Title: Line Defect Quantum Numbers and Anomalies

Speakers: Thomas Dumitrescu

Collection: Global Categorical Symmetries

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Abstract: I will consider four-dimensional gauge theories whose global symmetries admit certain discrete 't Hooft anomalies that are intimately related to the (fractionalized) global-symmetry quantum numbers of Wilson-'t Hooft line defects in the theory. Determining these quantum numbers is typically straightforward for Wilson lines, but requires a careful analysis of fermion zero modes for 't Hooft lines, which I will describe for several classes of examples. This in turn leads to a calculation of the anomaly. Along the way I will comment on how this understanding relates to some classic and recent examples in the literature.

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Perimeter Institute 06/10/22 (1.)
GCS Conference
Thomas Dumitrescu (UCLA)

"Line Defect Quantum Numbers and Anomalies"

to appear soon with Daniel Brennan and Clay Córdova

Some References (more to follow):

- Córdova - TD [1806.09592]
- Wang-Wen-Witten [1810.00844]
- "QFT & Geometry" Seminar (05/28/20)
- Lee-Ohmori-Tachikawa [2108.05369]
- Bhardwaj-Bullimore-Ferrari-Schäfer-Nameki [2205.15330]

I apologize in advance for any references I will fail to mention.

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(2.)

(Some) 't Hooft anomalies

$\updownarrow$

fractionalized Q.N. of certain defects.

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4d G gauge theories

defects ~ W, T lines.

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$G = U(1)$, $SU(2)$.

③

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$\underbrace{}$ free $G = U(1)$ Maxwell theory.

$U(1)$ $a^{(1)}$ connection

$$f^{(2)} = d a^{(1)}$$

$$S = \frac{1}{2e^2} \int f \wedge * f.$$

3.

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3.

$w_2(M_4) \neq 0$. $\underbrace{\text{(also } w_1(M_4)\neq 0 \text{ is ok).}}_{\text{T - symmetry.}}$

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T - symmetry.

<u>Key:</u> intrinsic 1-form symmetry [GKSW]

$$U(1)_e^{(1)} \times U(1)_m^{(1)}$$

$$Q_e(\Sigma_2) = \frac{1}{e^2}\int_{\Sigma_2} * f^{(2)} \qquad Q_m(\Sigma_2) = \frac{1}{2\pi}\int_{\Sigma_2} f^{(2)}$$

4

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4

b.g. fields $\quad B^{(2)}_{e,m} \to B^{(2)}_{e,m} + \underbrace{d\lambda^{(1)}_{e,m}}_{\text{fluxes in } 2\pi\mathbb{Z}}.$

(4)

b.g. fields $B^{(2)}_{e,m} \to B^{(2)}_{e,m} + \underbrace{d\lambda^{(1)}_{e,m}}_{\text{fluxes in } 2\pi\mathbb{Z}}$

$$S = \frac{i}{2e^2}\int_{M_4} (f^{(2)} - B_e^{(2)}) \wedge * (f^{(2)} - B_e^{(2)}) + \frac{i}{2\pi}\int B_m^{(2)} \wedge (f^{(2)} - B_e^{(2)})$$

$$\left[a^{(1)} \to a^{(1)} + \lambda_e^{(1)}\right]$$

't Hooft anomaly:

$$\delta S = -\frac{i}{2\pi}\int_{M_4} d\lambda_m^{(1)} \wedge B_e^{(2)}$$

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$$M_4 \qquad + \frac{i}{2\pi} \int B_m^{(2)} \wedge \left(f^{(2)} - B_e^{(2)} \right).$$

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$$\boxed{\mathcal{A}_5 = \frac{i}{2\pi} \int_{M_5} B_m^{(2)} \wedge dB_e^{(2)}}$$

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This talk: $\left(\mathbb{Z}_{2,e}^{(1)} \subseteq U(1)_e^{(1)}\right) \times \left(\mathbb{Z}_{2,m}^{(1)} \subseteq U(1)_m^{(1)}\right).$

5.

$\Rightarrow \quad dB_{e,m}^{(1}$

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5.

$$\Rightarrow \quad dB^{(2)}_{e,m} = 0.$$

$$\oint_{\Sigma_2} B^{(2)}_{e,m} \in \mathbb{R}/2\pi\mathbb{Z}.$$

$\mathbb{Z}^{(1)}_{2,e} \times \mathbb{Z}^{(1)}_{2,m}$ b.g. fields

$$b^{(2)}_{e,m} \in H^2(M_4, \mathbb{Z}_2).$$

$$[B^{(2)}_{e,m}] = \pi \, b^{(2)}_{e,m}$$

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6.

$b^{(2)}_{e,m} \in H^2(M_4, \mathbb{Z}_2)$.

$$[B^{(2)}_{e,m}] = \pi \, b^{(2)}_{e,m}$$

$$A_5 = \frac{i}{2\pi}\int_{M_5} \underbrace{B^{(2)}_m}_{\pi\, b^{(2)}_m} \underbrace{\wedge}_{\cup} \underbrace{dB^{(2)}_e}_{?}$$

$$\frac{dB^{(2)}_e}{2\pi} \longrightarrow \beta(b^{(2)}_e) \in H^3(M_4, \mathbb{Z}_2)$$

Bockstein map.

6.

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$2\pi \quad M_5$

$\pi \, b_m^{(2)} \qquad \cup \qquad ?$

$$\frac{dB_e^{(2)}}{2\pi} \longrightarrow \beta(b_e^{(2)}) \in H^3(M_5, \mathbb{Z}_2)$$

"$\frac{1}{2}\delta b_e^{(2)}$" Bockstein map.

$$\boxed{A_5 = i\pi \int_{M_5} b_m^{(2)} \cup \beta(b_e^{(2)})} \qquad (6.)$$

$$\frac{d\,\Sigma_e^{(1)}}{2\pi} \longrightarrow \beta(b_e^{(2)}) \in H^3(M_4, \mathbb{Z}_2)$$

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6.

$$\boxed{\mathcal{A}_5 = i\pi \int_{M_5} b_m^{(2)} \cup \beta(b_e^{(2)})}$$

Enrich Maxwell thy

~ couple to $G_f^{(0)}$ flavor symmetry.

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$\uparrow$ connected.

$G_f^{(0)}$ is independent of $U(1)_e^{(1)} \times U(1)_m^{(1)}$.

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Typical example: $G_f^{(0)} = SO(3)$.

(i

7.

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Two possibilities.

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(i) line transforms $R \in Rep(SO(3))$

$$\mathrm{Tr}_R \, P\exp\left(i \oint_L A^{(1)}\right)$$ b.g. field $A^{(1)}$

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local counterterm on line L.

(7.)

$\Longrightarrow$ |

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⑦

$\Rightarrow$ the choice of $R \in \mathrm{Rep}(SO(3))$ is scheme-dep / UV sensitive / ....

(ii) L can transform projectively.

$$SO(3) = \frac{SU(2)}{\mathbb{Z}_2}.$$

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"symmetry fractionalization"

obstructed by H^3 class defining a 2-gp.

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$$SO(3) = \frac{SU(2)}{\mathbb{Z}_2} \qquad \tilde{R} \in \mathrm{Rep}(SU(2))$$

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note:

only $\tilde{R}/R$.

$\Longleftrightarrow$ rep. of line under $\mathbb{Z}_2$.

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$\sim$ characterizes 't Hooft anomalies in 1d QM w SO(3) symmetry.

L

defining a 2-gp.

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$\partial \Sigma_2 = L$ $\qquad$ $\int_{\Sigma_2} w_2(SO(3))$.

L

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8.

Maxwell thy:

$b_e^{(2)} = w_2(SO(3))$ or 0.

W

$b_m^{(2)} = n$

T

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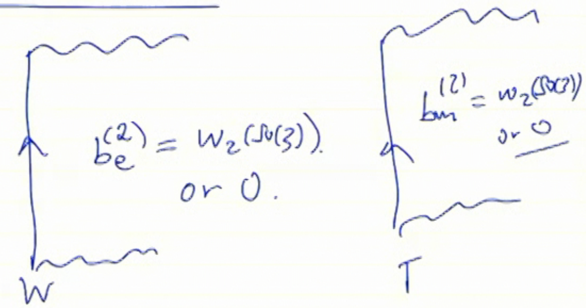
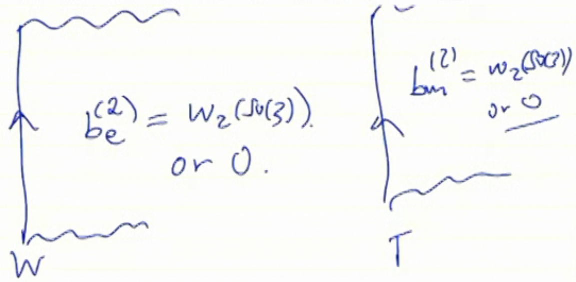


$$A_5 = i\pi \int_{M_5} \underbrace{w_2(SO(3))}_{b_e} \cup \underbrace{\beta(\underbrace{w_2(SO(3))}_{b_m})}_{w_3(SO(3))}$$



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$w_3(SO(3))$.

Two generalizations

(i) $SO(3)$ not. can fractionaliz

4.

be.

Two generalizations $\qquad w_3(SO(3))$.

(i) $SO(3)$ rot. can fractionalize

$$b^{(2)}_{e,m} \sim w_2(M_4).$$

[(ii) T : $L \sim$ Kramers doublet $T^2=-1$

$$b^{(2)}_{e,m} = w_1 \cup w_1$$]

(4.)

[(ii) $\underline{\underline{T}}$: $L \sim$ Kramers doublet $T^2 = -1$

$b^{(2)}_{e,m} = w_1 \cup w_1$]

(9.)

if W, T are fermions.

$$b_e^{(2)} = b_m^{(2)} = w_2 \, (M_4).$$

$$\Rightarrow \quad \mathcal{A}_5 = i \int_{M_5} w_2 \cup w_3$$

"all fermion electrodynamics".

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RG flow :

UV: non-abelian g.t. + matter

$\downarrow$

pure Maxwell.

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$G = SU(2)$.

One real Higgs

$\phi^{a=1,2,3}$ ~ adjoint.

$\lambda_\alpha^{a\,i}$, $i=1,2$. $N_f = 2$ of $SU(2)$-adjoint Weyl fermions.

$\alpha = 1,2$ spin index

(10)

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$$\mathcal{L}_y \sim \epsilon_{ij}\, \phi^a \lambda_\alpha^{bi} \lambda_\beta^{cj} \epsilon^{\alpha\beta} \epsilon_{abc}$$

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i, j ~ doublets of SU(2) flavour symmetry.

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$$\frac{Spin(4)_L \times SU(2)_R}{\mathbb{Z}_2}$$

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$i, j \sim$ doublets of $SU(2)$ flavour symmetry.

$$\frac{Spin(4)_L \times SU(2)_R}{\mathbb{Z}_2}$$

$SU(2)_R$ of pure $\mathcal{N}=2$ $SU(2)$ SYM.

$$(-1)^F = -\mathbb{1} \in SU(2)_R.$$

$SU(2)_R \implies$ consider $SO(3)_R$ bundles.

as long as $w_2(SO(3)_R) = w_2(M_4)$.

Symmetry fractionalization

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$\mathbb{Z}_2^{(1)}$ center symmetry.

$B^{(2)}_{UV} \sim \mathbb{Z}_2$ b.g. field.

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(T)

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(T) fractionalizes due to fermi zero modes,

monopole ~ "hyper"

$SU(2)_{spin} \times SU(2)_R$

$$\underbrace{(1/2, 0)}_{q_\alpha} \oplus \underbrace{(0, 1/2)}_{h^i}.$$

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add to pure SU(2)

$W=2$ g.t. 1 $\square$ hyper

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