Title: TQFT's and flat connections

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Collection: Global Categorical Symmetries

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Abstract: I will discuss some of the (higher) structure of TQFT's that can be deformed by flat connections for continuous global symmetries, focusing on examples coming from twists of 3d supersymmetric theories, and the manifestation of this structure in boundary VOA's.

Twists & (Flat) Connections

GCS @ PI June 2022

Tudor Dimofte, University of Edinburgh

cf. [Creutzig - TD - Garner - Geer '21]
[Ballin - Niu '22]
[Ballin - Niu + Creutzig - TD WIP]
[Creutzig - TD - Feigin - Lentner WIP]

Q: What extended TQFT's arise from twists of SUSY (gauge) theories in $d \geq 3$?

- d-category of boundary conditions
- $(d-1)$-cat of (hyper)surface ops

  ⋮
- state spaces & algebras of local ops
- M^d invariants (maybe)

Not fully known, in almost any example!

Exception: B-twist of 3d $\mathcal{N}=4$ σ-model to T^*X ['08]

There are many (famous...) partial constructions, and much exciting current work in progress.

Setting for this talk: twists of 3d $\mathcal{N}=4$ gauge theories

Data: G_c, T^*V (+ sometimes CS levels)

compact

[Gaiotto-Witten '08]

Twists: $A \xleftrightarrow{\text{3D MS}} B$

A: reduction of Witten's Donaldson twist ~ 2d A twist

B: Blau-Thompson, Rozansky-Witten ~ 2d B twist

Plan:
- Structure of expected TQFT's [dg!]
- Role of continuous global symmetry [connections!]
- Application: big & small quantum groups @ $q^{2h}=1$.

Structure

Slogan: everything is dg / derived

Not unitary, not semisimple.

Often infinite... but tamed by R-symmetry (coh. $\mathbb{Z}$ grading)

State spaces

Typically $\dim Z(\Sigma) = \infty$.

E.g. $Z_A(S^2) = \mathbb{C}[\mathcal{M}_{\text{Coulomb}}]$ $\quad Z_B(S^2) = \mathbb{C}[\mathcal{M}_{\text{Higgs}}]$

But $\mathbb{Z}$-graded $\sim U(1) \subset SU(2)_R$

CFT $\Rightarrow$ $$Z(\Sigma) = \bigoplus_{d \geq 0} \underbrace{Z(\Sigma)^{(d)}}_{\text{fin. dim}}$$

Better: $Z(\Sigma)$ are dg vector spaces w/ bounded-below cohomology...

Example: free hypermultiplet $\qquad G = 1 \qquad V = \mathbb{C}$

$$Z_A^{hyper}(S^1) \approx D\text{-mod}(\mathbb{C}((z)))$$

modules for mode algebra of $\beta(z), \gamma(z)$

$$\beta(z) = \sum_{n\in\mathbb{Z}} \beta_n z^{-n-1} \qquad \gamma(z) = \sum_{n\in\mathbb{Z}} \gamma_n z^{-n-1}$$

$$[\beta_n, \gamma_m] = \delta_{n+m+1}$$

$$\mathbb{1} = (\text{mode algebra}) / \begin{matrix}\beta_n = 0 \\ \gamma_n = 0\end{matrix} \quad n \geq 0$$

$\underline{\beta(z),\ \gamma(z) \text{ regular}}$

$$\text{End}^\bullet(\mathbb{1}) = \mathbb{C} = \mathbb{C}[\mathcal{M}_{coulomb}]$$

Flavor vortex lines:

$$V_\ell = (\text{mode alg.}) / \begin{matrix}\beta_{n+\ell} = 0 \\ \gamma_{n-\ell} = 0\end{matrix} \quad n \geq 0$$

$$\beta(z) \sim z^{-\ell}(\text{reg}) \qquad \gamma(z) \sim z^{\ell}(\text{reg})$$

- form nontrivial bound states!

$$\text{Hom}^\bullet(V_\ell, V_m) \neq 0 \ldots$$

Dually: SQED $\quad G_c = U(1) \quad V = \mathbb{C}$

$Z_B^{SQED}(S^1) \ni$ Wilson lines W_ℓ

w/ nontrivial bound states

$Z_A^{SQED}(S^1) \simeq \text{D-mod}\left(\mathbb{C}((z)) / \mathbb{C}((z))^*\right)$

has no more vortices (screened)

but $\mathbb{1} = \mathcal{O}_{\mathbb{C}[[z]] / \mathbb{C}[[z]]^*}$

reproduces $\text{End}(\mathbb{1}) = \mathbb{C}[T^*\mathbb{C}]$

[Braverman-Finkelberg-Nakajima '15]

Rough summary:

	B	A
matter + flavor	sheaves	(flavor) vortex lines
gauge	Wilson lines	"dual sheaves"

MS

Holomorphic boundary conditions exist!

[Gaiotto '16]
[Costello - Gaiotto '18]

Analogous to WZW b.c. for Chern-Simons ($A_{\bar{z}}|_{\partial} = 0$)

- support VOA's $\mathcal{V}$ — not rational
- $Z(\Sigma) \simeq$ derived conformal blocks $(\mathcal{V}, \Sigma)$
  (chiral homology)
- $Z(S^1) \simeq D^b\, \mathcal{V}\text{-mod}$

[Costello - Creutzig - Gaiotto '19]
[Ballin - Niu '22]

Examples:

$\mathcal{V}_B^{hyper} \simeq$ symplectic fermions

$$\chi(z)\,\psi(w) \sim \frac{1}{(z-w)^2}$$

$\mathcal{V}_A^{hyper} =$ symplectic bosons

$$\beta(z)\,\gamma(w) \sim \frac{1}{z-w}$$

$\mathcal{V}_B^{SQED} \simeq WZW[\mathfrak{gl}(1|1)]$

$\mathcal{V}_A^{SQED} =$ symplectic bosons $\otimes\, bc /\!/ U(1)$

Global symmetry

extending [Gaiotto '16], [Creutzig-TD-Garner-Geer '21]

cf [Bhardwaj-Bullimore-Ferrari-Schafer-Nameki '22]
0 + higher in untwisted 3d $\mathcal{N}=4$

Consider continuous, 0-form flavor sym Γ_c
that (WLOG) acts on matter.

$\Gamma := (\Gamma_c)_{\mathbb{C}}$

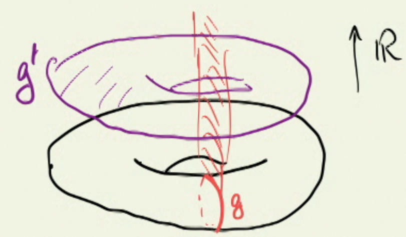
Leads to enhanced (& higher) structures upon twisting.

B-twist

Γ acts on everything and the TQFT is enhanced by $\boxed{\text{flat } \Gamma \text{ connections}}$.

cf. discrete symmetry in physical QFT

- state spaces $\quad \underset{\substack{\uparrow \\ \text{flat } \Gamma\text{-conn on } \Sigma}}{Z(\Sigma, A)} \circlearrowright \mathrm{Stab}_\Gamma(A)$



$g g' = g' g$

globally: $\quad \begin{array}{c} Z(\Sigma) \\ \downarrow \\ \mathrm{Flat}_\Gamma(\Sigma)/\text{gauge} \end{array} \quad$ coh. sheaf

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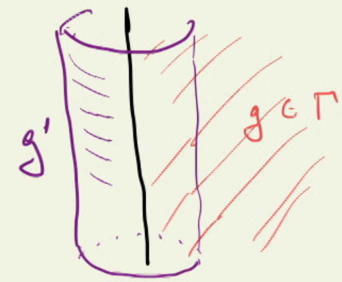
- state spaces $\quad Z(\Sigma, A) \circlearrowleft \mathrm{Stab}_\Gamma(A)$

  $\uparrow$ flat Γ-conn on Σ

- line operators $\quad Z(S^1, A) \circlearrowleft \mathrm{Stab}_\Gamma(A)$

  $\uparrow$ flat Γ-conn on S^1

  $\sim$ monodromy $g \in \Gamma$



globally $\quad Z(S^1)$

$\downarrow$ coh. sheaf of cats

$\Gamma/_{\mathrm{ad}}\Gamma$

$\mathrm{Hom}^\cdot(L_g, L_{\tilde{g}}) = 0$

if $g, \tilde{g}$ not conjugate

$\sim$ 1-form symmetry

generated by flavor Wilson lines.

B-twist

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  A: flat Γ-conn on Σ

- line operators $Z(S^1, A) \circlearrowleft \mathrm{Stab}_\Gamma(A)$

  A: flat Γ-conn on S^1

  $\sim$ monodromy $g \in \Gamma$

<u>Example</u> : free hypermultiplet $\qquad \Gamma_c = Sp(1)$, $\quad \Gamma = SL(2,\mathbb{C})$

$$Z(\Sigma_g, A=0) \simeq \mathbb{C}[\overbrace{x, y}^{\text{bos}}, \overbrace{\chi_1, \psi_1, \dots, \chi_g, \psi_g}^{\text{fermi}}]$$

$$Z(S^1, A=0) \simeq D^b\mathrm{Coh}(T^*\mathbb{C})$$

∞, non-semisimple

$$Z(\Sigma_g, A \text{ generic}) = \mathbb{C}[\overbrace{\chi_1, \psi_1, \dots, \chi_{g-1}, \psi_{g-1}}^{\text{all fermi}}]$$

$$Z(S^1, A \text{ generic}) = D^b\mathrm{Coh}(0) \simeq dg\,\mathrm{Vect}$$

finite, semisimple

B-twist $\boxed{\text{flat } \Gamma \text{ connections}}$

$\Gamma_c = Sp(1)$

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∞, non-semisimple

finite, semisimple

VOA

$$\chi(z)\psi(w) \sim \frac{1}{(z-w)^2} \quad \rightsquigarrow \quad \chi(z)\psi(w) \sim \frac{1}{(z-w)^2} + \frac{A(w)}{z-w}$$

$\circlearrowleft$

$SL(2,\mathbb{C})$ global symmetry

$$\begin{pmatrix} \chi \\ \psi \end{pmatrix} \mapsto g \begin{pmatrix} \chi \\ \psi \end{pmatrix}$$

$\simeq$ free fermions, rational VOA

promote to a holomorphic action

$$A(z) = g(z)^{-1} \partial_z g(z)$$

A-twist

The TQFT is enhanced by Γ-monopoles — $\boxed{\text{holomorphic } \Gamma\text{-conn}^s}$ in $d \le 2$.

The infinitessimal Γ (Lie Γ) action is trivialized (Q-exact) and an ∞ 1-form symmetry emerges.

- State spaces $\quad Z(\Sigma, A) \circlearrowleft \mathrm{Lie}(\Gamma)$ trivial $\qquad H^\bullet(\mathrm{Stab}_\Gamma(A))$ may still act...

  hol^c connection, on a hol^c bundle

  $A \in \mathrm{Bun}_G(\Sigma)$

  globally
  $$\begin{array}{c} Z(\Sigma) \\ \downarrow \\ \mathrm{Bun}_G(\Sigma) \end{array}$$
  flat sheaf

- line ops $\quad Z(S^1, A) \circlearrowleft \mathrm{Lie}(\Gamma)$ trivial $\qquad$ but there's more!

  hol^c conn^n on S^1 $\sim$ holonomy $g \in \Gamma/_{\mathrm{ad}\,\Gamma}$

  globally
  $$\begin{array}{c} Z(S^1, A) \\ \downarrow \\ \Gamma /\!/_{\mathrm{ad}}\, \Gamma \end{array}$$
  flat sheaf of categories

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The infinitessimal Γ action is $\underline{\text{trivialized}}$ and an ∞ 1-form symmetry emerges.
(Lie Γ) (Q-exact)

- line ops $\quad Z(S^1, A) \supset \mathrm{Lie}(\Gamma)$ trivial

hol^c conn^n on S^1 $\sim$ holonomy $g \in \Gamma/_{ad\,\Gamma}$

globally $\quad \begin{matrix} Z(S^1, A) \\ \downarrow \\ \Gamma/\!/_{ad\,\Gamma} \end{matrix}$ flat sheaf of categories

Claim: $\exists$ emergent 1-form symmetry $\Lambda^* := \mathrm{cochar}\,\Gamma = \mathrm{Hom}(U(1), \mathrm{torus}(\Gamma))$

w/ "charges" $\Gamma/_{ad\,\Gamma} \simeq \mathrm{torus}(\Gamma)/_{Weyl}$

Its generators are torus(Γ) flavor vortex lines $\quad V_\ell \quad \ell \in \Lambda^*$

$$V_\ell \otimes V_{\ell'} \simeq V_{\ell+\ell'}$$

A-twist

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Mirror-dual perspective: B-twist of SQED $G_c = U(1)$ $V = \mathbb{C}$

- topological (A-type) $\Gamma_c = U(1)$ flavor symmetry

  $\rightsquigarrow$ $\mathbb{Z}$ 1-form symmetry

- generators are Wilson lines W_ℓ $\qquad W_\ell \otimes W_{\ell'} = W_{\ell+\ell'}$
- eigenobjects are vortex lines $\mathbb{V}_\alpha$ $\quad \alpha \in \mathbb{C}/_{2\pi i \mathbb{Z}}$

  s.t. $\mathrm{hypers}\big|_{line} = 0$ $\qquad$ gauge connection$\big|_{line} \sim \alpha\, d\theta$

Global symmetry & twists: summary

given matter-like flavor symmetry Γ_c ...

	$Z(\Sigma)$	Lie(Γ) 0-form
A	flat sheaf over $Bun_\Gamma(\Sigma)$	trivial
B	coh sheaf over $Flat_\Gamma(\Sigma)$	nontrivial

	$Z(S^1)$	emergent 1-form
A	flat sheaf over $\Gamma/_{Ad\,\Gamma}$	Λ^* internal
B	coh sheaf over $\Gamma/_{Ad\,\Gamma}$	Λ "external" (flavor Wilson lines)

Application

Review: $U_q(\mathfrak{g}) = \mathbb{C}\langle E_i, F_i, K_i \rangle / _{\text{quantum of relations}}$

$$U_q(\mathfrak{sl}_2) = \mathbb{C}\langle E, F, K \rangle \Big/ \begin{matrix} KE = q^2 EK \\ KF = q^{-2} FK \end{matrix} \quad [E,F] = \frac{K - K^{-1}}{q - q^{-1}}$$

At $q = e^{\frac{i\pi}{k}}$, $U_q(\mathfrak{g})$ has an exceptional center,

generated by E_i^k, F_i^h, K_i^{2h} $\sim$ coords on $G^\vee$

$$\Rightarrow \quad \begin{array}{c} U_q(\mathfrak{g})\text{-mod} \\ \downarrow \\ G^\vee \approx \mathrm{Spec}(\text{center}) \end{array}$$

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Fiber over $1 \in G^\vee$ = reps of $u_q(\mathfrak{g}) = U_q(\mathfrak{g}) / (E^h = 0, F^h = 0, K^{2h} = 1)$
"small quantum group"

- $u_q(\mathfrak{g})$-mod is famously non-semisimple
- a small, semisimple part of it appears in Chern-Simons / RT

Application

$$U_q(\mathfrak{g})\text{-mod}$$
$$\downarrow$$
$$G^\vee \approx \mathrm{Spec}(\text{center})$$

Fiber over $1 \in G^\vee$ = reps of $u_q(\mathfrak{g}) = U_q(\mathfrak{g}) / (E^h = 0, F^h = 0, K^{2h} = 1)$

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Conjecture For $\mathfrak{g}$ type A, $\exists\ \mathcal{T}_{\mathfrak{g},k}$ 3d $N=4$

st. $Z_A(S^1) \simeq D^b\, u_q(\mathfrak{g})\text{-mod}$

$\simeq \qquad \simeq$

Feigin-Tipunin VOA-mod

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$\simeq$ $\qquad$ $\simeq$

Feigin-Tipunin VOA-mod

- $\mathcal{T}_{\mathfrak{g},k} := G_k \backslash T[G] \circlearrowleft G^\vee$

- Roughly, G_k Chern-Simons theory w/ $T[G]$ matter. A-twisted.

- Global (B-type) symmetry $G^\vee$ $\Rightarrow$ $\begin{array}{c} Z_A(S^1) \\ \downarrow \\ G^\vee / \mathrm{Ad}\, G^\vee \end{array}$, TQFT enhanced by flat $G^\vee$ connections