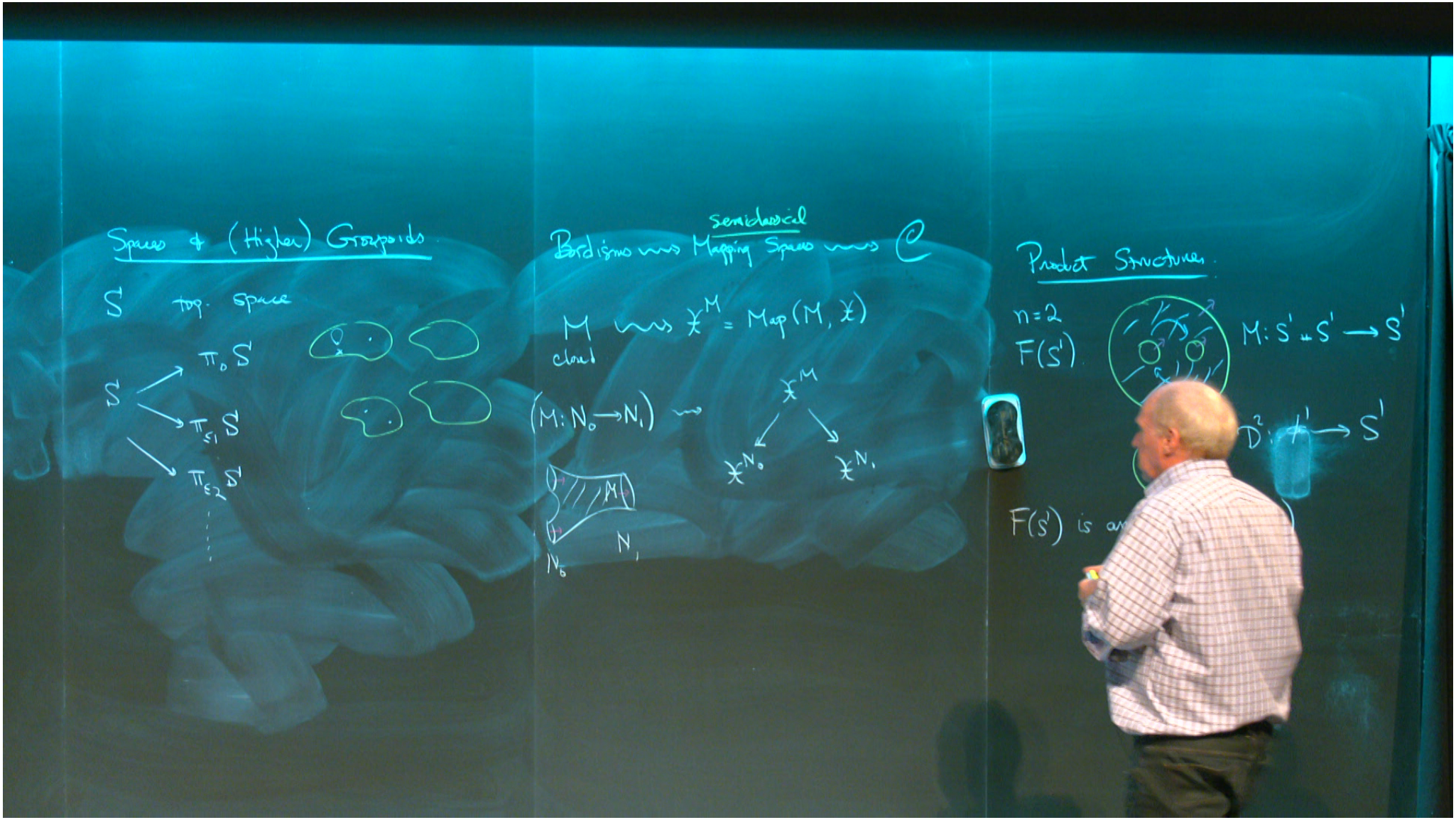
$F(S^1)$ is an algebra (in Vect)

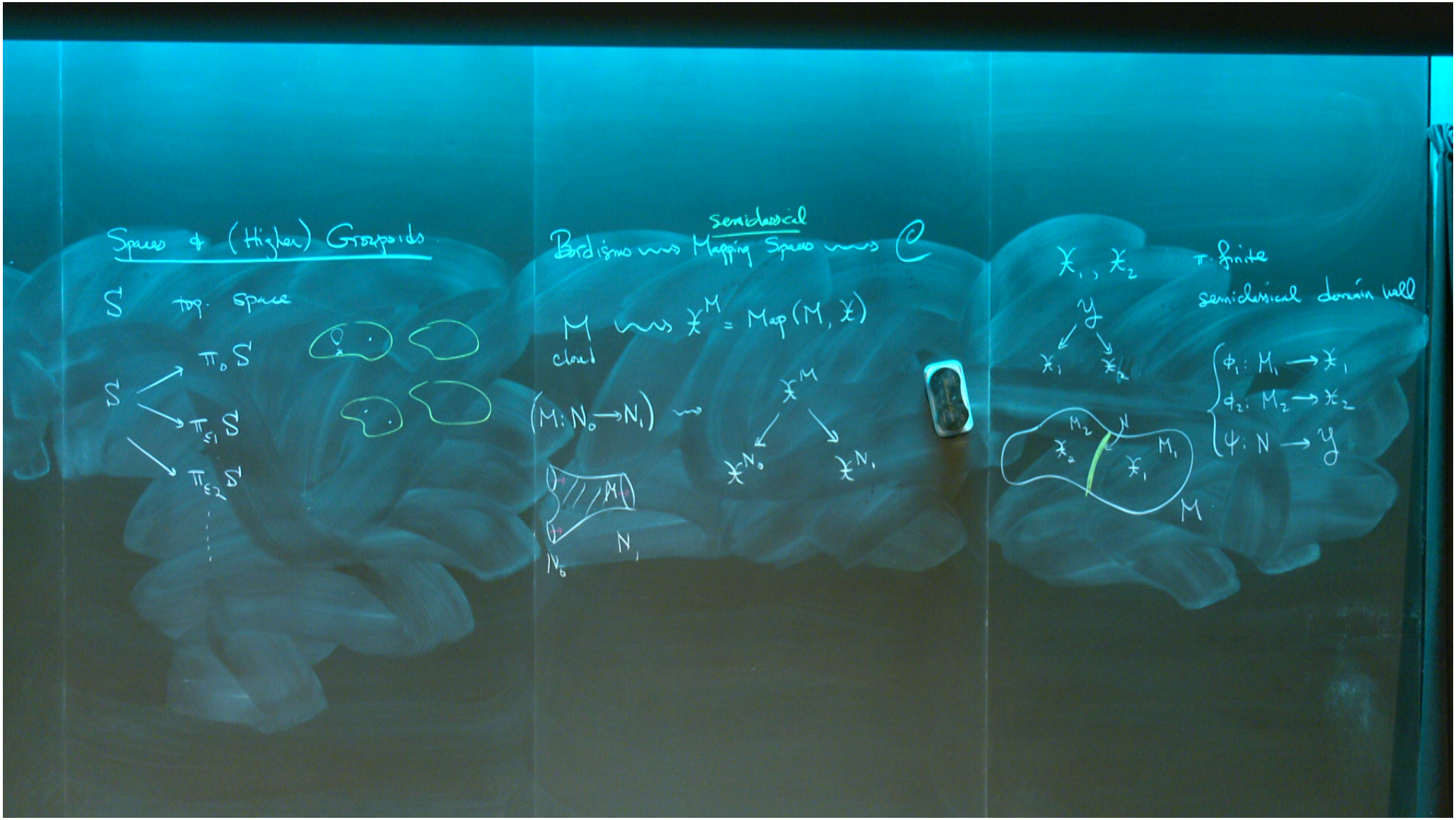
Commutative


Finite Homotopy Theories
Defn: $\mathcal{X}$ topological space
$\mathcal{X}$ is π-finite if
• $\pi_0\mathcal{X}$ is finite
• $(\forall x \in \mathcal{X})$ $\pi_q(\mathcal{X}, x)$ is finite, $q \geq 1$
• $(\exists Q)$ $\pi_q(\mathcal{X}, x) = 0$ $(\forall q > Q)$.
Domain: $\mathrm{Bord}_n(\mathcal{T})$ n-category
Symmetric monoidal
Codomain: closed n-mfld
(n-1)-
(n-2)-
(topological)
$\mathbb{C}$-vect
$\mathbb{C}$-algebra
$\mathcal{C}$ ~ symmetric monoidal
$M: N_0$
N_0
N_1


Finite Homotopy Theories
Defn: 1) $\mathcal{X}$ topological space
$\mathcal{X}$ is π-finite if
• $\pi_0\mathcal{X}$ is fi
• $(\forall x \in \mathcal{X})$ π
• $(\exists Q)$
2) π-finite
Examples: G finite group
A abelian finite gp
• $K(1,G) = BG$
$(q>1)$ $K(q,A) = B^q A = B(B^{q-1}A)$
• Extensions: $B^2A \to \mathcal{X} \to BG$
$\pi_1 \cong G$
$\pi_2 \cong A$
Domain: $\mathrm{Bord}_n(\mathcal{T})$ n-category
Symmetric monoidal
Codomain: closed n-mfld $\rightsquigarrow$ $\mathbb{C}$
$(n-1)$- $\rightsquigarrow$ $\mathbb{C}$-(topological) vect space / $\mathbb{Z}/2$-grading
$(n-2)$- $\rightsquigarrow$ $\mathbb{C}$-linear category
$\mathbb{C}$-algebra
$\mathcal{C}$ ~ symmetric monoidal n-category.
$M: N_0 \to N_1$
N_0 N_1


Examples: G finite group
A abelian finite gp
$B^2A \to \mathcal{X} \to BG$
$\pi_1 \cong G$
$\pi_2 \cong A$
Spaces & (Higher) Groupoids
S top. space
$\pi_0 S$
$\pi_{\leq 1} S$
Defn: A field thy is a homomorphism
$F: Bord_n(\mathcal{T}) \to \mathbb{C}$
$\hat{M}: \coprod S_i \to \phi^{n-1}$
M^n closed
$F(M) \in \mathbb{C}$
$F(\hat{M}): \otimes F(S_i) \to \mathbb{C}$
correlation fns


Spaces & (Higher) Groupoids
S top. Space
$\pi_0 S$
$\pi_{\leq 1} S$
$\pi_{\leq 2} S$
Bordisms ⇝ Mapping Spaces ⇝ C
semiclassical
M ⇝ $\mathcal{X}^M$ = Map(M, $\mathcal{X}$)
closed
$(M: N_0 \to N_1)$ ⇝
$\mathcal{X}^M$
$\mathcal{X}^{N_0}$
$\mathcal{X}^{N_1}$
M
N_1
N_0
Product Structures
n=2
F(S')
$M: S' \amalg S' \to S'$
$D^2: \emptyset' \to S'$
F(S') is an


Spaces & (Higher) Groupoids
S top. space
$\pi_0 S$
$\pi_{\leq 1} S$
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Bordisms ⇝ Mapping Spaces ⇝ C
semiclassical
M closed
$\mathcal{X}^M = \mathrm{Map}(M, \mathcal{X})$
$(M: N_0 \to N_1)$ ⇝
$\mathcal{X}^M$
$\mathcal{X}^{N_0}$
$\mathcal{X}^{N_1}$
M
N_1
N_0
$\mathcal{X}_1, \mathcal{X}_2$
$\mathcal{Y}$
π-finite
semiclassical domain wall
$\phi_1: M_1 \to \mathcal{X}_1$
$\phi_2: M_2 \to \mathcal{X}_2$
$\psi: N \to \mathcal{Y}$
M_2
N
M_1
M

$\mathcal{X} = BG$

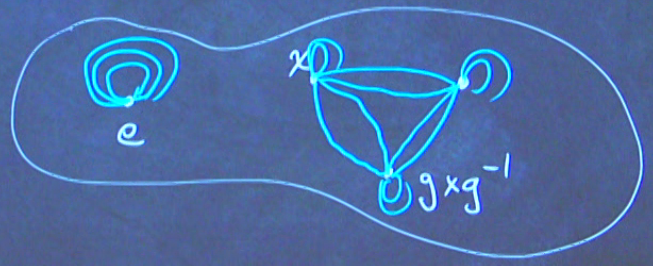
$\mathcal{X}^M \sim \mathrm{Bun}_G(M)$

$\pi_0 =$ iso classes of principal G-bundles $P \downarrow M$

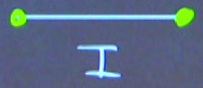
$\pi_1 \text{ at } P = \mathrm{Aut}_0\begin{pmatrix} P \\ \downarrow \\ M \end{pmatrix}$ gauge transformations

Ex: $M = S^1$

$\mathcal{X}^{S^1} = \mathcal{L}\mathcal{X} \sim \mathrm{Bun}_G(S^1) \sim G/\!/G$



$* \downarrow BG$ (semiclassical) regular boundary thy



$\mathcal{X}^M = \mathrm{Map}\left((I, \partial I), (\mathcal{X}, *)\right)$

$= \Omega\mathcal{X}$

$\sim G$

$\mathcal{X} = BG$

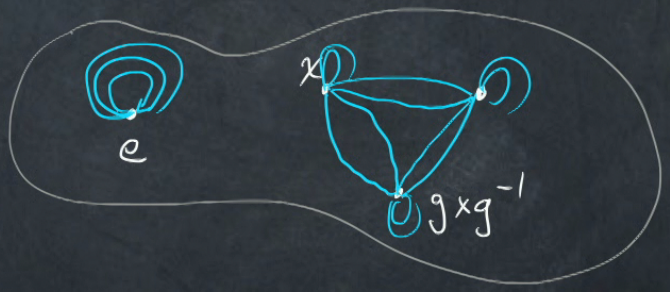
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$\pi_1 \mathrm{at}\, P = \mathrm{Aut}_0\begin{pmatrix} P \\ \downarrow \\ M \end{pmatrix}$ gauge transformations

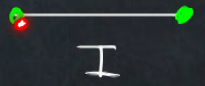
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$= \Omega\mathcal{X}$

$\sim G$

Quantization

M closed

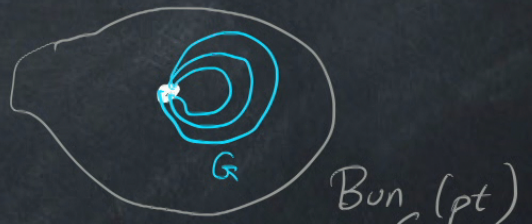
Codim	$\sigma_{\mathcal{X}}(M)$
0	"$\#\mathcal{X}^M$" $= \sum_{[\phi]\in\pi_0\mathcal{X}^M} \frac{\#\pi_2(\mathcal{X}^M,\phi)}{\#\pi_1(\mathcal{X}^M,\phi)} \frac{\#\pi_4(\mathcal{X}^M,\phi)}{\#\pi_3(\mathcal{X}^M,\phi)} \cdots$
1	$\mathrm{Fun}(\mathcal{X}^M) = \mathrm{Fun}(\pi_0\mathcal{X}^M)$
2	$\mathrm{Vect}(\mathcal{X}^M) = \mathrm{Vect}(\pi_0\mathcal{X}^M)$

Ex: $m = 2$

$\mathcal{X} = BG$

$\sigma(pt)$: $\mathcal{X}^{pt} = \mathcal{X} \sim */\!/G$

$\mathrm{Vect}(*/\!/G) = \mathrm{Rep}\,G$



$\mathrm{Bun}_G(pt)$

$\sigma(S^1)$: $\mathcal{X}^{S^1} = \mathcal{L}\mathcal{X} \sim G/\!/G$

$\mathrm{Fun}(G/\!/G) = \mathrm{Fun}(G)^G$

central functions on G.

$\sigma($ —— $)$: $\mathcal{X}^{\text{——}} = \Omega\mathcal{X} \sim G$

$\mathrm{Fun}(G)$

all functions on G

$\mathcal{X} = BG$

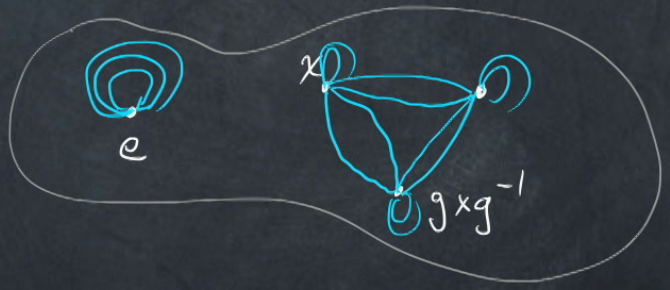
$\mathcal{X}^M \sim \mathrm{Bun}_G(M)$

π_0 = iso classes of principal G-bundles $\quad P \downarrow M$

$\pi_1 \mathrm{Aut}_P = \mathrm{Aut}_0\begin{pmatrix} P \\ \downarrow \\ M \end{pmatrix}$ gauge transformations

Ex: $M = S^1$

$\mathcal{X}^{S^1} = \mathcal{L}\mathcal{X} \sim \mathrm{Bun}_G(S^1) \sim G/\!/G$



$* \downarrow BG$ (semiclassical) regular boundary thy



$\mathcal{X}^M = \mathrm{Map}\left((I, \partial I), (\mathcal{X}, *)\right)$

$= \Omega\mathcal{X}$

$\sim G$

Quantization

M closed

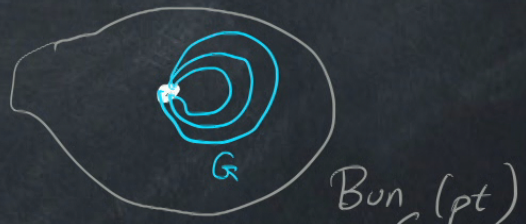
Codim	$\sigma_{\mathcal{X}}(M)$
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1	$\mathrm{Fun}(\mathcal{X}^M) = \mathrm{Fun}(\pi_0\mathcal{X}^M)$
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Ex: $n = 2$

$\mathcal{X} = BG$

$\sigma(pt)$: $\mathcal{X}^{pt} = \mathcal{X} \sim */\!/G$

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$\mathrm{Fun}(G/\!/G) = \mathrm{Fun}(G)^G$

central functions on G.

$\sigma(\longrightarrow)$: $\mathcal{X}^{\longrightarrow} = \Omega\mathcal{X} \sim G$

$\mathrm{Fun}(G)$

all functions on G

$\mathcal{X} = B^2 A$ (A finite abelian)

$m = 5$

M^3 closed

$\pi_0 \mathcal{X}^M = H^2(M;A) \ni m$

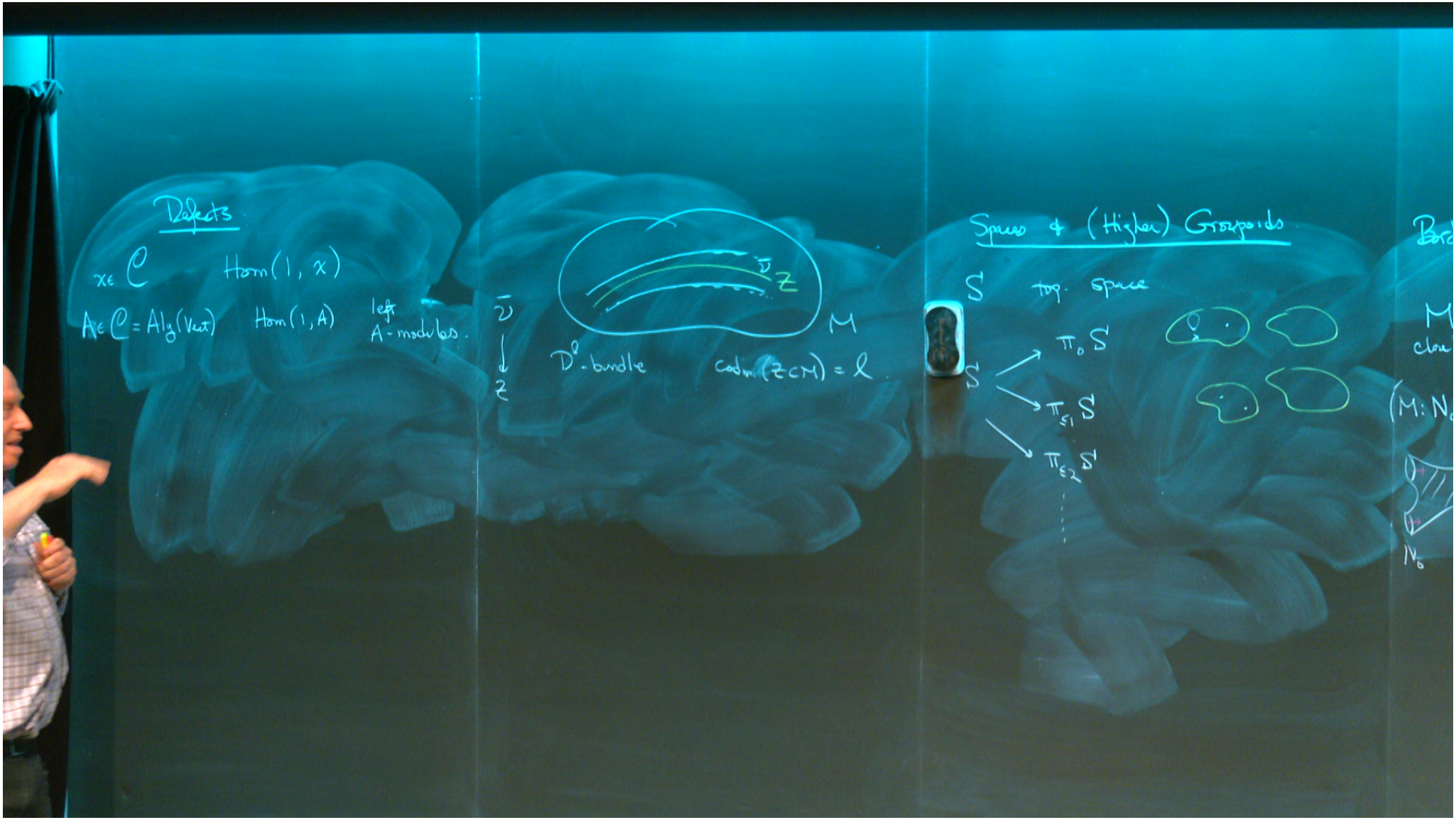
$\pi_1 \mathcal{X}^M = H^1(M;A)$

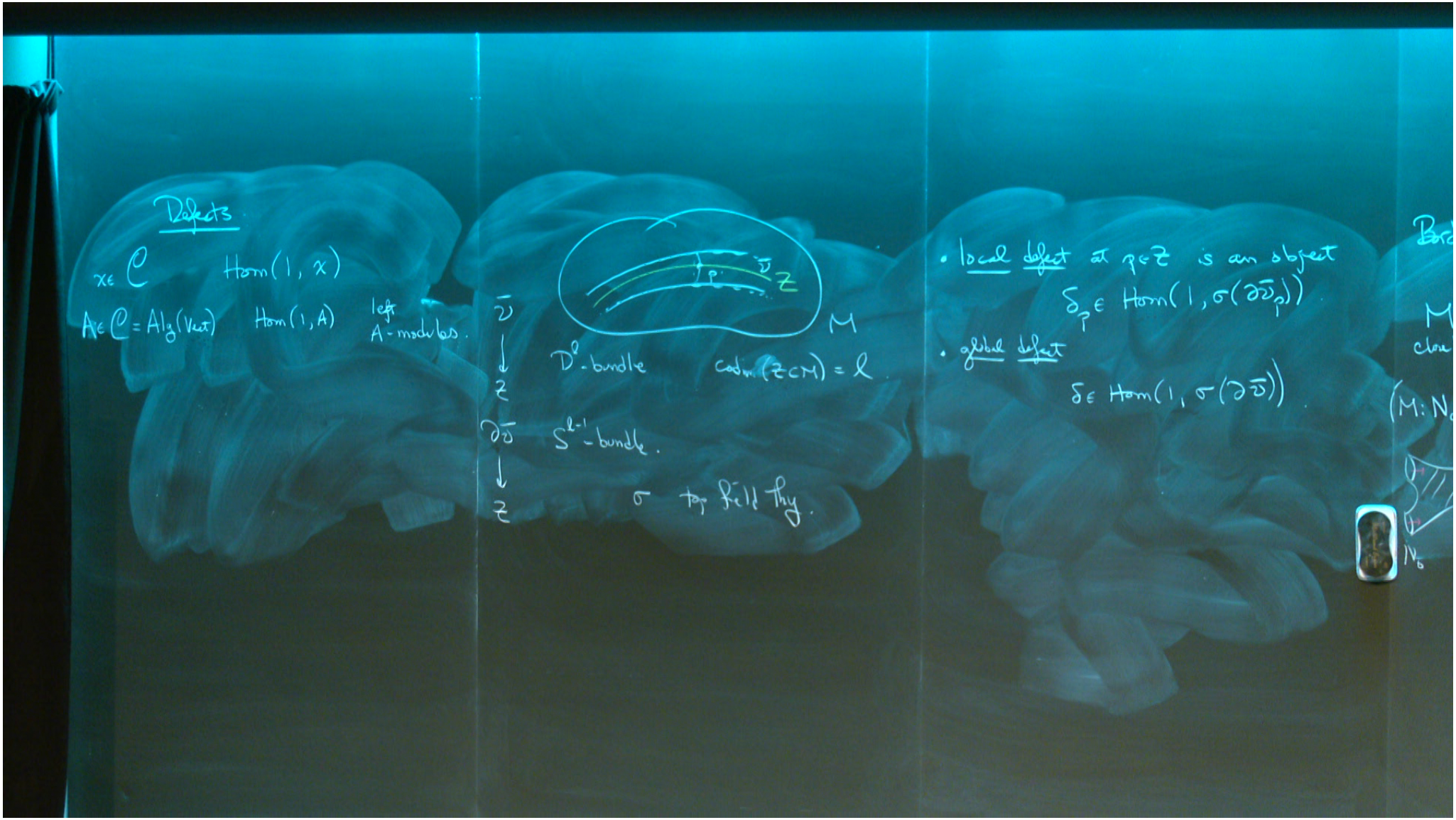


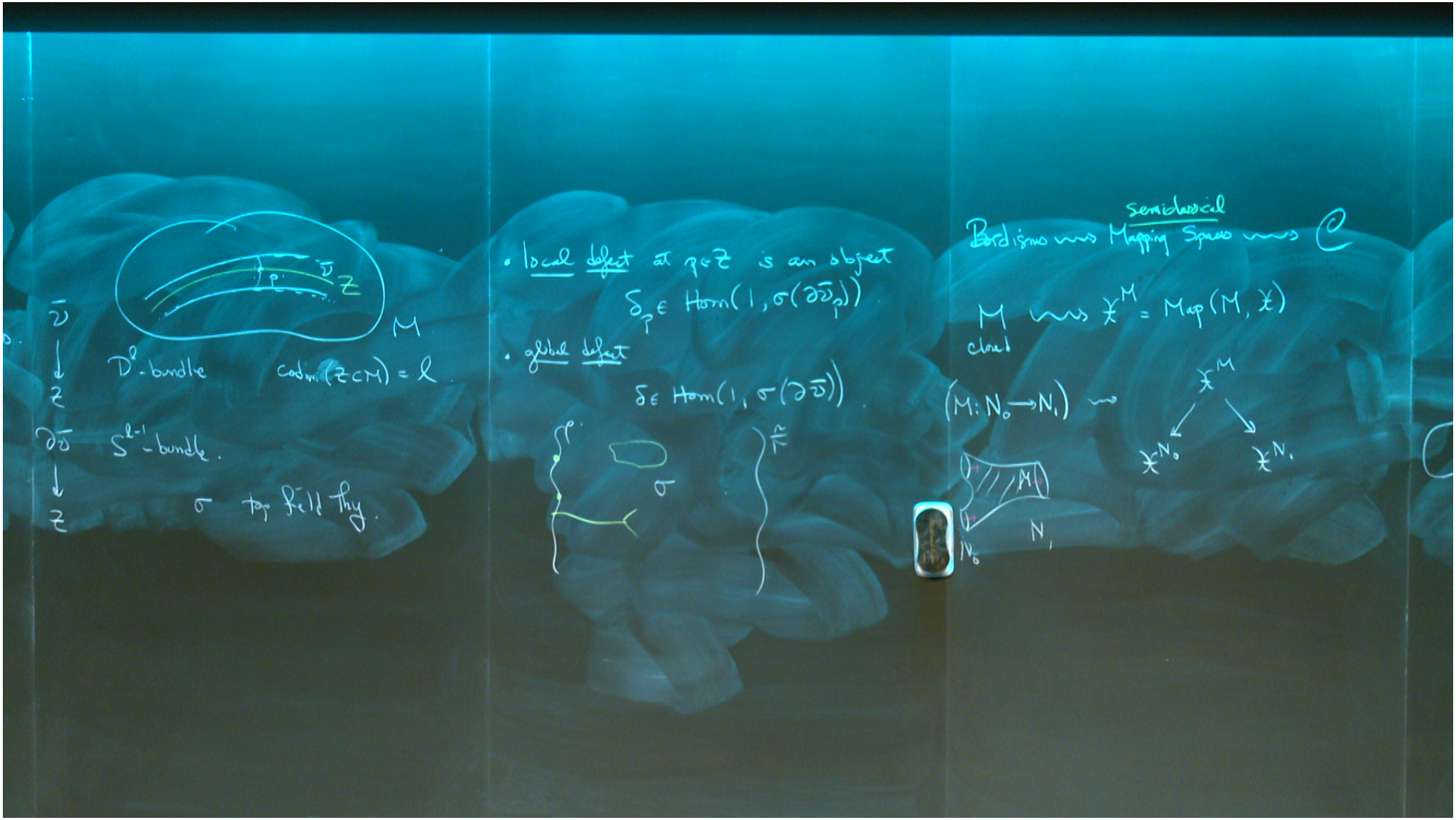
$\pi_{\leq 1} \mathcal{X}^M$

$$W_m = \bigoplus_{e \in H^1(M;A)^\vee} W_{m,e} \otimes \mathbb{C}_e$$

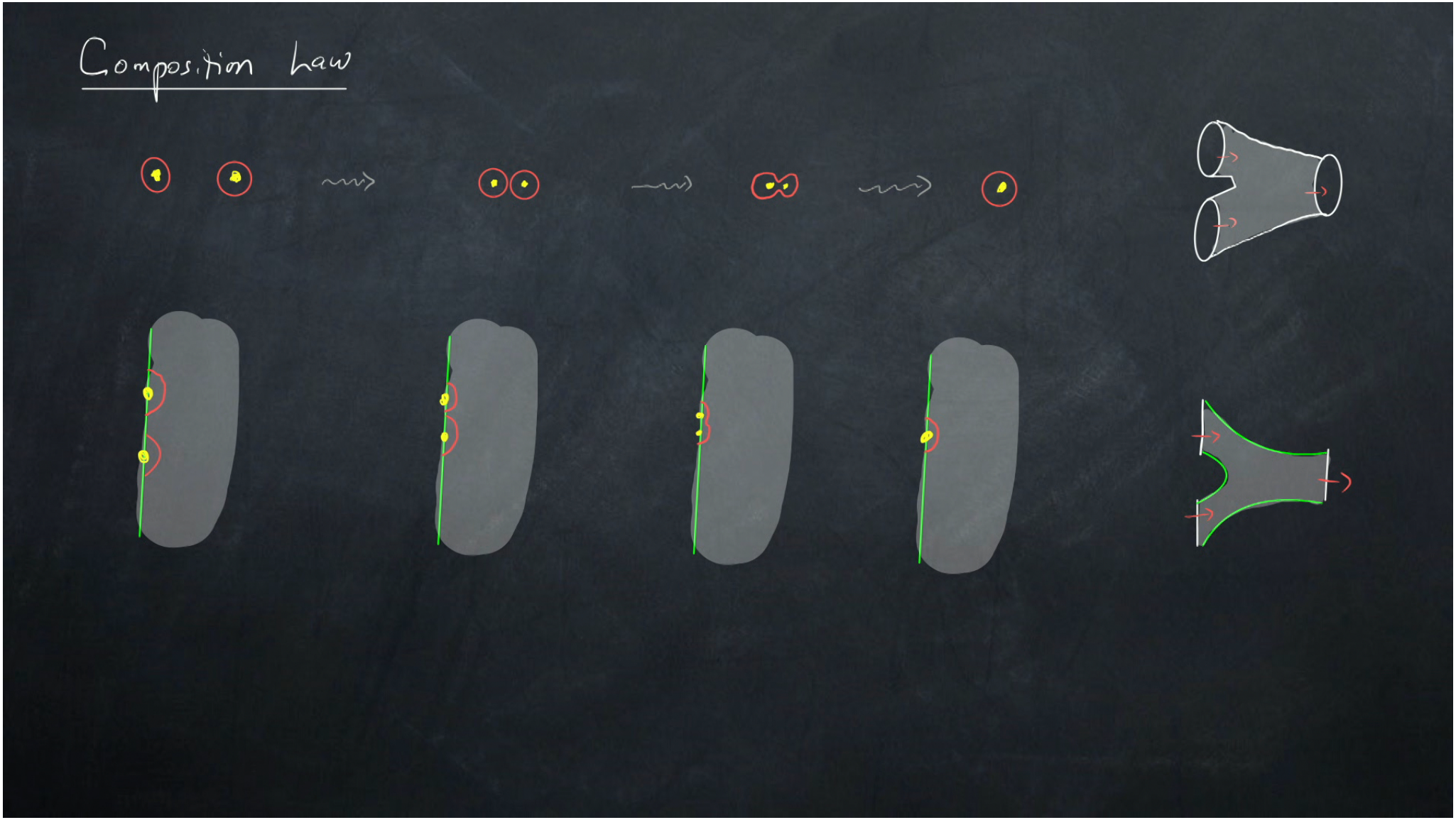
$\sigma(M)$ has simple objects labeled by $(m,e) \in H^2(M;A) \times H^1(M;A)^\vee$

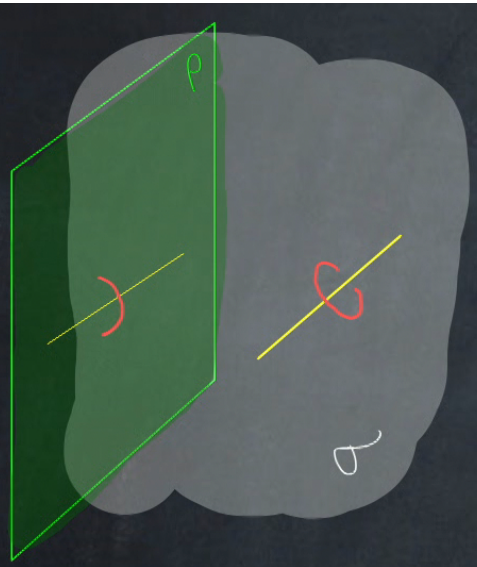

Defects
$x \in \mathcal{C}$
$\mathrm{Hom}(1, x)$
$A \in \mathcal{C} = \mathrm{Alg}_1(\mathrm{Vect})$
$\mathrm{Hom}(1, A)$
left
A-modules.
$\bar{v}$
Z
D^ℓ-bundle
M
$\mathrm{codim}(Z \subset M) = \ell$
Spaces & (Higher) Groupoids
S
top. space
$\pi_0 S$
$\pi_{S^1} S$
$\pi_{S^2} S$


Defects
$x \in \mathcal{C}$ $\mathrm{Hom}(1, x)$
$A \in \mathcal{C} = \mathrm{Alg}(\mathrm{Vect})$ $\mathrm{Hom}(1, A)$ left A-modules.
$\bar{v}$
Z
$\partial\bar{v}$
Z
M
D^ℓ-bundle $\mathrm{codim}(Z \subset M) = \ell$
$S^{\ell-1}$-bundle.
σ top field thy.
• local defect at $p \in Z$ is an object
$\delta_p \in \mathrm{Hom}(1, \sigma(\partial\bar{v}_p))$
• global defect
$\delta \in \mathrm{Hom}(1, \sigma(\partial\bar{v}))$


$\bar{v}$
Z
M
D^ℓ-bundle
$\mathrm{codim}(Z \subset M) = \ell$
$\partial\bar{v}$
$S^{\ell-1}$-bundle.
σ top field thy.
local defect at $p \in Z$ is an object
$\delta_p \in \mathrm{Hom}(1, \sigma(\partial\bar{v}_p))$
global defect
$\delta \in \mathrm{Hom}(1, \sigma(\partial\bar{v}))$
σ
$\tilde{Z}$
Bordisms
semiclassical
Mapping Spaces
$\mathcal{C}$
M
cloud
$\mathcal{X}^M = \mathrm{Map}(M, \mathcal{X})$
$(M: N_0 \to N_1)$
$\mathcal{X}^M$
$\mathcal{X}^{N_0}$
$\mathcal{X}^{N_1}$
M
N_1
N_0

Composition Law





$m = 3$

$\mathcal{X} = BG$

local line ρ-defect :

Link =

$\mathcal{X}^{\longrightarrow} = \Omega\mathcal{X} \sim G$

$\sigma(\longrightarrow) = \mathrm{Vect}(G)$

Fusion product : $\mathrm{Vect}[G]$

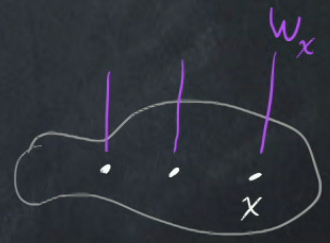
local line σ-defect :

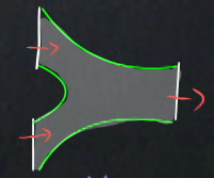
Link $= S^1$

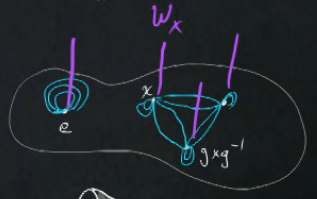
$\mathcal{X}^{S^1} = \mathcal{L}\mathcal{X} \sim G/\!/G$

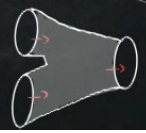
$\sigma(S^1) = \mathrm{Vect}_G(G)$

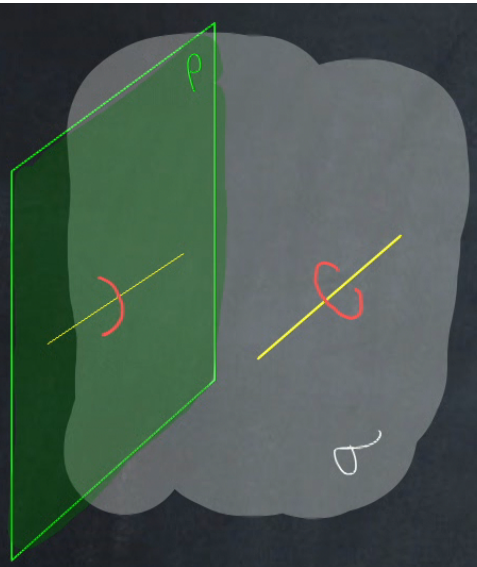
Fusion product











$m = 3$

$\mathcal{X} = BG$

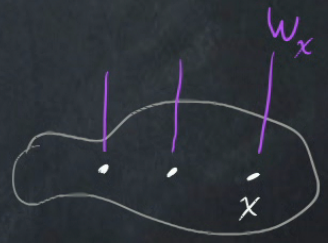
local line ρ-defect :

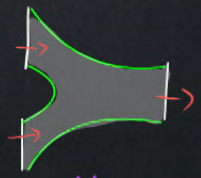
Link =

$\mathcal{X}^{\longrightarrow} = \Omega\mathcal{X} \sim G$

$\sigma(\longrightarrow) = \mathrm{Vect}(G)$

Fusion product: $\mathrm{Vect}[G]$





local line σ-defect :

Link = S^1

$\mathcal{X}^{S^1} = \mathcal{L}\mathcal{X} \sim G/\!/G$

$\sigma(S^1) = \mathrm{Vect}_G(G)$

Fusion product

