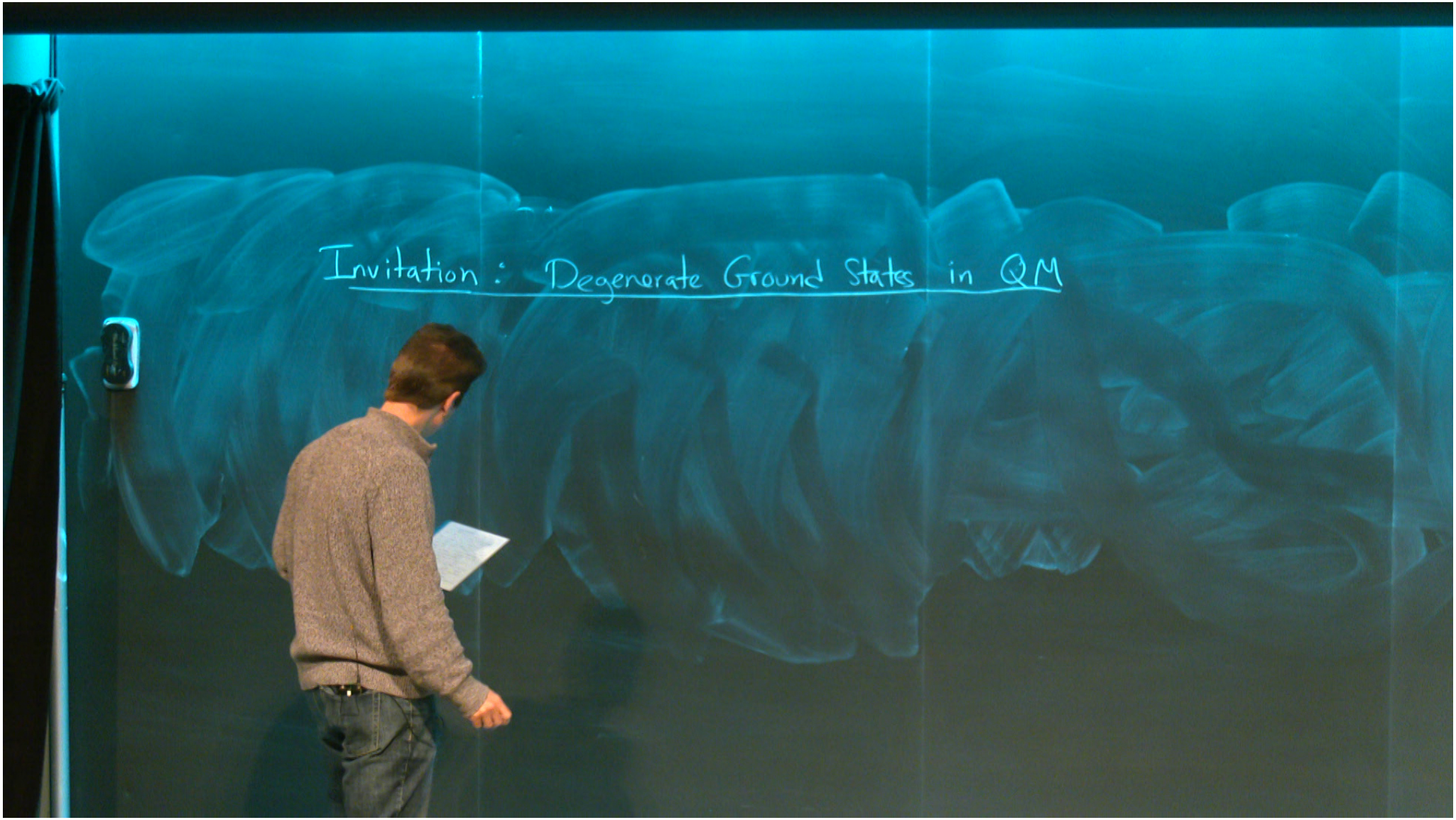
Title: Introduction to Anomalies in Quantum Field Theory

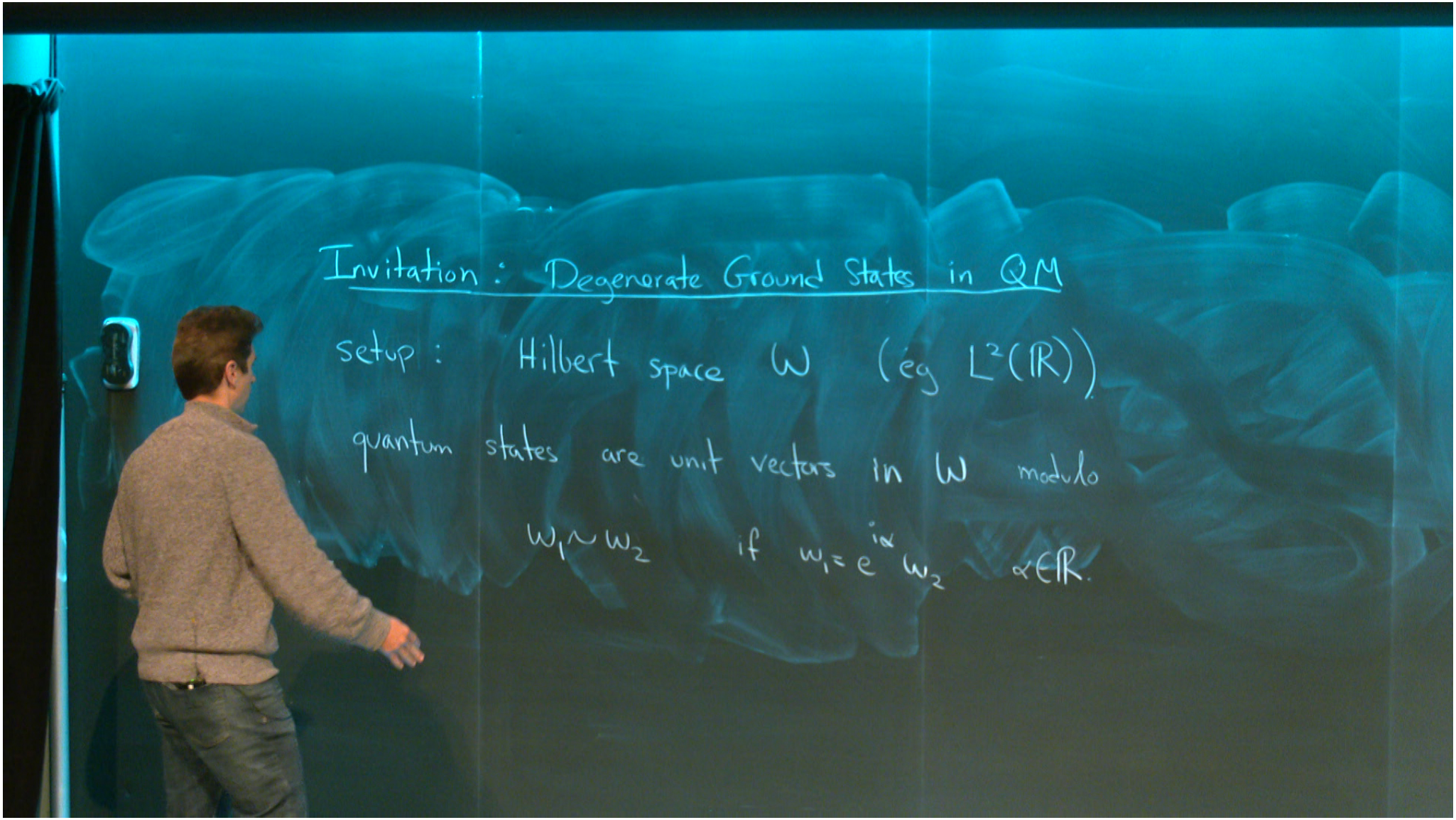
Speakers:

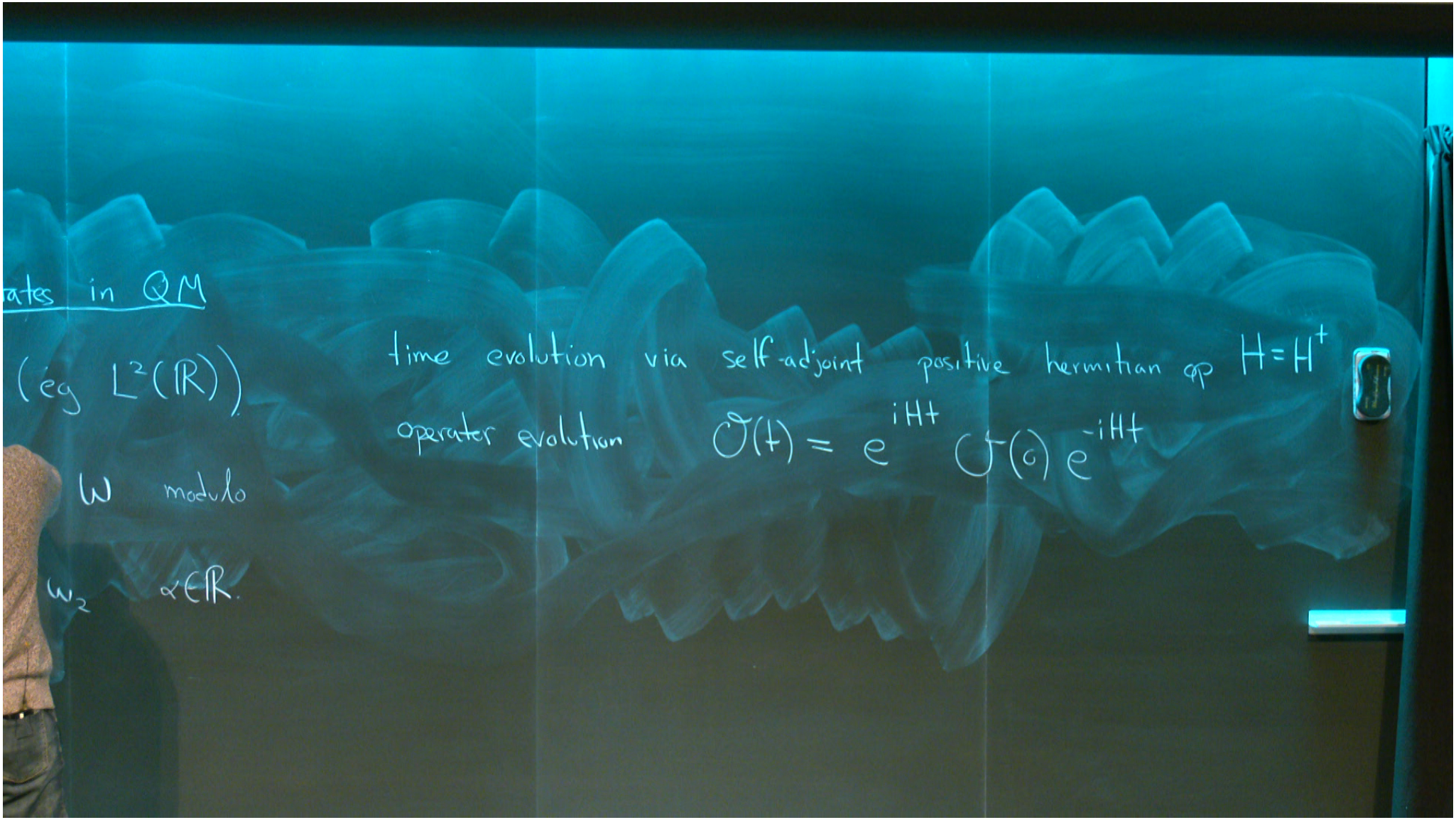
Collection: Global Categorical Symmetries

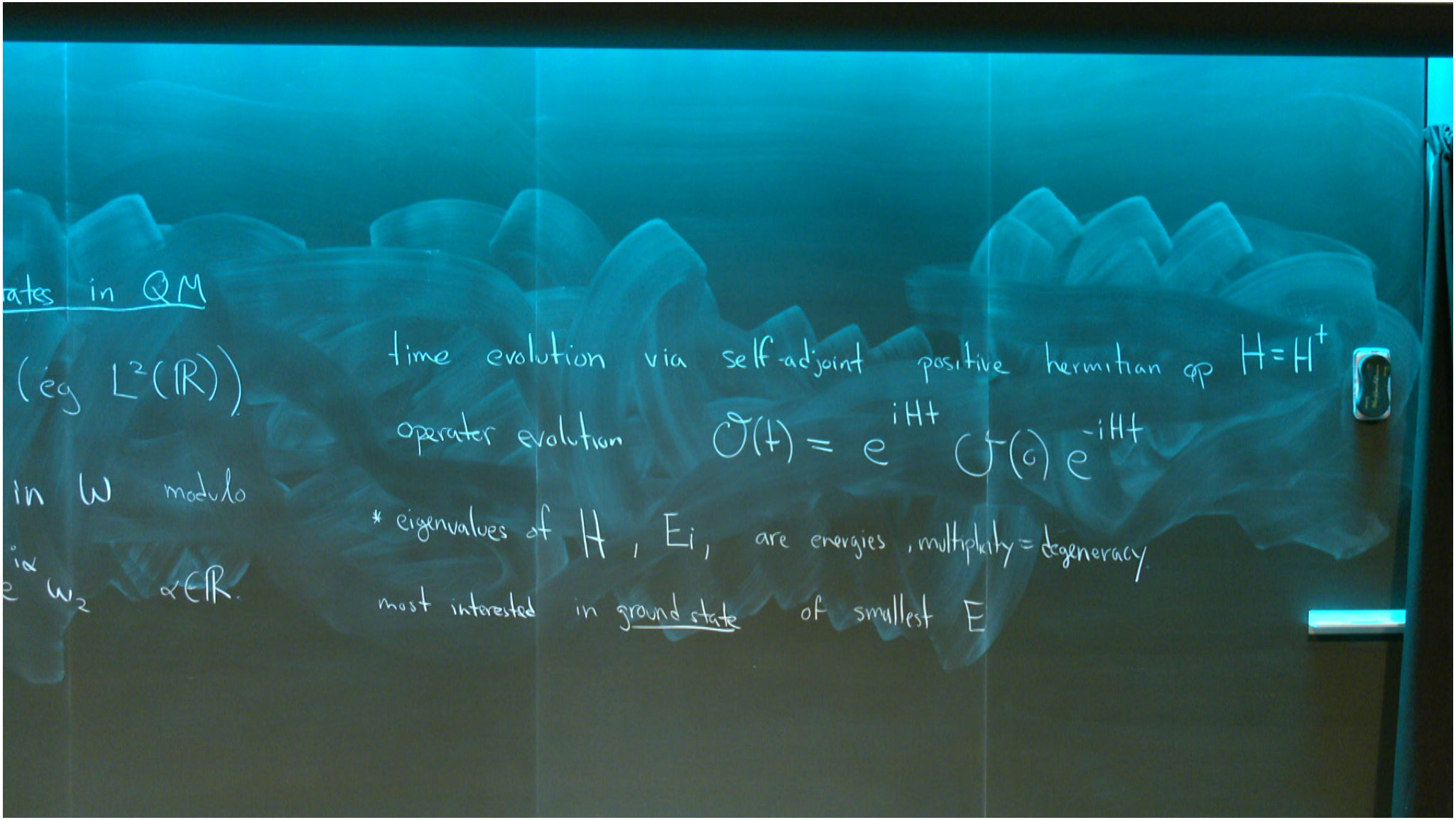
Date: June 13, 2022 - 2:00 PM

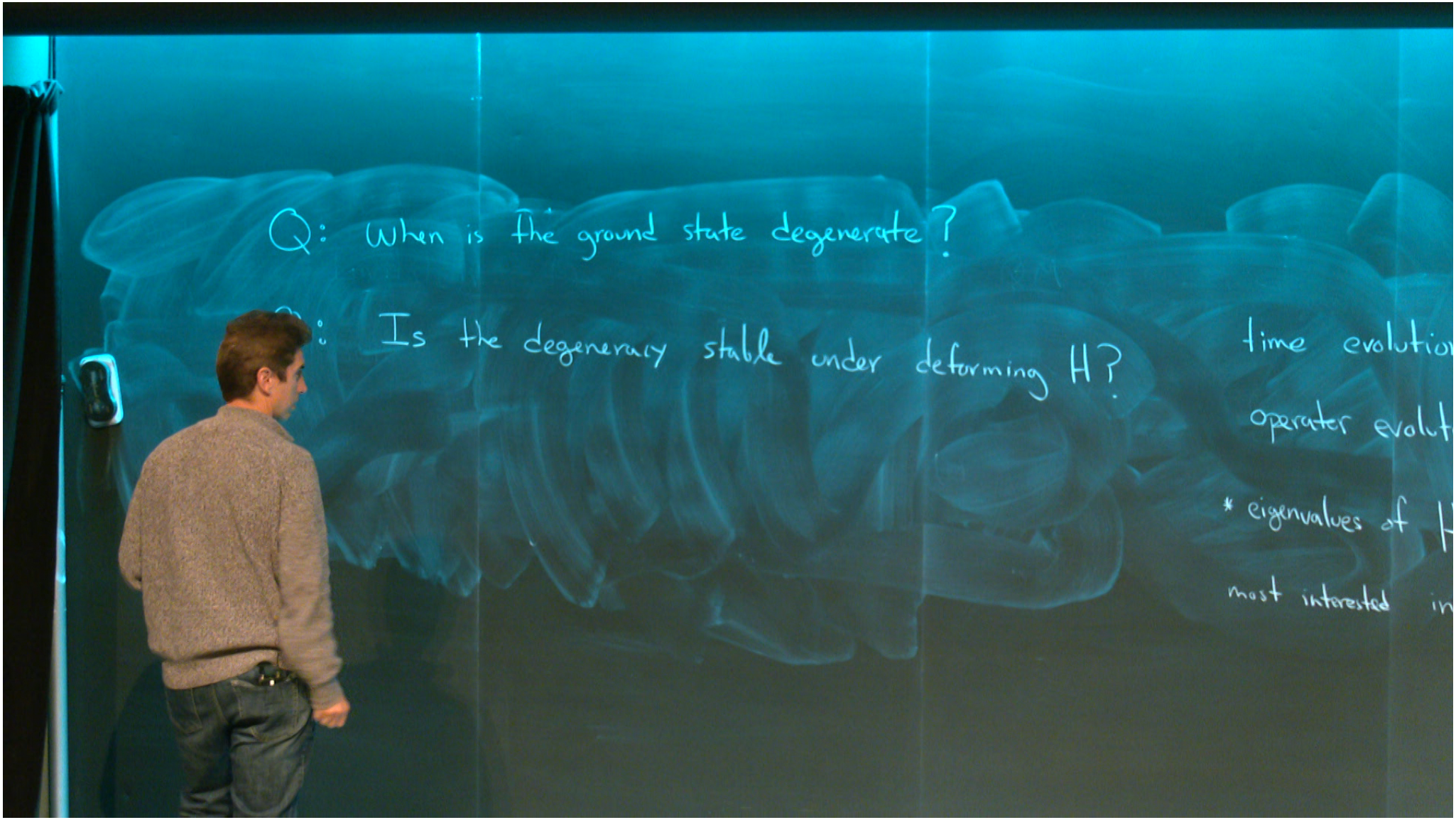
URL: https://pirsa.org/22060025

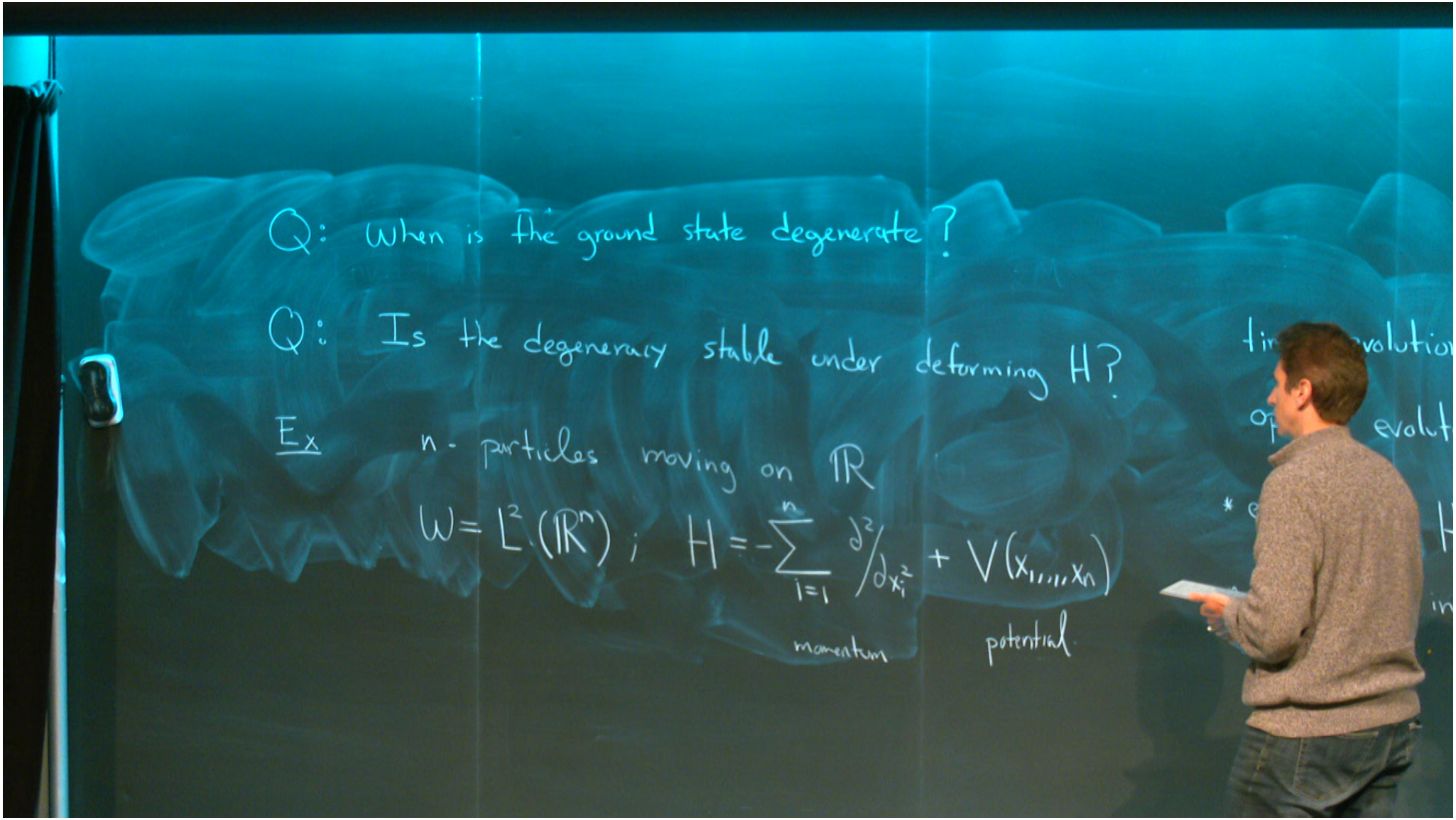

Invitation: Degenerate Ground States in QM

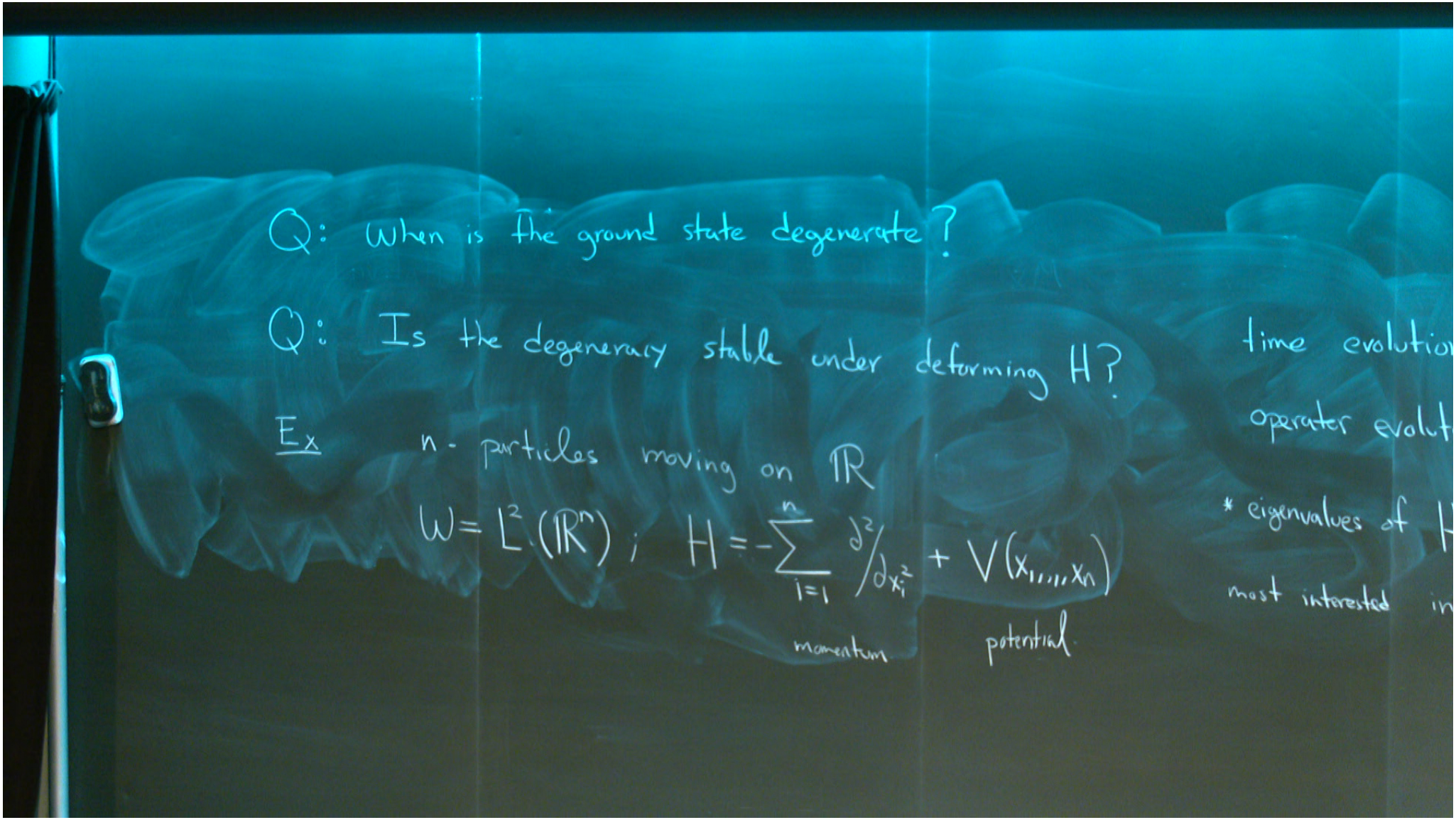

Invitation: Degenerate Ground States in QM
setup: Hilbert space $\mathcal{W}$ (eg $L^2(\mathbb{R})$)
quantum states are unit vectors in $\mathcal{W}$ modulo
$w_1 \sim w_2$ if $w_1 = e^{i\alpha} w_2$ $\alpha \in \mathbb{R}$.


ates in QM
(eg $L^2(\mathbb{R})$)
W modulo
W_2 $\alpha \in \mathbb{R}$
time evolution via self-adjoint positive hermitian op $H = H^\dagger$
operator evolution $\mathcal{O}(t) = e^{iHt}\,\mathcal{O}(0)\,e^{-iHt}$


ates in QM
(eg $L^2(\mathbb{R})$)
in W modulo
$e^{i\alpha} W_2$ $\alpha \in \mathbb{R}$.
time evolution via self-adjoint positive hermitian op $H = H^\dagger$
operator evolution $\mathcal{O}(t) = e^{iHt}\, \mathcal{O}(0)\, e^{-iHt}$
* eigenvalues of H, E_i, are energies, multiplicity = degeneracy
most interested in ground state of smallest E


Q: When is the ground state degenerate?
Q: Is the degeneracy stable under deforming H?
time evolution
operator evolut
* eigenvalues of
most interested


Q: When is the ground state degenerate?
Q: Is the degeneracy stable under deforming H?
Ex n-particles moving on $\mathbb{R}$
$\mathcal{H} = L^2(\mathbb{R}^n)$; $H = -\sum_{i=1}^{n} \partial^2/\partial x_i^2 + V(x_1,\ldots,x_n)$
momentum
potential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momentum
potential
time evolution
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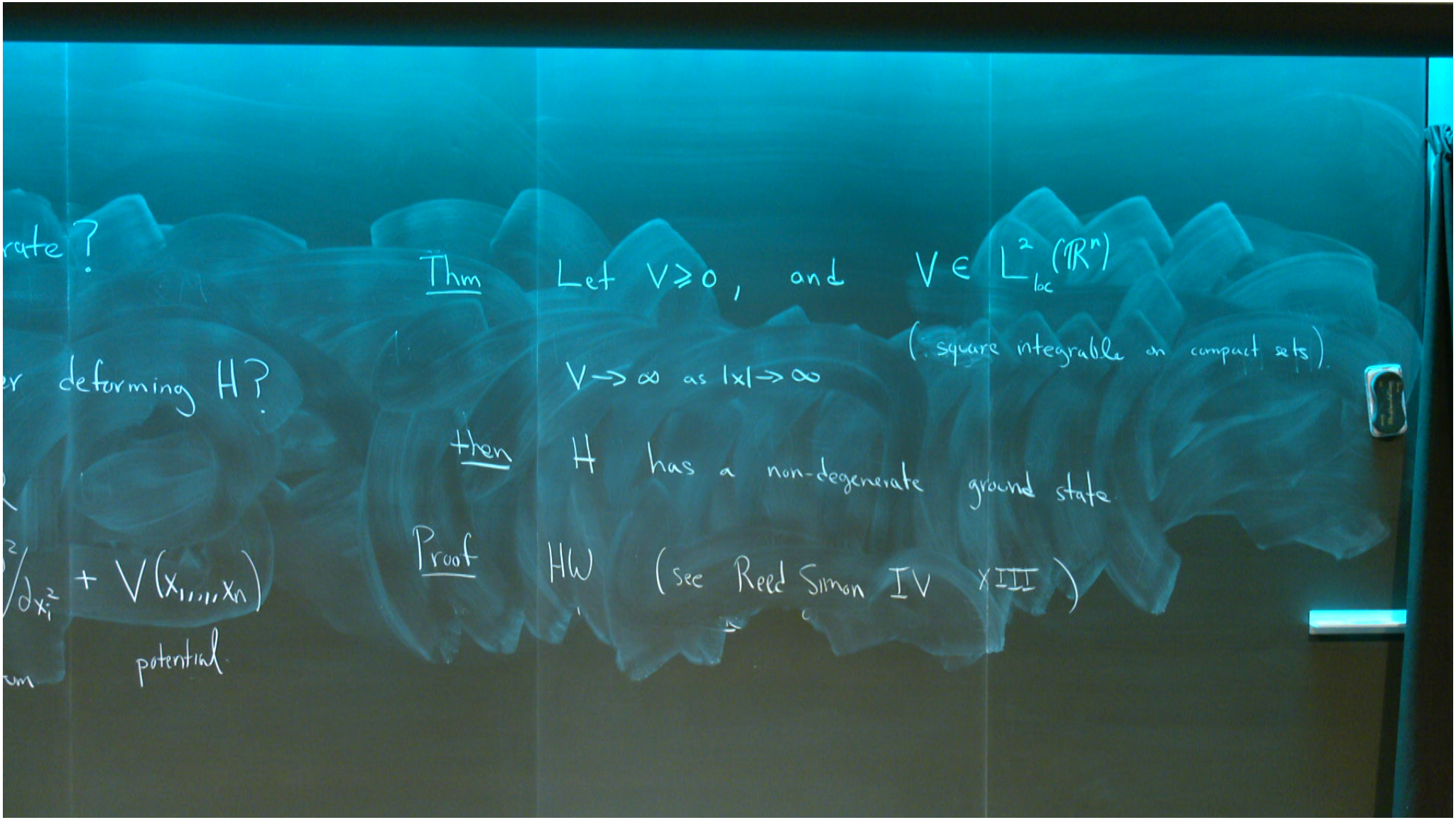
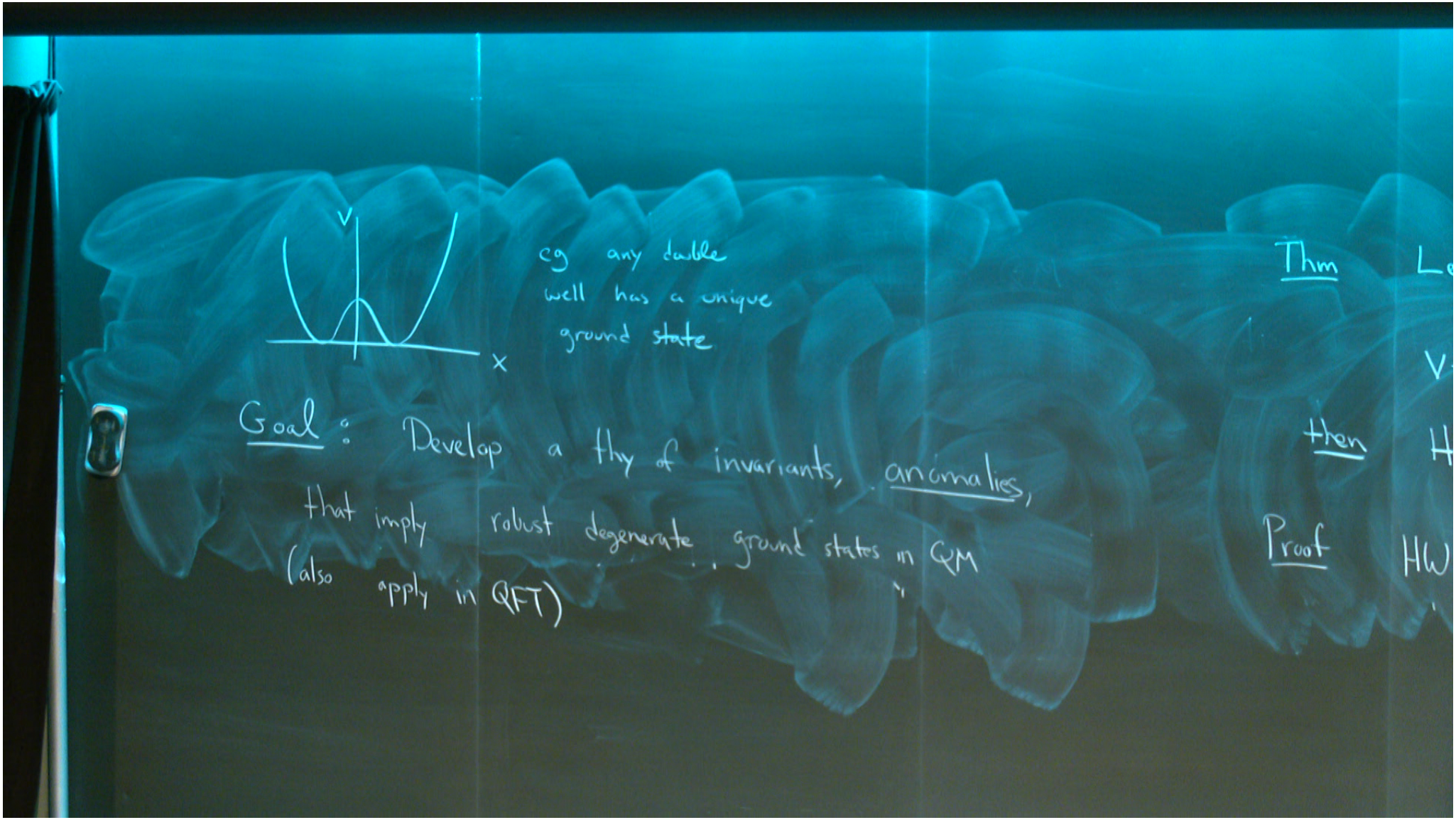
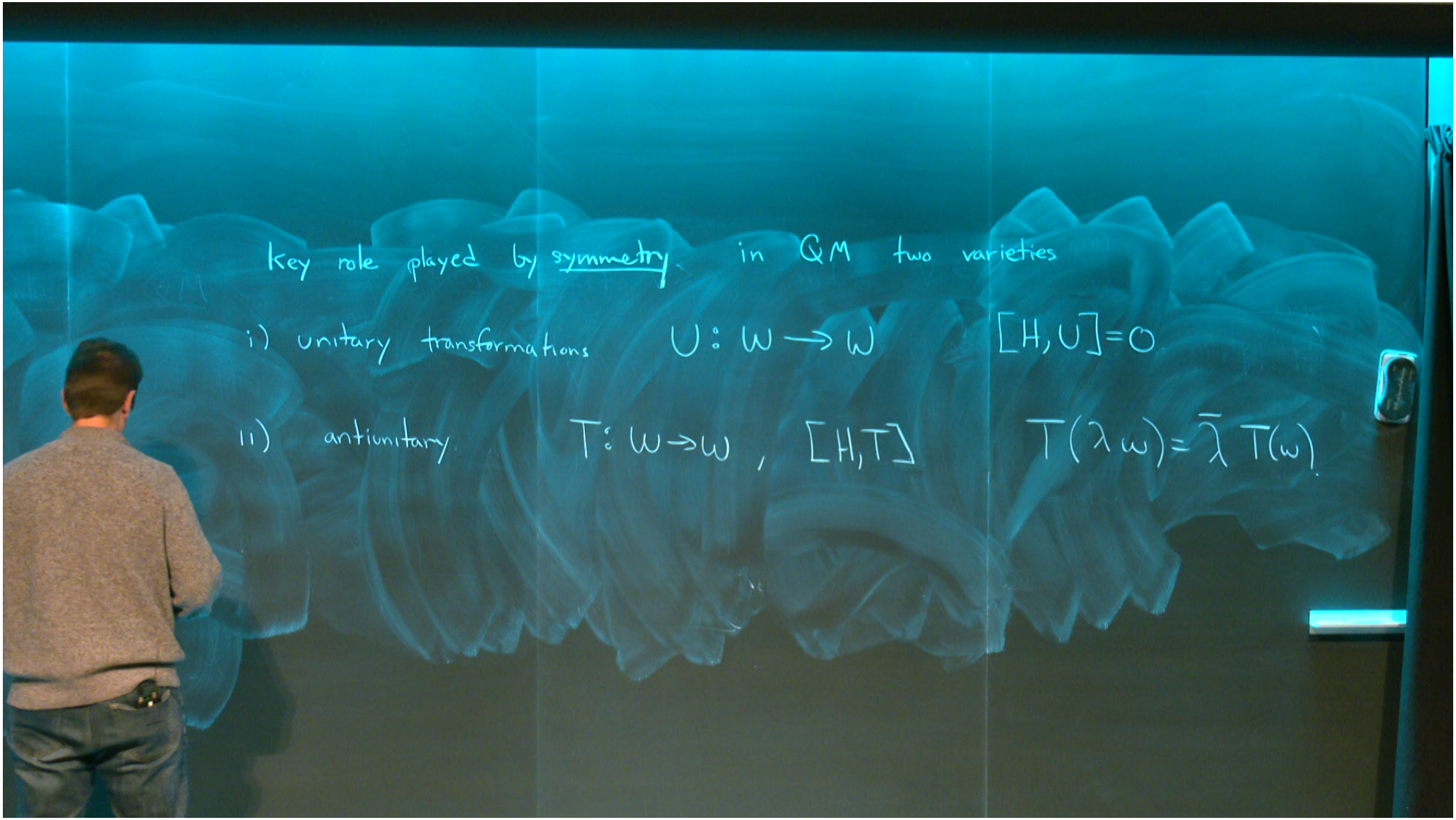
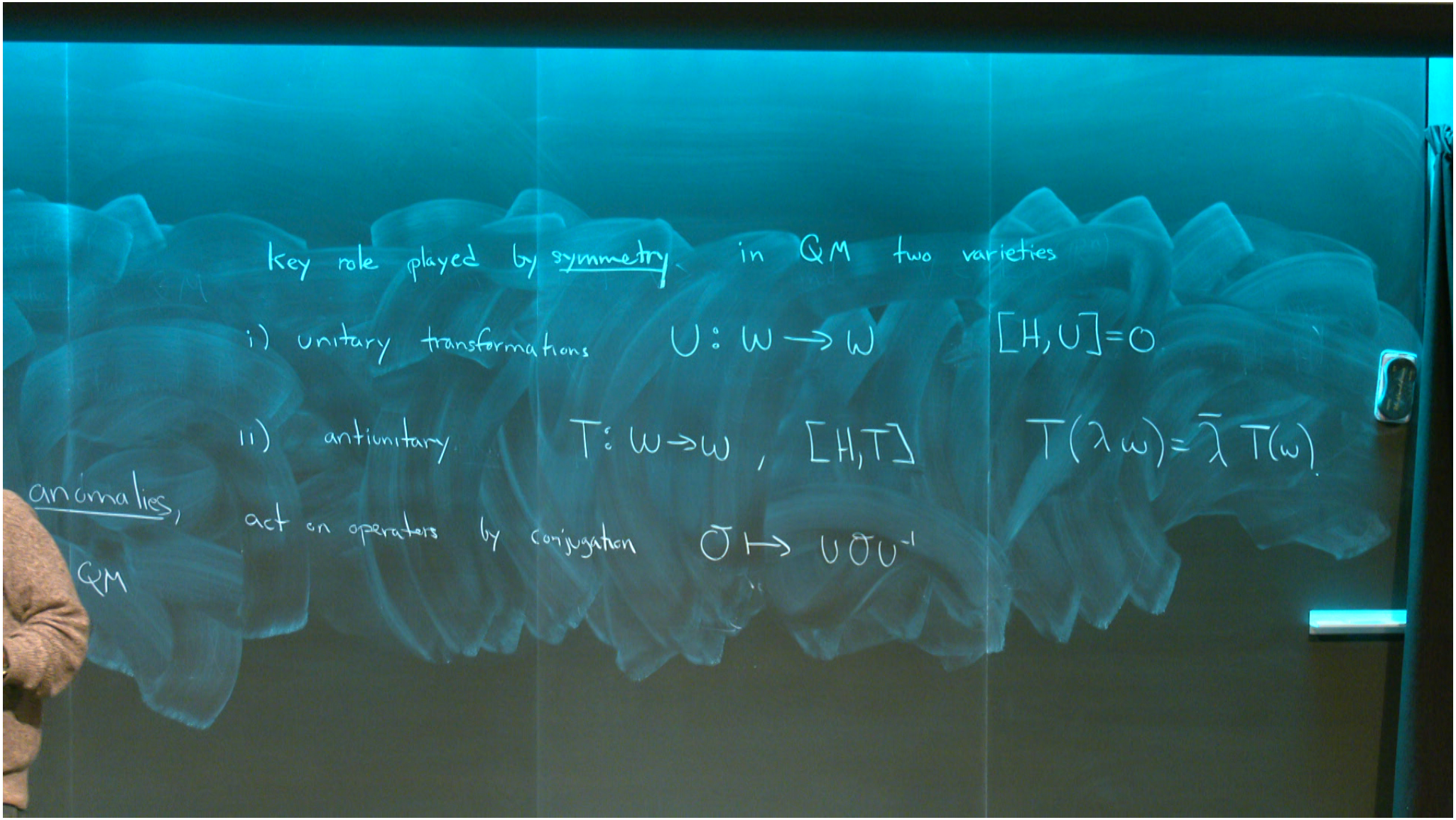
* eigenvalues of H
most interested in


rate?
er deforming H?
$\partial^2/\partial x_i^2 + V(x_1,\ldots,x_n)$
potential
um
Thm Let $V \geq 0$, and $V \in L^2_{loc}(\mathbb{R}^n)$
(: square integrable on compact sets)
$V \to \infty$ as $|x| \to \infty$
then H has a non-degenerate ground state
Proof HW (see Reed Simon IV XIII)


V
x
eg any double well has a unique ground state
Thm
then
Proof
HW


V
x
eg any double well has a unique ground state
Goal: Develop a thy of invariants, anomalies, that imply robust degenerate ground states in QM (also apply in QFT)
Thm
then
Proof
HW


key role played by symmetry in QM two varieties
i) unitary transformations $U: W \to W$ $[H, U] = 0$
ii) antiunitary $T: W \to W$, $[H, T]$ $T(\lambda w) = \bar{\lambda} T(w)$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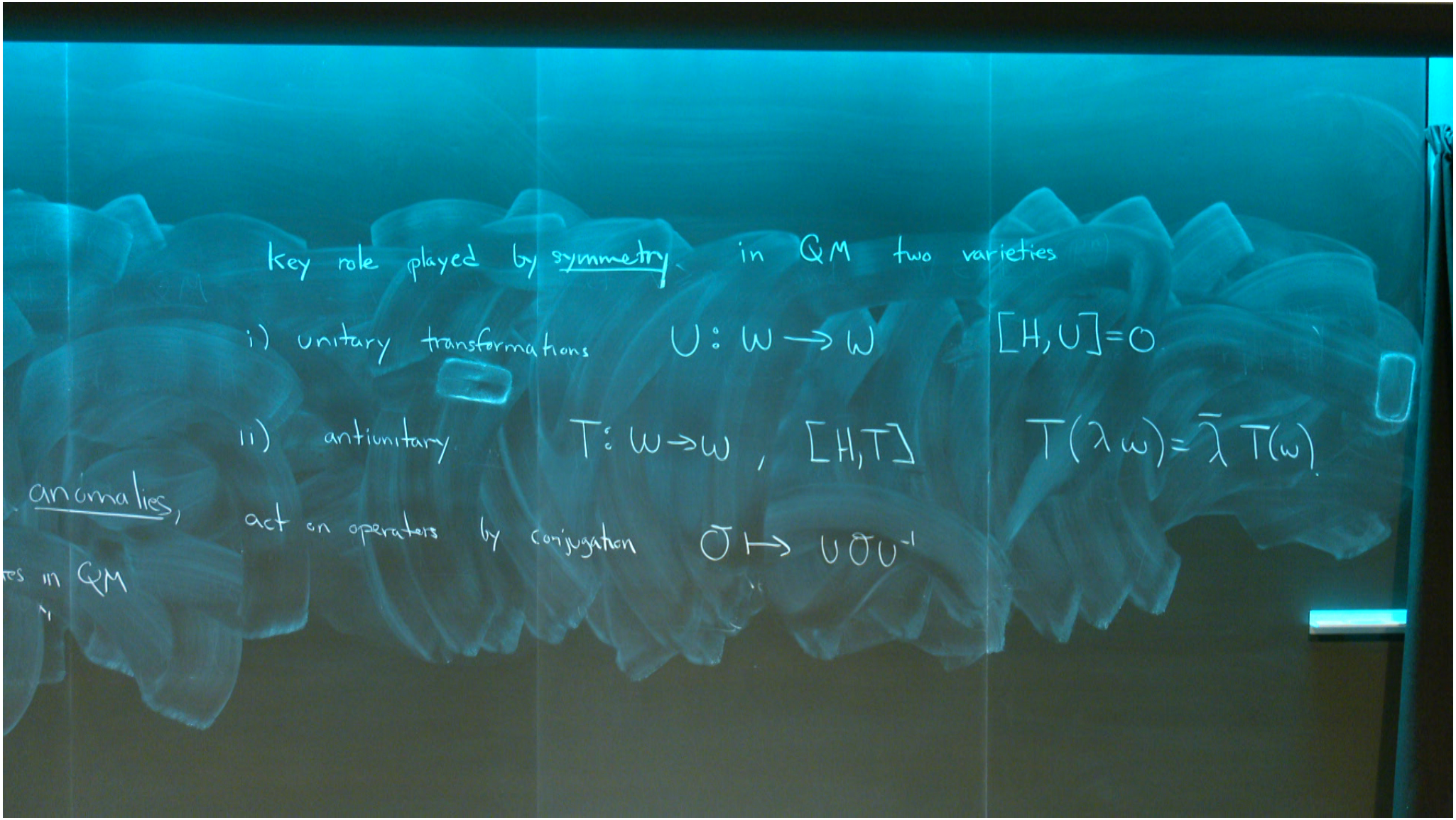
anomalies,
act on operators by conjugation $\mathcal{O} \mapsto U \mathcal{O} U^{-1}$
QM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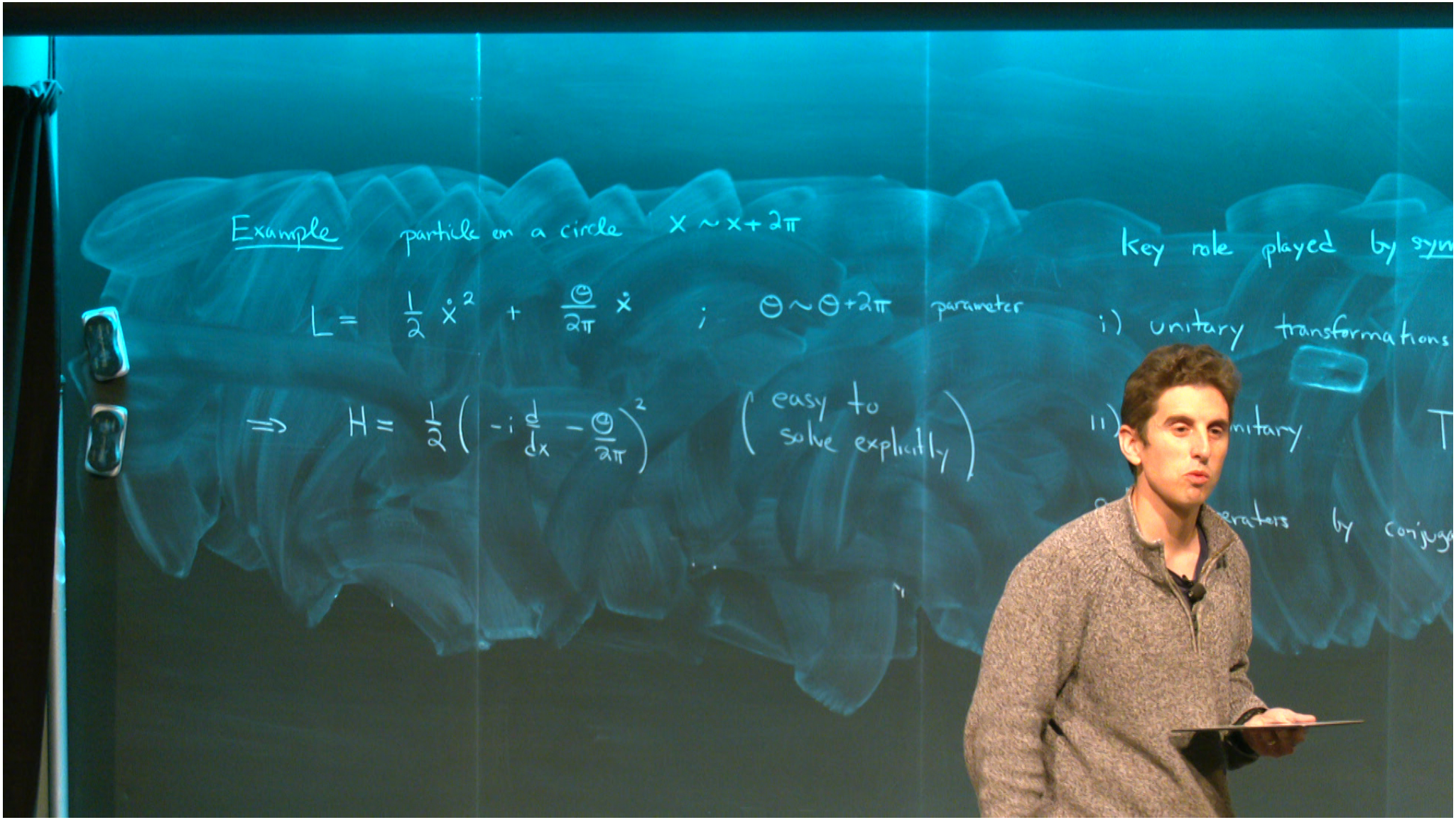
Example particle on a circle $x \sim x + 2\pi$
$L = \frac{1}{2}\dot{x}^2 + \frac{\theta}{2\pi}\dot{x}$; $\theta \sim \theta + 2\pi$ parameter
$\Rightarrow H = \frac{1}{2}\left(-i\frac{d}{dx} - \frac{\theta}{2\pi}\right)^2$
(easy to solve explicitly)
Key role played by
i) unitary transformations
ii)
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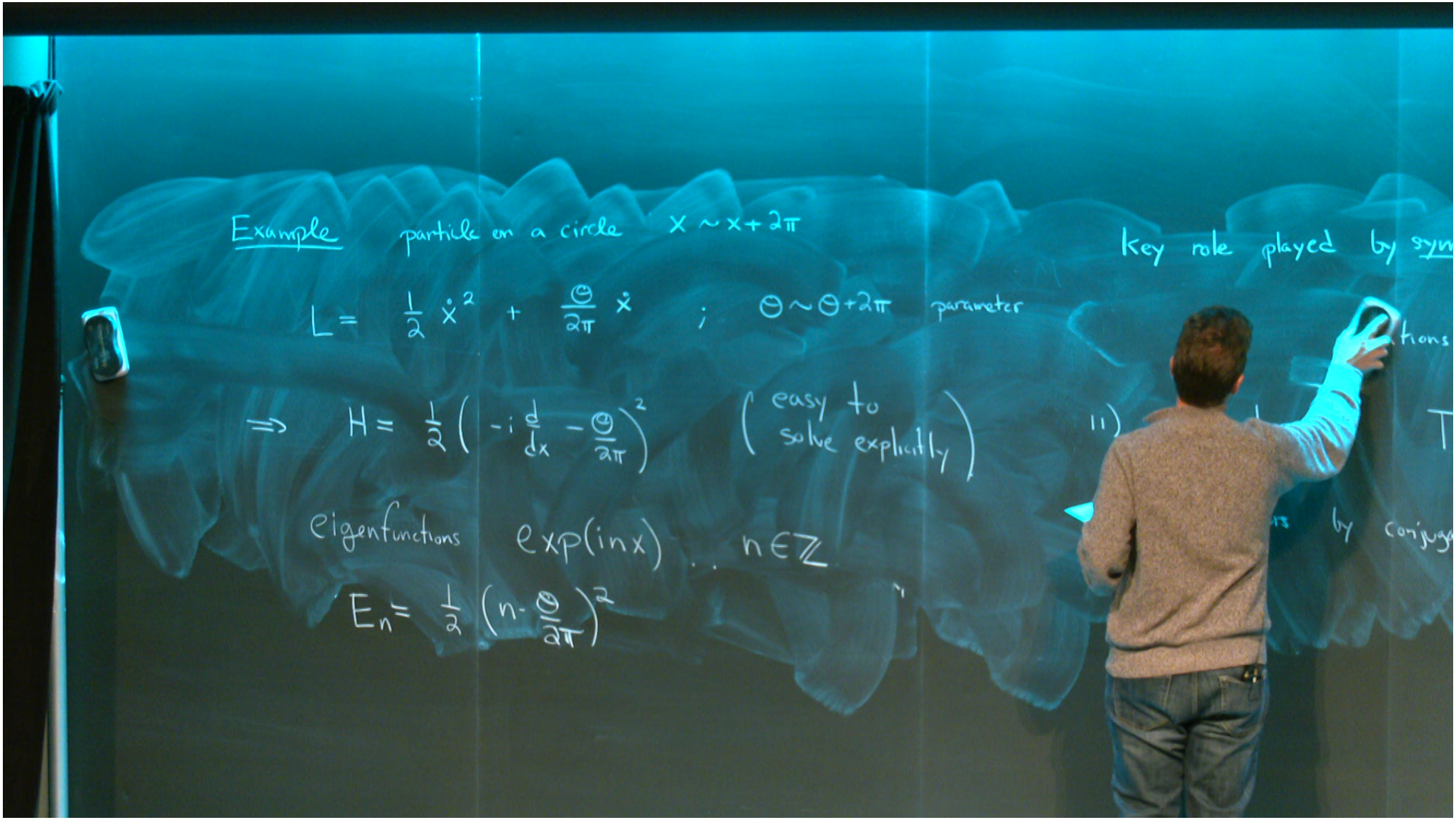
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$\Rightarrow H = \frac{1}{2}\left(-i\frac{d}{dx} - \frac{\Theta}{2\pi}\right)^2$
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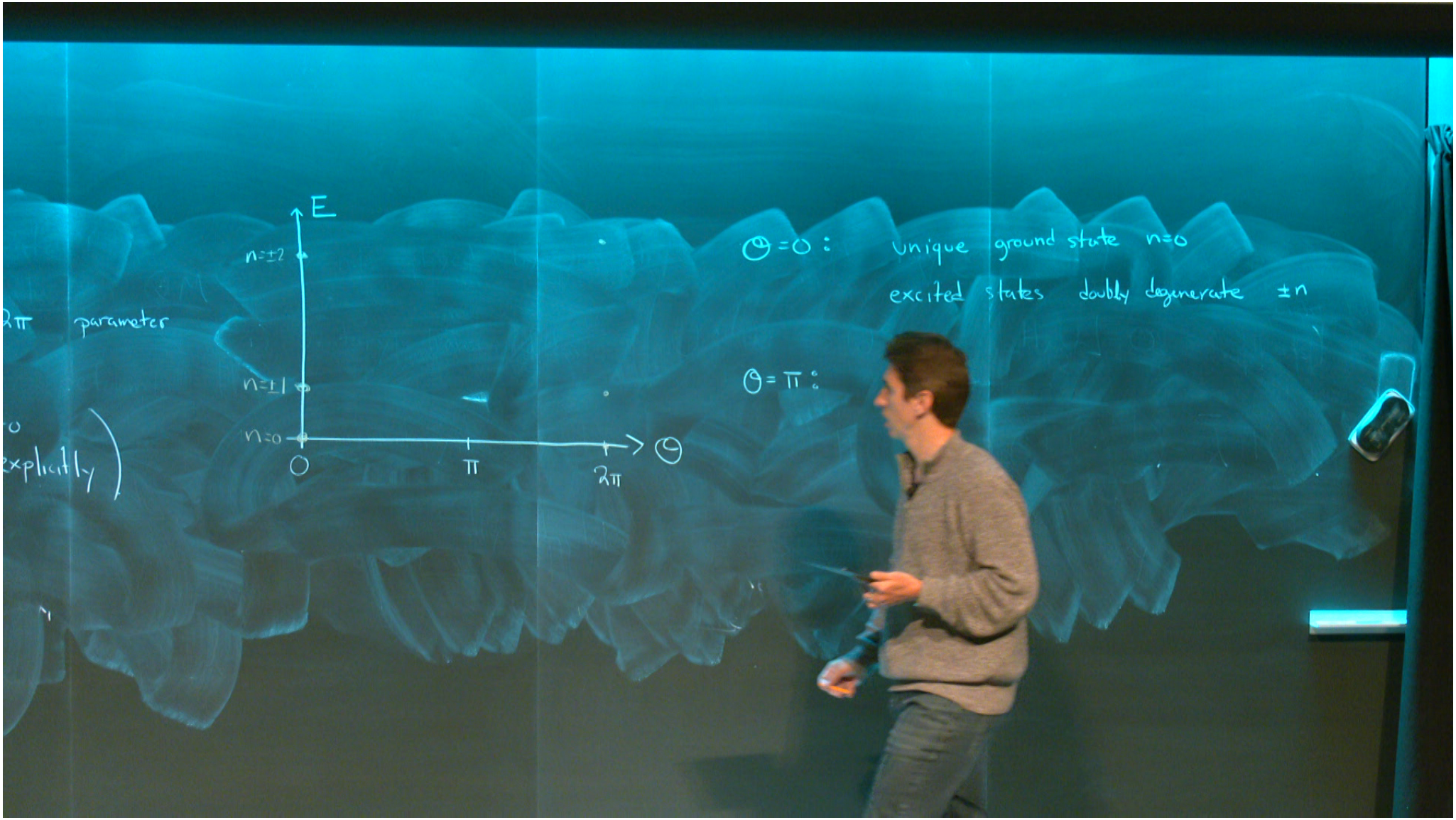
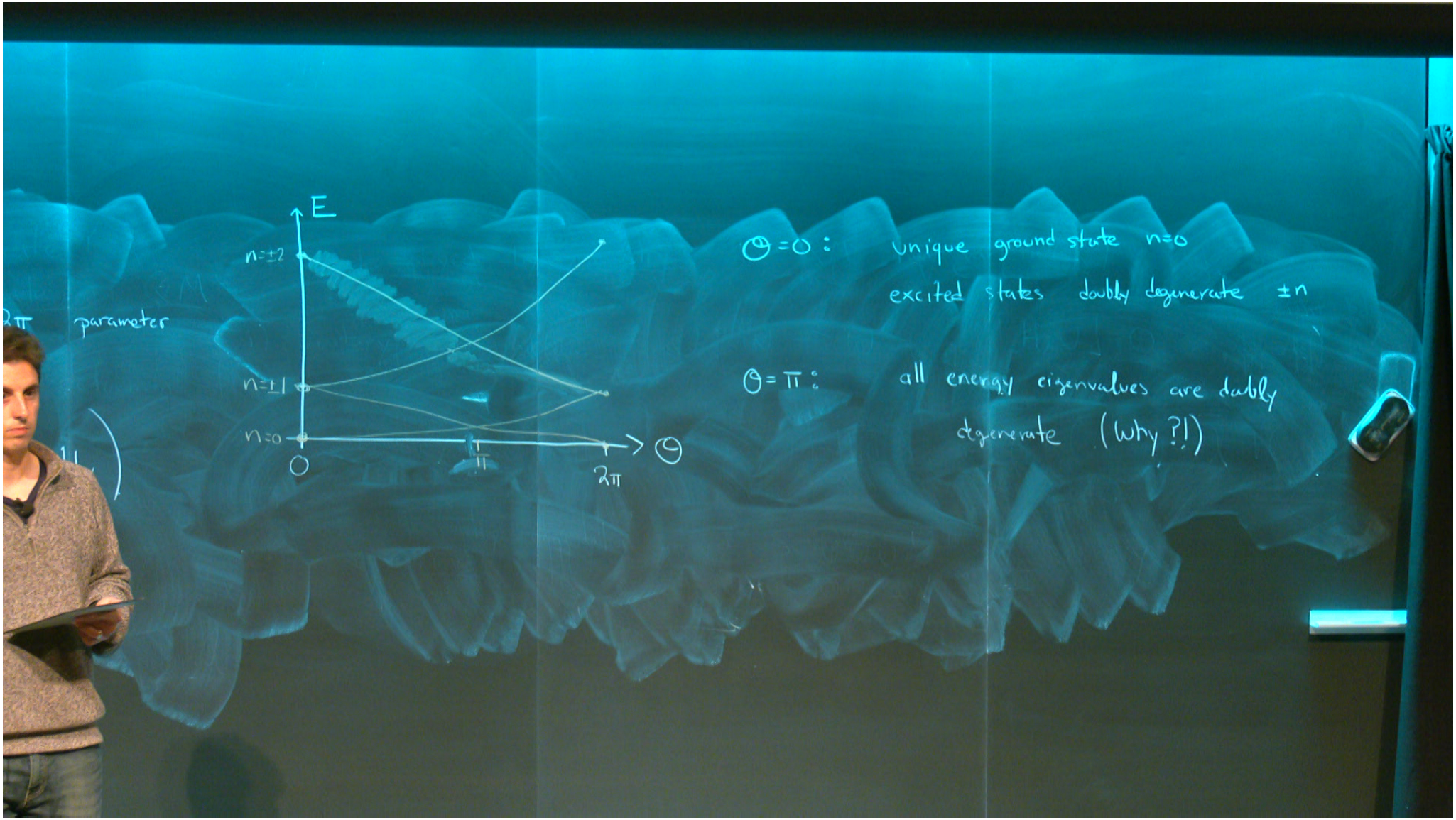
eigenfunctions $\exp(inx)$ . . $n \in \mathbb{Z}$
$E_n = \frac{1}{2}\left(n - \frac{\Theta}{2\pi}\right)^2$


E
n=±2
n=±1
n=0
0
π
2π
θ
θ=0 : unique ground state n=0
excited states doubly degenerate ±n
θ=π :
parameter
explicitly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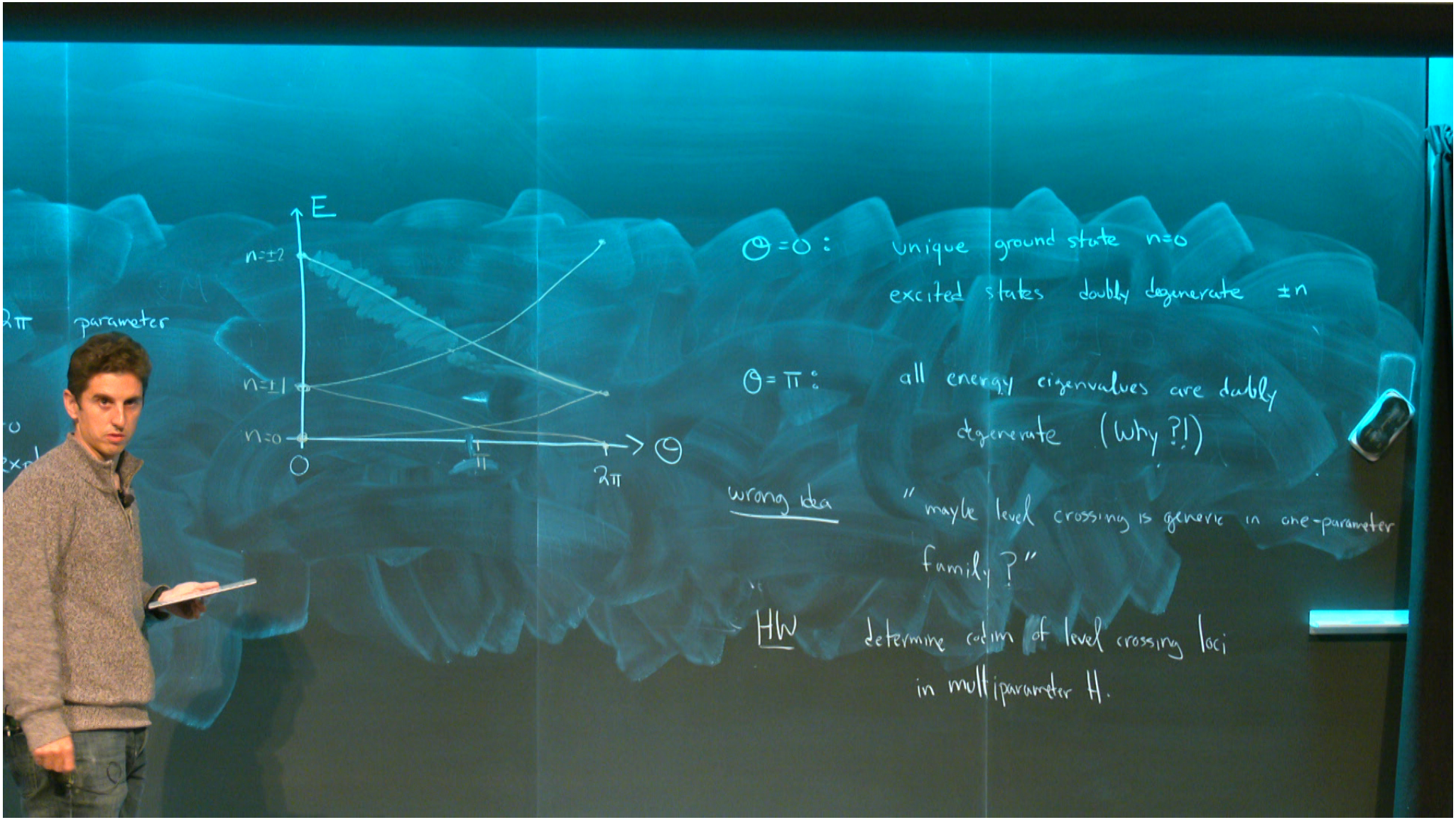
θ=π : all energy eigenvalues are doubly degenerate (Why ?!)
parameter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0
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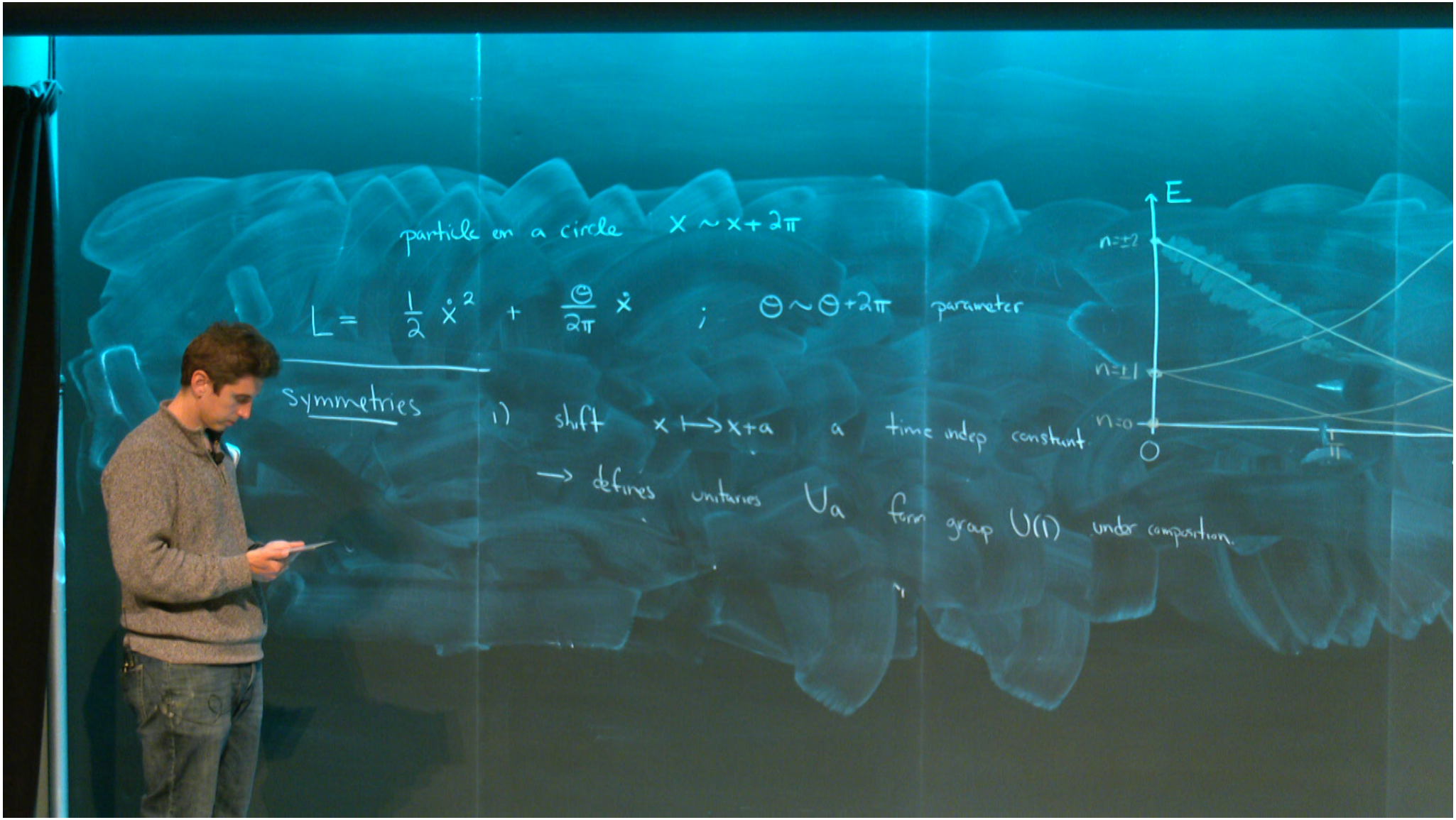
θ=0 : unique ground state n=0
excited states doubly degenerate ±n
θ=π : all energy eigenvalues are doubly degenerate (Why ?!)
wrong idea "maybe level crossing is generic in one-parameter family ?"
HW determine codim of level crossing loci in multiparameter H.


particle on a circle $X \sim X + 2\pi$
$L = \frac{1}{2}\dot{X}^2 + \frac{\Theta}{2\pi}\dot{X}$; $\Theta \sim \Theta + 2\pi$ parameter
Symmetries
1) shift $x \longmapsto x + a$ a time indep constant
$\longrightarrow$ defines unitaries U_a form group $U(1)$ under composition.
E
$n=\pm 2$
$n=\pm 1$
$n=0$
O
$\frac{1}{\pi}$

particle on a circle $X \sim X + 2\pi$

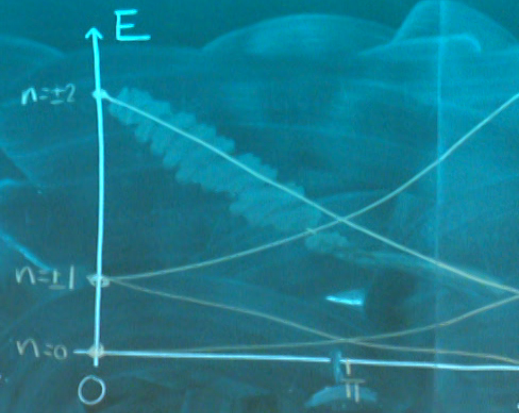
$$L = \frac{1}{2}\dot{X}^2 + \frac{\Theta}{2\pi}\dot{X} \quad ; \quad \Theta \sim \Theta + 2\pi \quad \text{parameter}$$

Symm

i) shift $x \longmapsto x + a$ a time indep constant

$\longrightarrow$ defines unitaries U_a form group $U(1)$ under composition.

ii) reflections $C : x \longmapsto -x$ symmetry only at $\Theta = 0, \pi$



particle on a circle $X \sim X + 2\pi$

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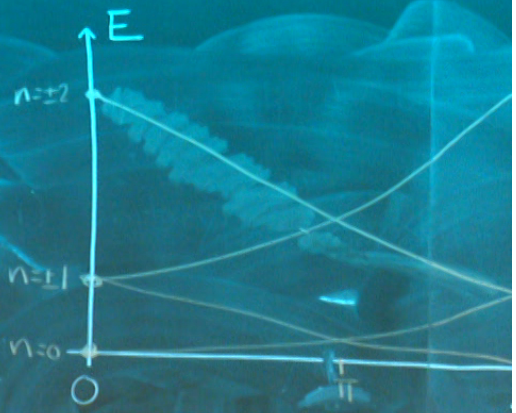

Symmetries

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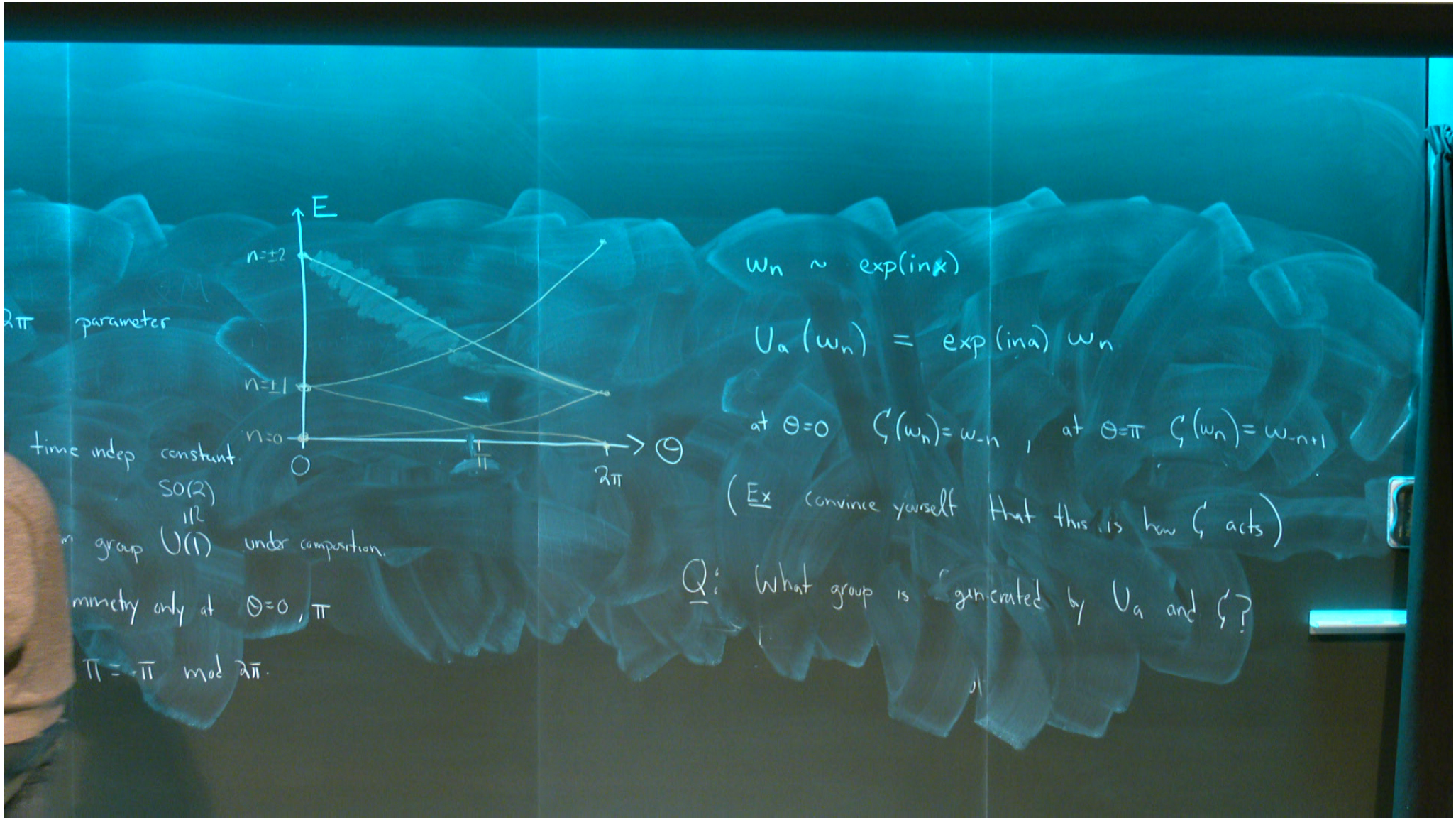
symmetry at $\Theta = \pi$ since $\pi = -\pi \mod 2\pi$.


E
n=±2
n=±1
n=0
O
π
2π
θ
2π parameter
time indep constant.
SO(2)
≅
group U(1) under composition.
mmetry only at θ=0, π
π = -π mod 2π.
$w_n \sim \exp(in\theta)$
$U_a(w_n) = \exp(ina)\, w_n$
at θ=0 $\zeta(w_n) = w_{-n}$, at θ=π $\zeta(w_n) = w_{-n+1}$
(Ex convince yourself that this is how ζ acts)
Q: What group is generated by U_a and ζ?

A_1 : by construction when acting on operators ($\sin x$, $\partial/\partial x$)

realize $O(2)$

Symmetries

i) shift $x \longmapsto x+a$ a time indep constant

$\longrightarrow$ defines unitaries U_a form group $SO(2) \cong U(1)$ under composition.

ii) reflections $C : x \longmapsto -x$ symmetry only at $\theta = 0, \pi$

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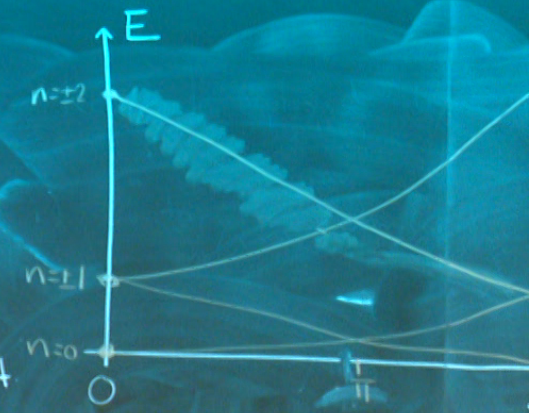
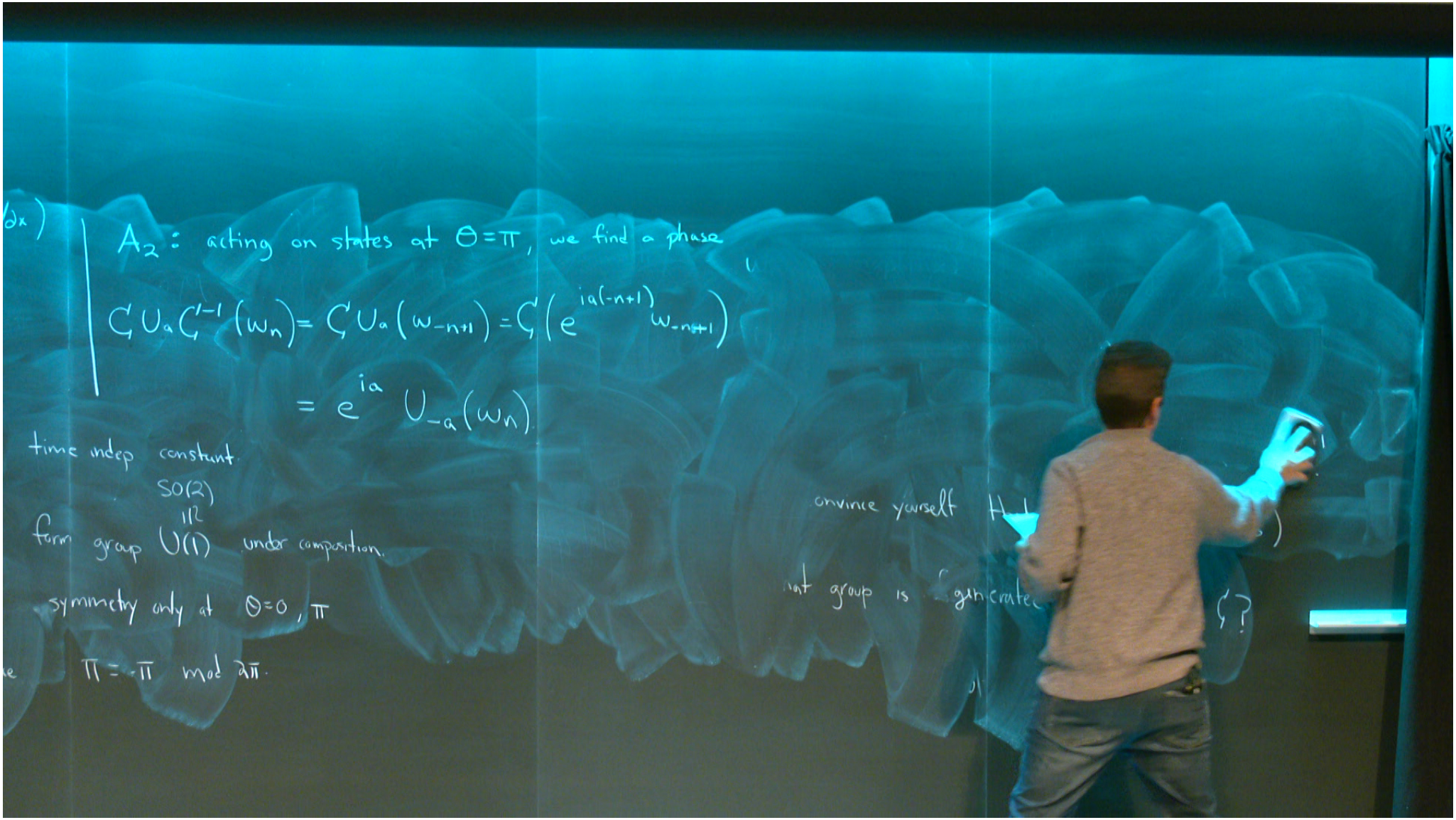



A_2: acting on states at $\theta=\pi$, we find a phase
$C U_a C^{-1}(\omega_n) = C U_a(\omega_{-n+1}) = C\left(e^{ia(-n+1)}\omega_{-n+1}\right)$
$= e^{ia} U_{-a}(\omega_n)$
time indep constant
SO(2)
form group U(1) under composition.
symmetry only at $\theta=0, \pi$
$\pi = -\pi \mod 2\pi$.
onvince yourself
at group is generated
C?

A_2: acting on states at $\Theta = \pi$, we find a phase

$$C U_a C^{-1}(w_n) = C U_a(w_{-n+1}) = C\left(e^{ia(-n+1)} w_{-n+1}\right)$$

$$= e^{ia} U_{-a}(w_n)$$

this is a projective representation associated to a double cover

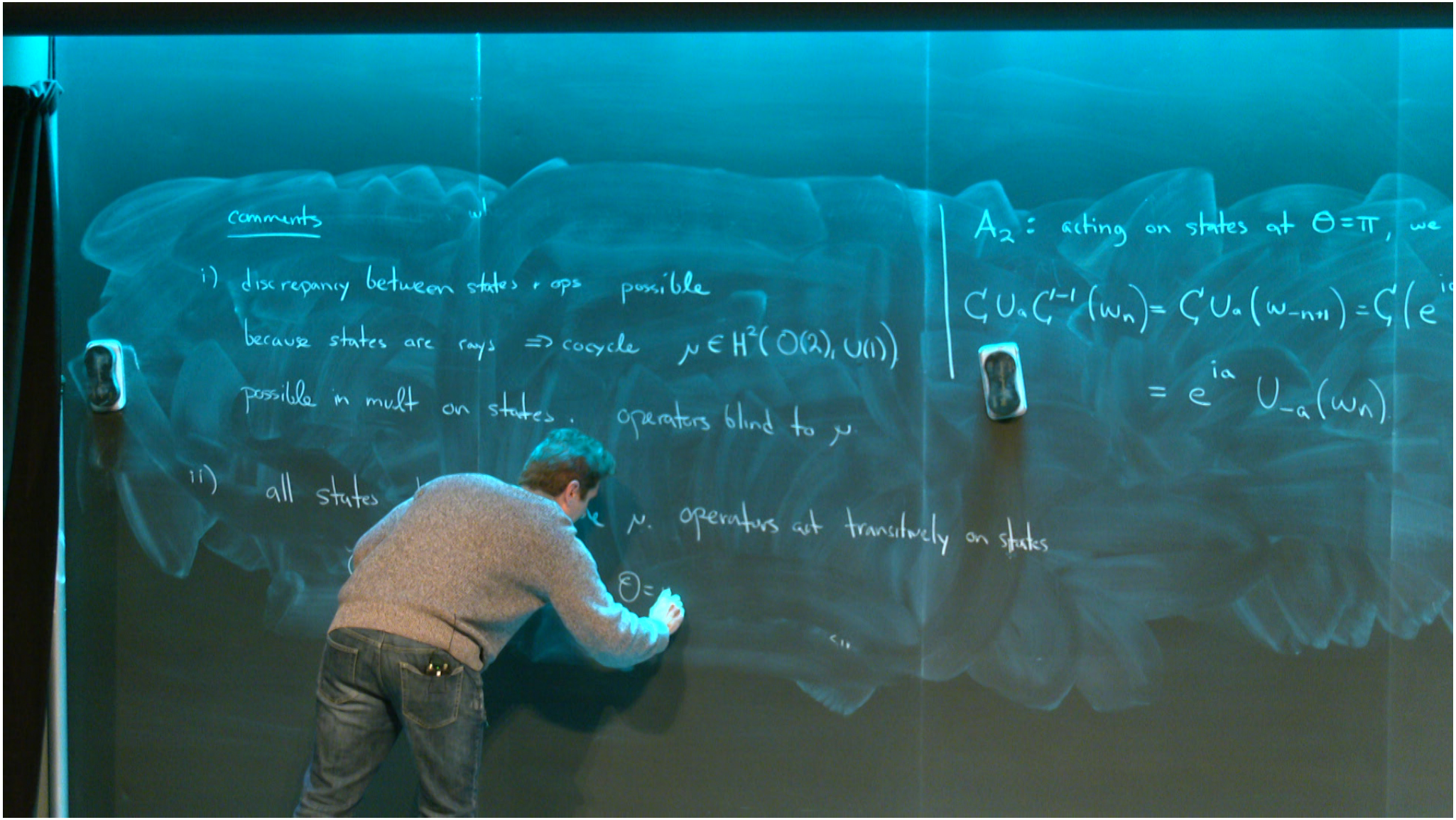
$$1 \to \mathbb{Z}_2 \to \mathrm{Pin}^-(2) \to O(2) \to 1$$

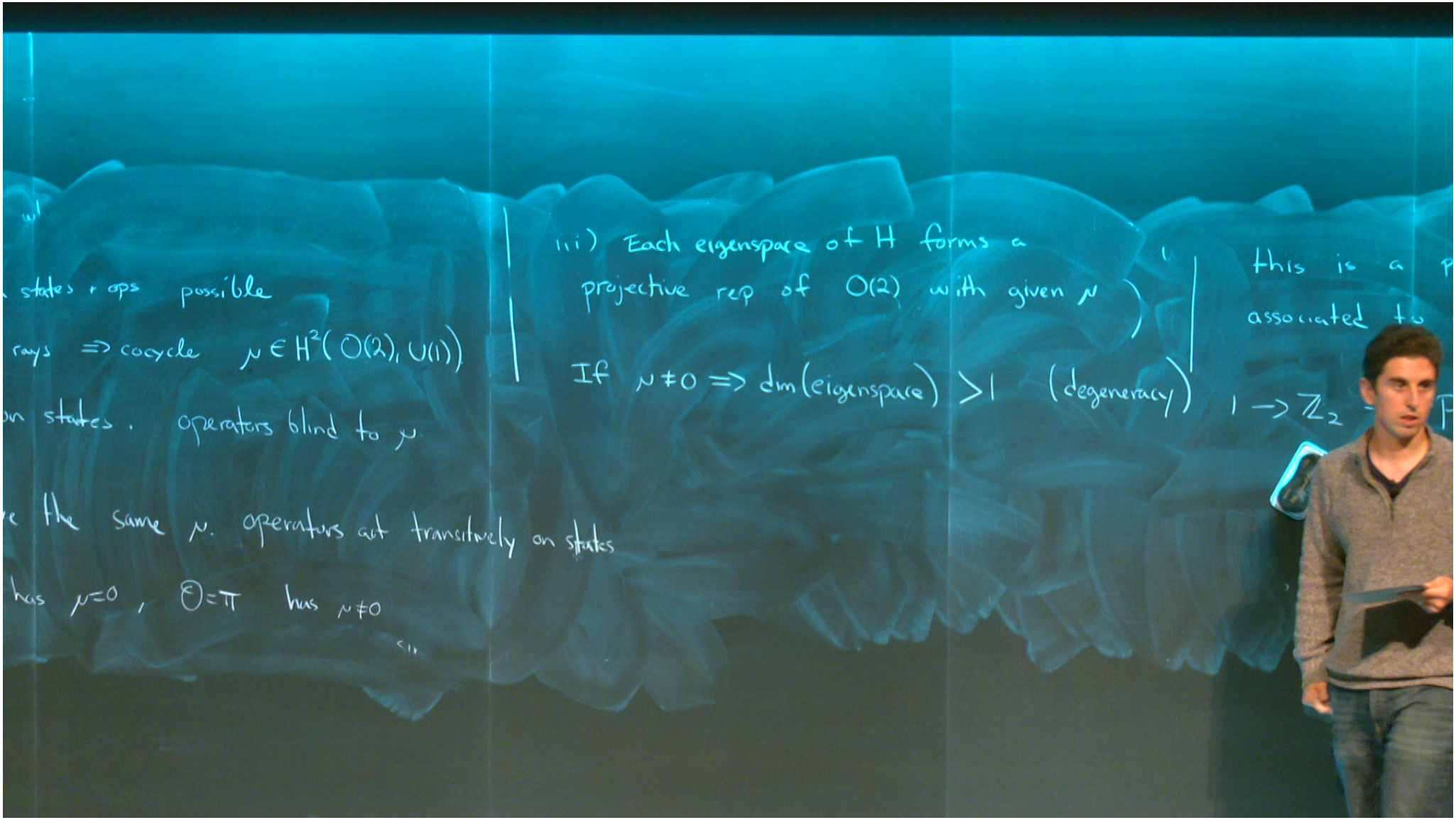
time indep constant.

$SO(2)$
$\cong$
form group $U(1)$ under composition.

symmetry only at $\Theta = 0, \pi$

$\pi = -\pi \mod 2\pi$.


comments
i) discrepancy between states + ops possible
because states are rays ⇒ cocycle $\nu \in H^2(O(2), U(1))$
possible in mult on states. operators blind to ν.
ii) all states
ν. operators act transitively on states
$\Theta =$
A_2: acting on states at $\Theta = \pi$, we
$C U_a C^{-1}(w_n) = C U_a(w_{-n}) = C(e^{i}$
$= e^{ia} U_{-a}(w_n)$


states + ops possible
rays ⇒ cocycle $\nu \in H^2(O(2), U(1))$
on states. operators blind to ν
the same ν. operators act transitively on states
has $\nu=0$, $\theta=\pi$ has $\nu\neq 0$
iii) Each eigenspace of H forms a projective rep of O(2) with given ν
If $\nu \neq 0 \Rightarrow \dim(\text{eigenspace}) > 1$ (degeneracy)
this is a
associated to
$1 \to \mathbb{Z}_2$

states + ops possible

rays $\Rightarrow$ cocycle $\mu \in H^2(O(2), U(1))$

on states. operators blind to μ.

e the same μ. operators act transitively on states

has $\mu = 0$, $\theta = \pi$ has $\mu \neq 0$

iii) Each eigenspace of H forms a projective rep of $O(2)$ with given μ

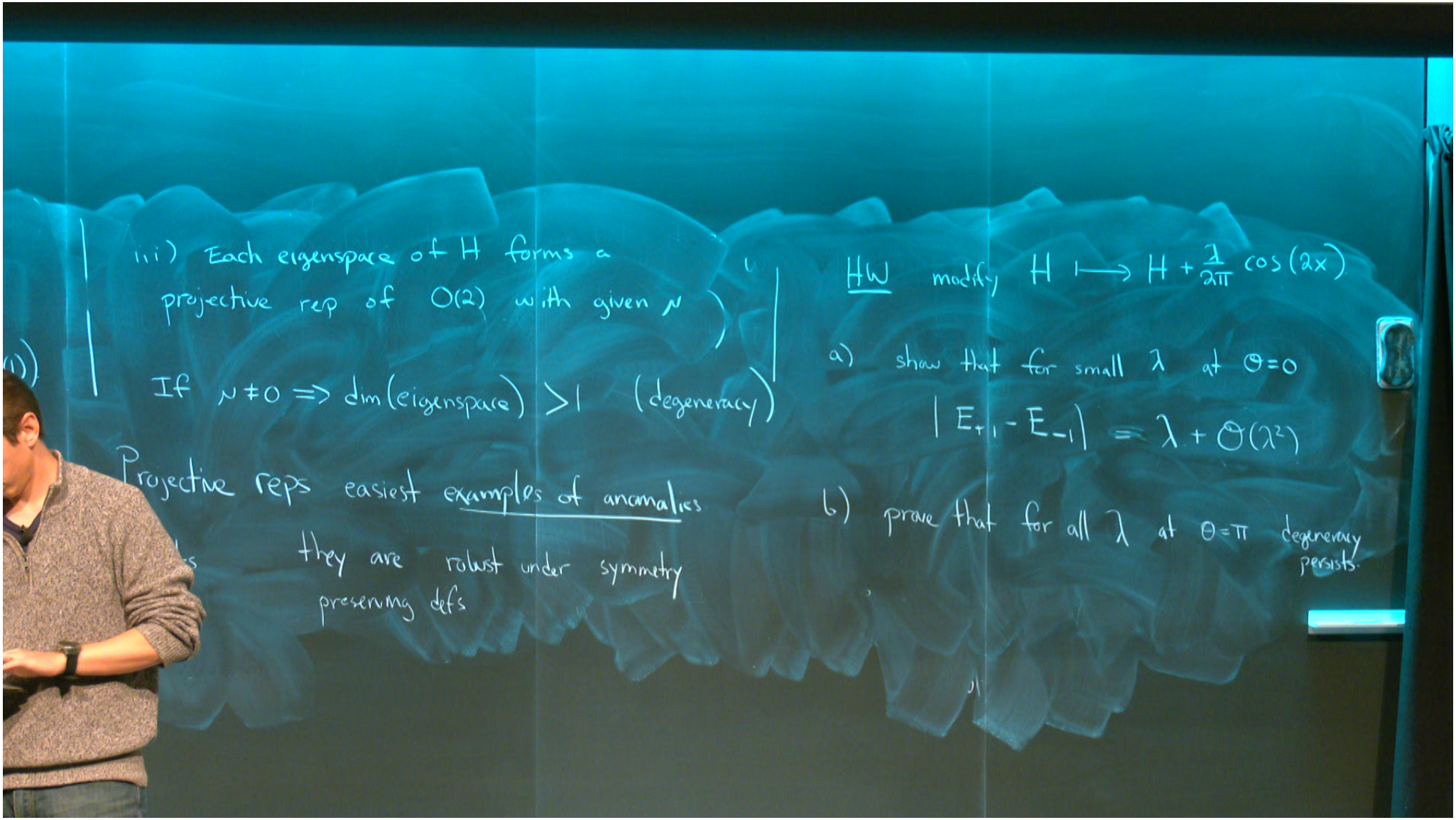
If $\mu \neq 0 \Rightarrow \dim(\text{eigenspace}) > 1$ (degeneracy)

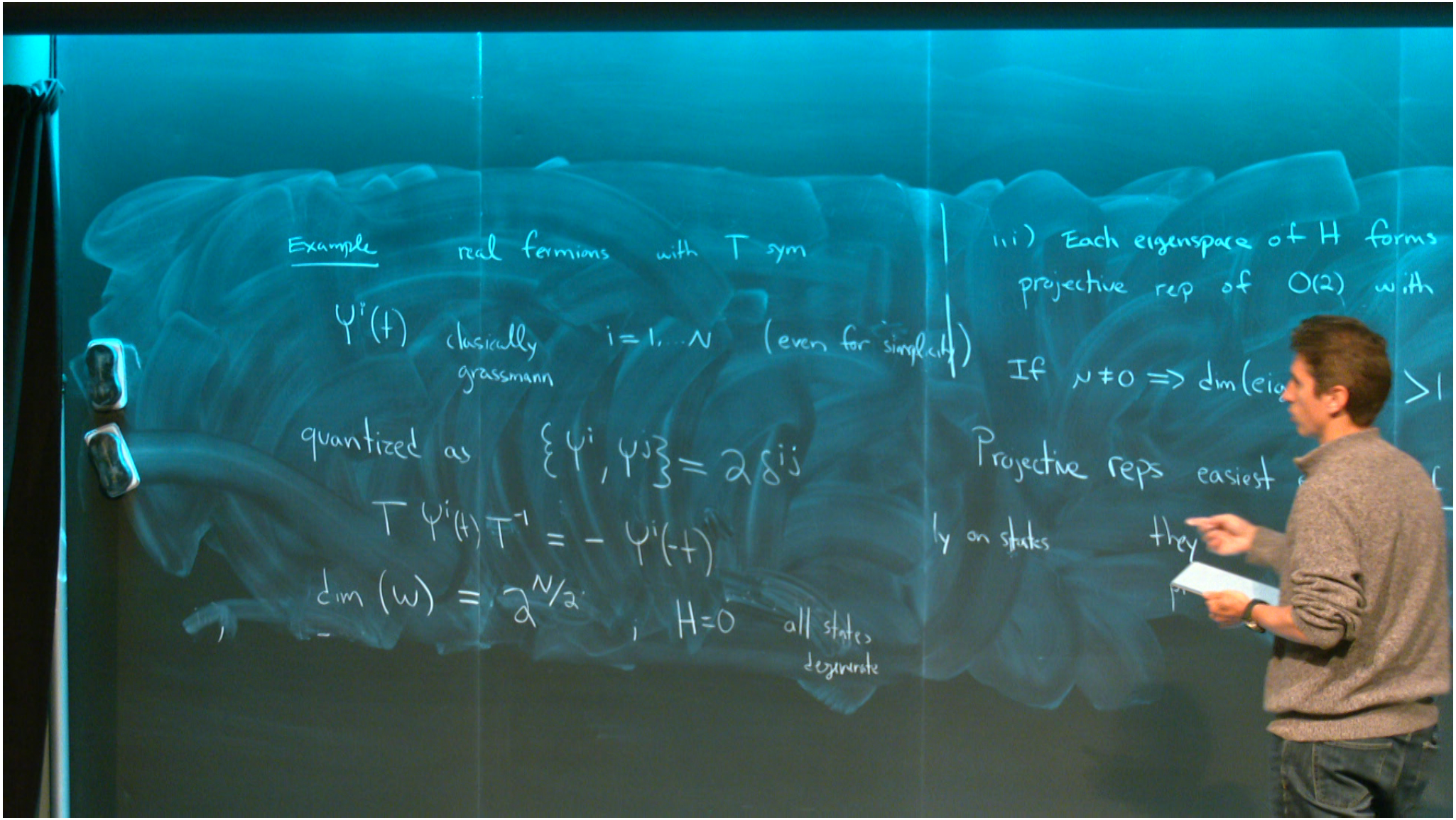
Projective reps easiest examples of anomalies

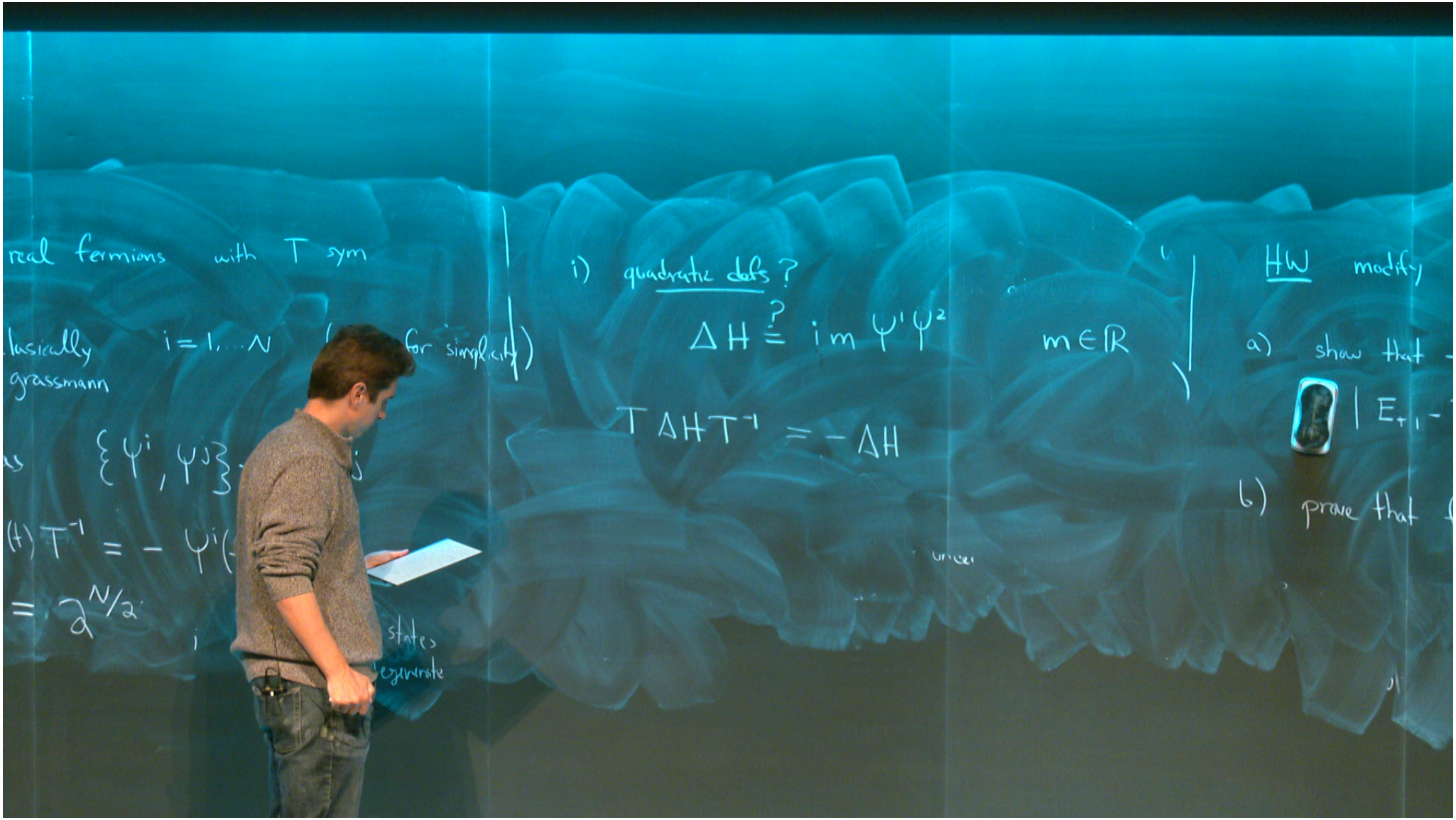
they are robust under symmetry preserving defs

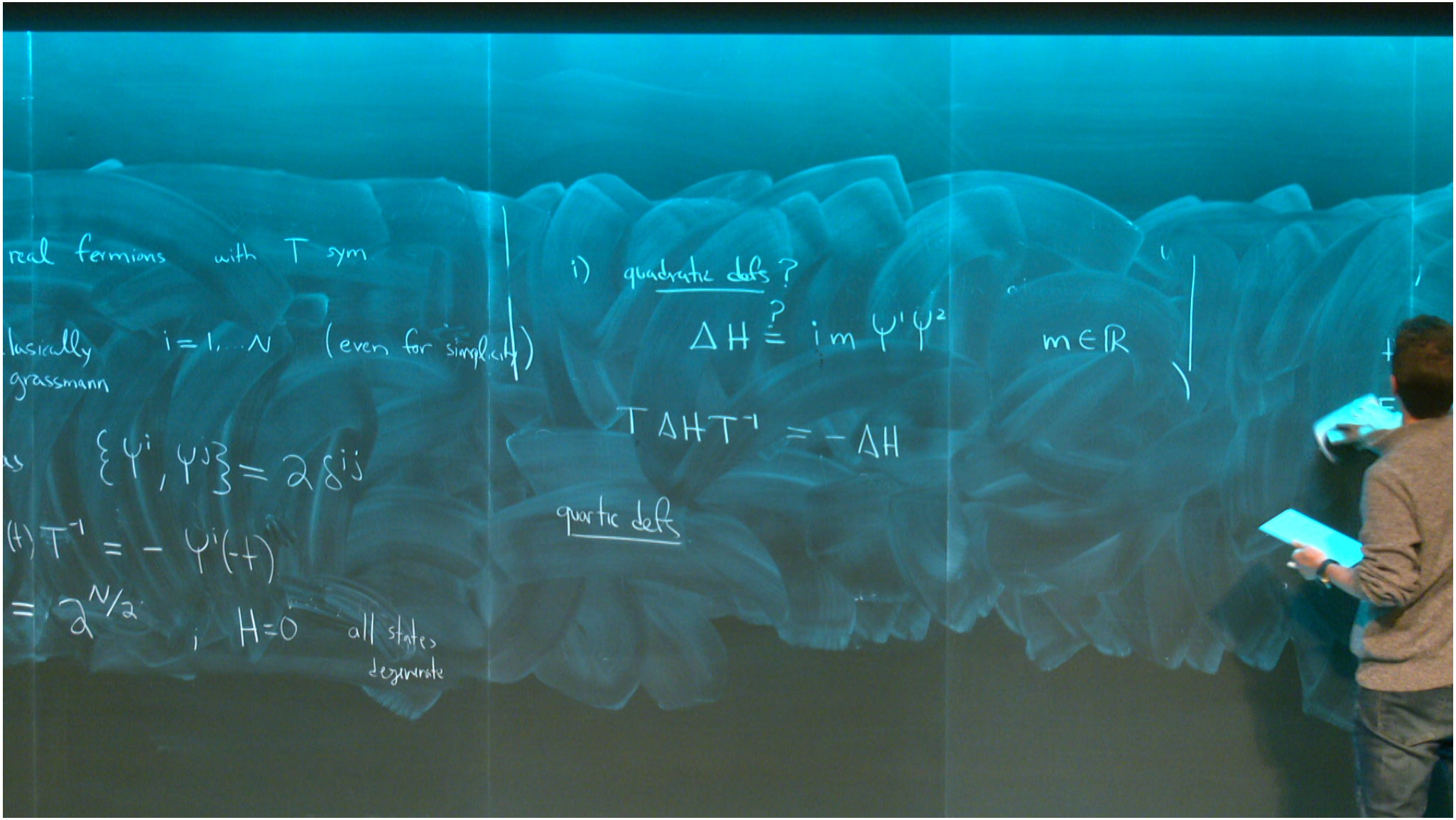
this is a p

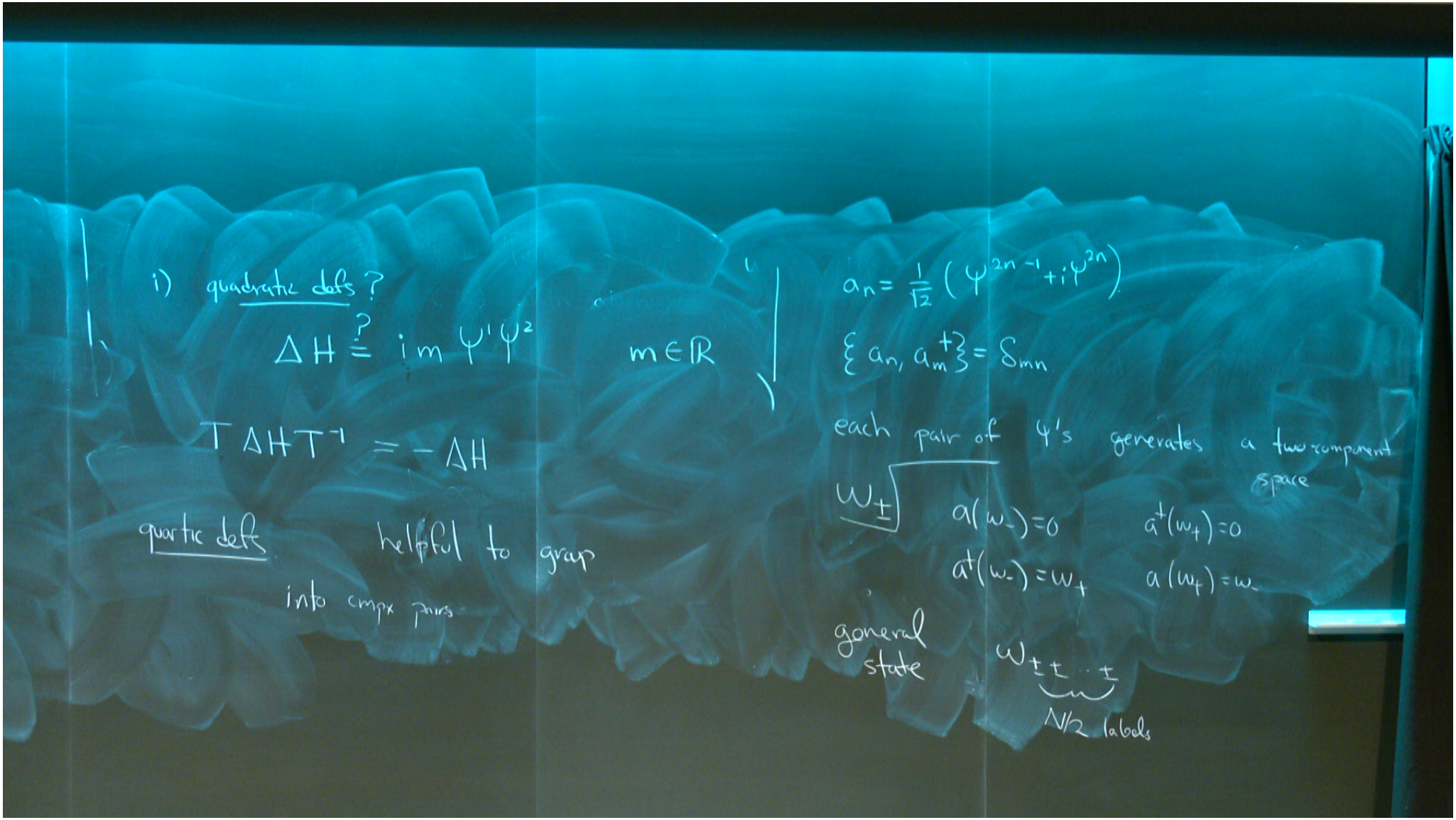
associated to

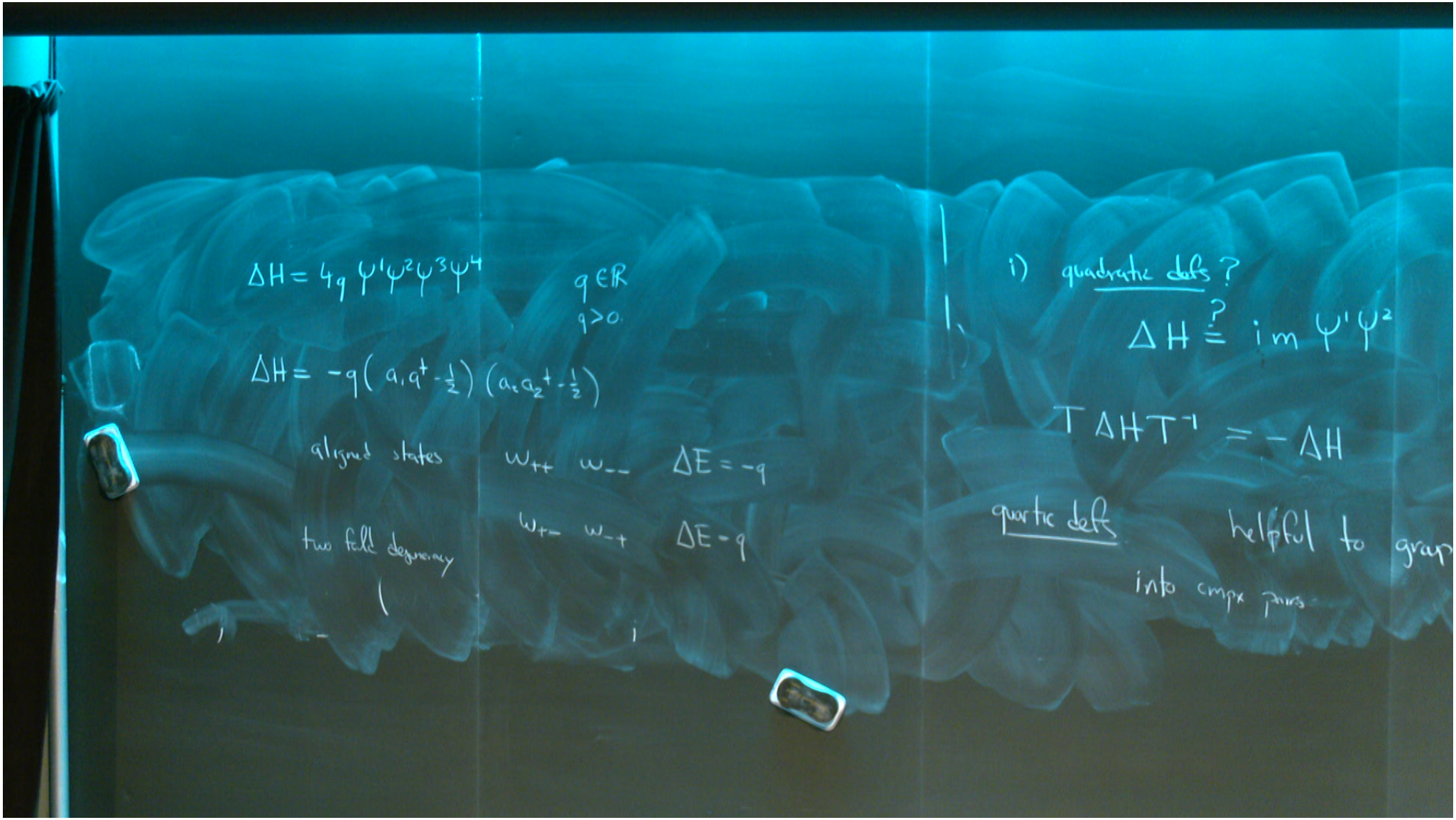

iii) Each eigenspace of H forms a projective rep of O(2) with given ν
If ν ≠ 0 ⇒ dim(eigenspace) > 1 (degeneracy)
Projective reps easiest examples of anomalies
they are robust under symmetry preserving defs
HW modify H ⟼ H + λ/2π cos(2x)
a) show that for small λ at θ=0
|E+1 - E-1| = λ + O(λ²)
b) prove that for all λ at θ=π degeneracy persists.

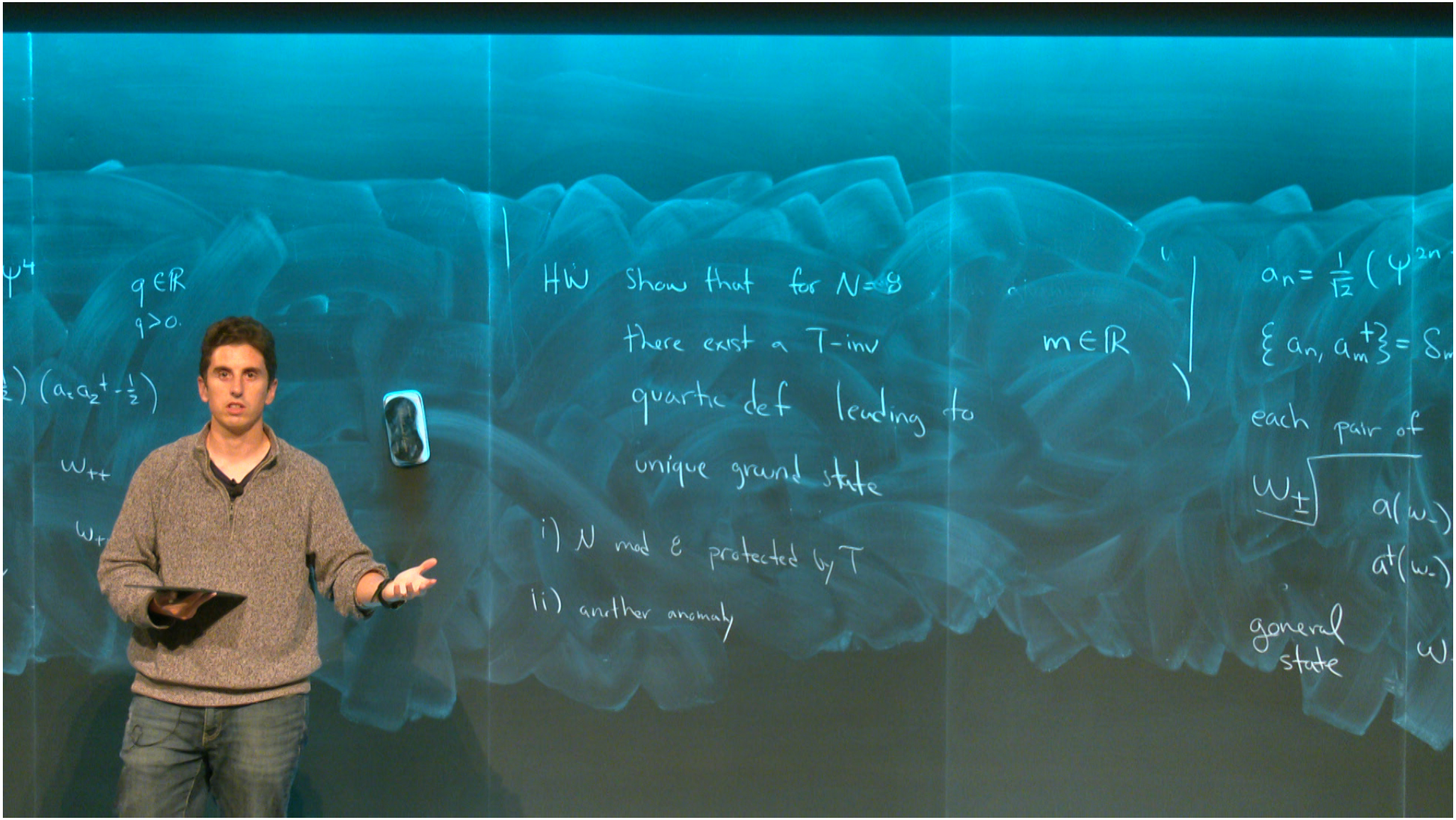

Example real fermions with T sym
$\psi^i(t)$ classically grassmann $i = 1, \ldots N$ (even for simplicity)
quantized as $\{\psi^i, \psi^j\} = 2\delta^{ij}$
$T\,\psi^i(t)\,T^{-1} = -\psi^i(-t)$
$\dim(w) = 2^{N/2}$; $H = 0$ all states degenerate
iii) Each eigenspace of H forms projective rep of O(2) with
If $N \neq 0 \Rightarrow \dim(\text{eig}$ > 1
Projective reps easiest
ly on states they


real fermions with T sym
i) quadratic defs ?
$\Delta H \stackrel{?}{=} im\,\psi^1\psi^2$
$m \in \mathbb{R}$
$T\,\Delta H\,T^{-1} = -\Delta H$
HW modify
a) show that
b) prove that
$= 2^{N/2}$


real fermions with T sym
i=1,...N (even for simplicity)
grassmann
$\{\psi^i, \psi^j\} = 2\delta^{ij}$
$T^{-1} = -\psi^i(-t)$
$= 2^{N/2}$; $H=0$ all states degenerate
i) quadratic defs ?
$\Delta H \stackrel{?}{=} im\psi^1\psi^2$ $m \in \mathbb{R}$
$T\Delta H T^{-1} = -\Delta H$
quartic defs


i) quadratic defs ?
$\Delta H \stackrel{?}{=} im\psi^1\psi^2$
$m \in \mathbb{R}$
$T\Delta H T^{-1} = -\Delta H$
quartic defs helpful to group into cmpx pairs
$a_n = \frac{1}{\sqrt{2}}(\psi^{2n-1} + i\psi^{2n})$
$\{a_n, a_m^\dagger\} = \delta_{mn}$
each pair of ψ's generates a two-component space
$\omega_\pm$
$a(\omega_-) = 0$
$a^\dagger(\omega_+) = 0$
$a^\dagger(\omega_-) = \omega_+$
$a(\omega_+) = \omega_-$
general state
$\omega_{\pm\pm\cdots\pm}$
N/2 labels


$\Delta H = 4q\,\psi^1\psi^2\psi^3\psi^4$
$q \in \mathbb{R}$
$q > 0$
$\Delta H = -q\left(a_1 a_1^\dagger - \frac{1}{2}\right)\left(a_2 a_2^\dagger - \frac{1}{2}\right)$
aligned states ω_{++} ω_{--} $\Delta E = -q$
ω_{+-} ω_{-+} $\Delta E = q$
two fold degeneracy
i) quadratic defs ?
$\Delta H \stackrel{?}{=} i m \psi^1\psi^2$
$T\,\Delta H\,T^{-1} = -\Delta H$
quartic defs
helpful to grou
into cmpx pairs


ψ^4
$q \in \mathbb{R}$
$q > 0$.
$(a_1 a_2^\dagger - \frac{1}{2})$
w_{++}
HW Show that for $N=8$ there exist a T-inv quartic def leading to unique ground state
$m \in \mathbb{R}$
i) N mod 8 protected by T
ii) another anomaly
$a_n = \frac{1}{\sqrt{2}} (\psi^{2n}$
$\{a_n, a_m^\dagger\} = \delta_n$
each pair of
$w_\pm$
$a(w_-)$
$a^\dagger(w_-)$
general state