Title: Non-Invertible Higher-Categorical Symmetries

Speakers: Sakura Schafer-Nameki

Collection: Global Categorical Symmetries

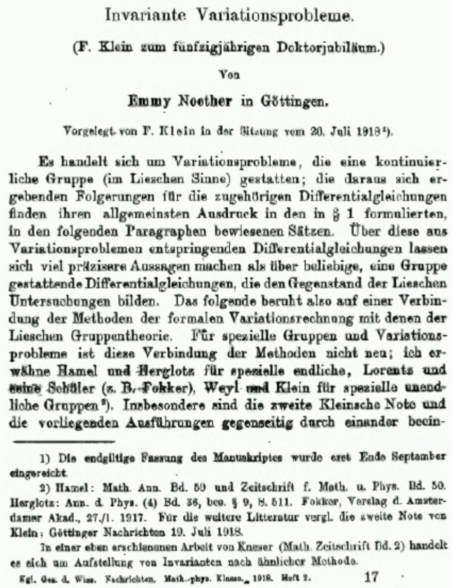
Date: June 10, 2022 - 9:00 AM

URL: https://pirsa.org/22060019

Abstract: "I will discuss a proposal for generating non-invertible symmetries in QFTs in d>2, by gauging outer automorphisms. First this will be illustrated in 3d, where the framework is relatively well established, and then extended to higher dimensions. For 4d gauge theories, a comparison to other approaches to non-invertible symmetries is provided, in particular the map to gauging theories with mixed anomalies. This talk is based on work that appeared in 2204.06564 and in progress, with Lakshya Bharwaj (Oxford), Lea Bottini (Oxford)
and Apoov Tiwari (Stockholm)."

Symmetries from Topological Operators

2022: Topological defects in a QFT should be interpreted as symmetries. This is a long way from Noether's 1918 continuous "Lieschen" type symmetries, though the core idea is the same:



Invariante Variationsprobleme.

(F. Klein zum fünfzigjährigen Doktorjubiläum.)

Von

Emmy Noether in Göttingen.

Vorgelegt von F. Klein in der Sitzung vom 26. Juli 1918[1]).

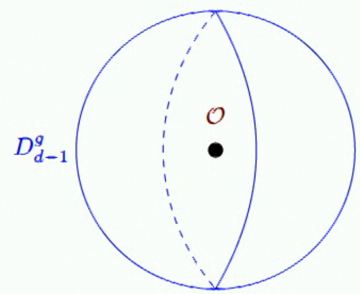
Es handelt sich um Variationsprobleme, die eine kontinuierliche Gruppe (im Lieschen Sinne) gestatten; die daraus sich ergebenden Folgerungen für die zugehörigen Differentialgleichungen finden ihren allgemeinsten Ausdruck in den in § 1 formulierten, in den folgenden Paragraphen bewiesenen Sätzen. Über diese aus Variationsproblemen entspringenden Differentialgleichungen lassen sich viel präzisere Aussagen machen als über beliebige, eine Gruppe gestattende Differentialgleichungen, die den Gegenstand der Lieschen Untersuchungen bilden. Das folgende beruht also auf einer Verbindung der Methoden der formalen Variationsrechnung mit denen der Lieschen Gruppentheorie. Für spezielle Gruppen und Variationsprobleme ist diese Verbindung der Methoden nicht neu; ich erwähne Hamel und Herglotz für spezielle endliche, Lorentz und seine Schüler (z. B. Fokker), Weyl und Klein für spezielle unendliche Gruppen[2]). Insbesondere sind die zweite Kleinsche Note und die vorliegenden Ausführungen gegenseitig durch einander beein-

1) Die endgiltige Fassung des Manuskriptes wurde erst Ende September eingereicht.

2) Hamel: Math. Ann. Bd. 59 und Zeitschrift f. Math. u. Phys. Bd. 50. Herglotz: Ann. d. Phys. (4) Bd. 36, bes. § 9, S. 511. Fokker, Verslag d. Amsterdamer Akad., 27./1. 1917. Für die weitere Litteratur vergl. die zweite Note von Klein: Göttinger Nachrichten 19. Juli 1918.

In einer eben erschienenen Arbeit von Kneser (Math. Zeitschrift Bd. 2) handelt es sich um Aufstellung von Invarianten nach ähnlicher Methode.

Kgl. Ges. d. Wiss. Nachrichten. Math.-phys. Klasse. 1918. Heft 2. 17



Higher Form, Higher Group, Higher Cat

Recent explosion of types of symmetries:

1. Higher-form symmetries $\Gamma^{(p)}$:
   p-dimensional charged defects, whose charge is measured by topological operators $D^g_{q=d-(p+1)}$, $g \in \Gamma^{(p)}$ [Gaiotto, Kapustin, Seiberg, Willett, 2014]

$$D_q^g \otimes D_q^h = D_q^{gh} \,, \qquad g, h \in \Gamma^{(p)}$$

2. Higher-group symmetries: [Sharpe][Tachikawa][Benini, Cordova, Hsin]...
   Higher-form symmetries might not form product groups, but a type of group extension. E.g. 0-form $\mathcal{F}^{(0)}$ and $\Gamma^{(1)}$ form a 2-group

$$\delta B_2 = B_1^* \Theta$$

   where $\Theta \in H^2(B\mathcal{F}^{(0)}, \Gamma^{(1)})$, and $B_1 : M_d \to B\mathcal{F}^{(0)}$.

3. Non-invertible symmetries:
   relax group law $\Rightarrow$ fusion algebra

$$D_p^i \otimes D_p^j = \bigoplus_k N_k^{ij} D_p^k$$

   This is very well developed in 2d and to some extent 3d, but unchartered until recently in $d > 3$.

4. Higher-categorical symmetries:
   topological operators of dimensions $0, \cdots, d-1$, with non-invertible fusion.

   $\Rightarrow$ Formulation in terms of objects and higher-morphisms to capture the full structure

The main (surprising?) point to remember is:
these are symmetries that occur in vanilla 4d Yang-Mills theories (no susy, no matter).

Non-Invertible Symmetries in QFTs

We are interested in the symmetries of QFTs in $d \geq 3$ and mostly $d > 3$, generated by topological operators of dimension p. We will label these by

$$D_p^i$$

with i in a suitable label-set.

What is the fusion structure of such topological operators, i.e. what is the "higher fusion category"?

Instead of developing the mathematical framework, we will directly start with field theory implementations, and put forward a proposal that is a working definition.

See talk by David Reutter.

Non-invertible Symmetries in $d > 3$:

In physics within the last year

[Heidenreich, McNamara, Monteiro, Reece, Rudelius, Valenzuela]

[Koide, Nagoya, Yamaguchi]

[Kaidi, Ohmori, Zheng]

[Choi, Cordova, Hsin, Lam, Shao]

[Roumpedakis, Seifnashri, Shao]

[Bhardwaj, Bottini, SSN, Tiwari]

[Choi, Cordova, Hsin, Lam, Shao]

[Kaidi, Zafrir, Zheng]

[Choi, Lam, Shao]

[Cordova, Ohmori]

And in this conference:

[Talks by Kaidi, Shao, Reutter] and

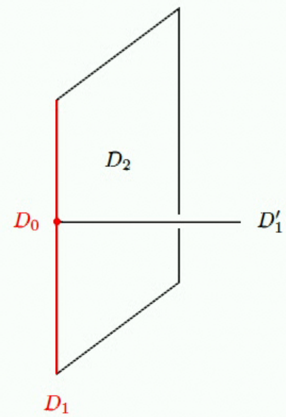
[Poster and Gong Show by Lea Bottini]

Plan

1. Symmetries from Topological Defects and Higher Cats
2. Gauging Outer Automorphisms in Higher Cats
3. Gauging Outer Automorphisms in 3d and 4d QFTs
4. Global Fusion

1. Symmetries from Topological Defects and Higher Cats

Consider a d-dimensional QFT $\mathfrak{T}$. Then the set of all topological defects will form a $(d-1)$-category.



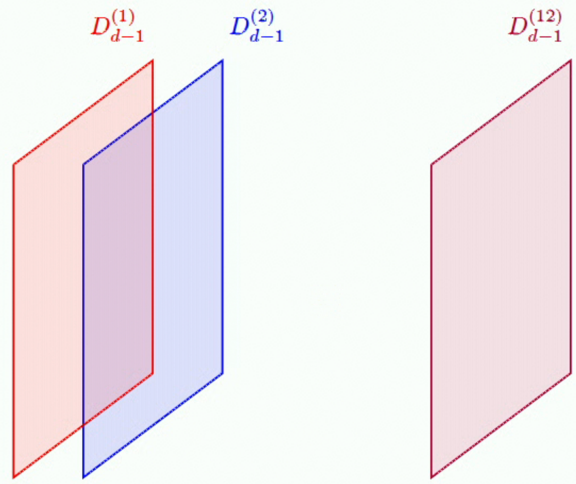
- Objects: D_{d-1}
- 1-morphisms D_{d-2} between objects
- 2-morphism D_{d-3} between 1-morphisms
- $\cdots$
- $(d-2)$-morphisms: local operators

Topological operators can be **genuine** or non-genuine, i.e. exist only at the ends of other topological operators.

The symmetry category $\mathcal{C}_{\mathfrak{T}}$ encodes the local fusion of these topological defects.

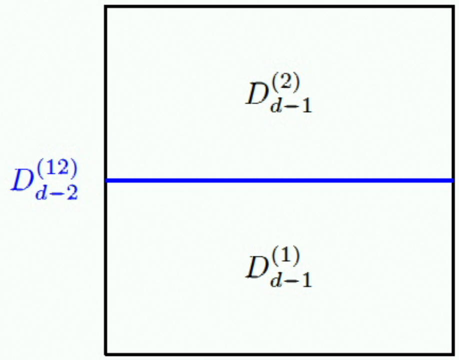
Objects: D_{d-1}

On objects we can define a fusion structure $D^{(1)}_{d-1} \otimes D^{(2)}_{d-1} = D^{(1,2)}_{d-1}$

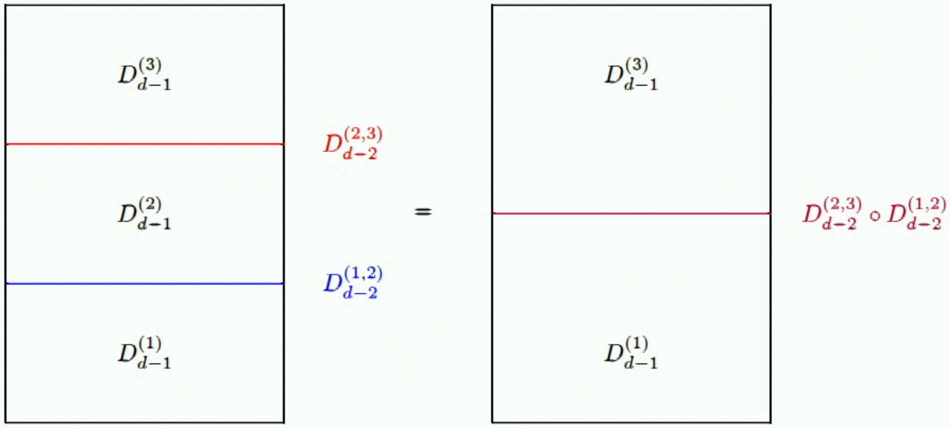


1-Morphisms

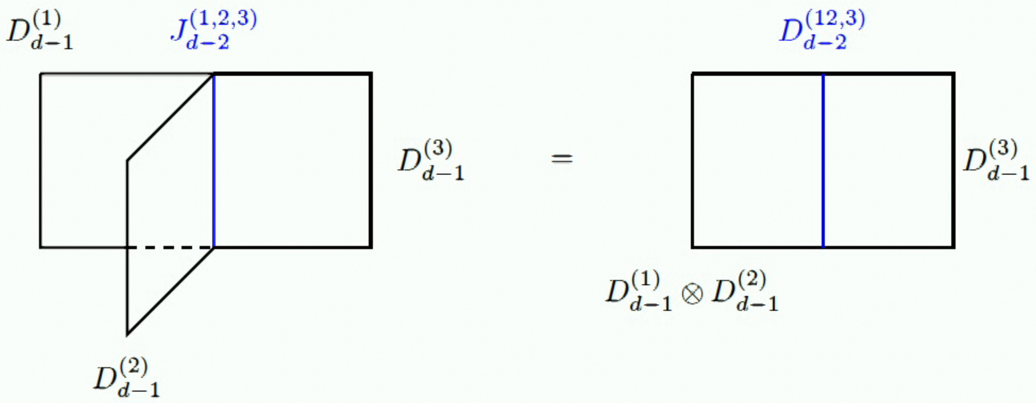
1-morphism D_{d-2} maps between two objects $D^{(i)}_{d-1}$:



And can be composed:

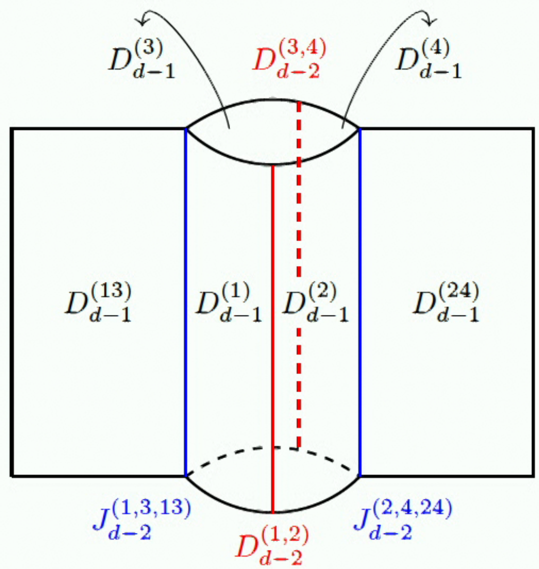


Codimension-2 Junctions



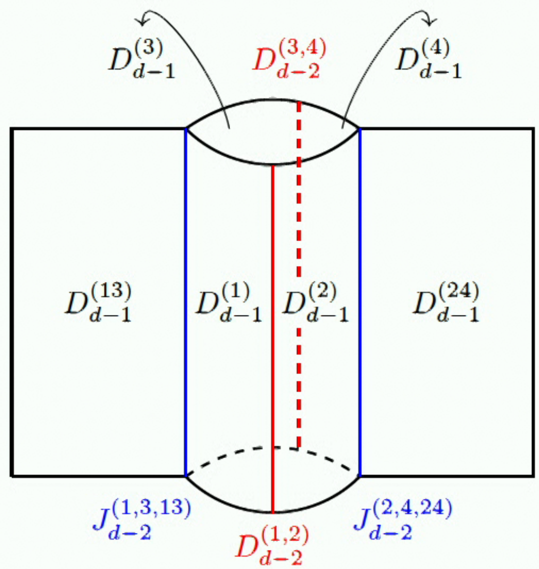
Fusion of 1-Morphisms

The 1-morphism $D_{d-2}^{(1,2)} \otimes D_{d-2}^{(3,4)}$ from $D_{d-1}^{(13)}$ to $D_{d-1}^{(24)}$ defined as a limit of the following combination of junctions:



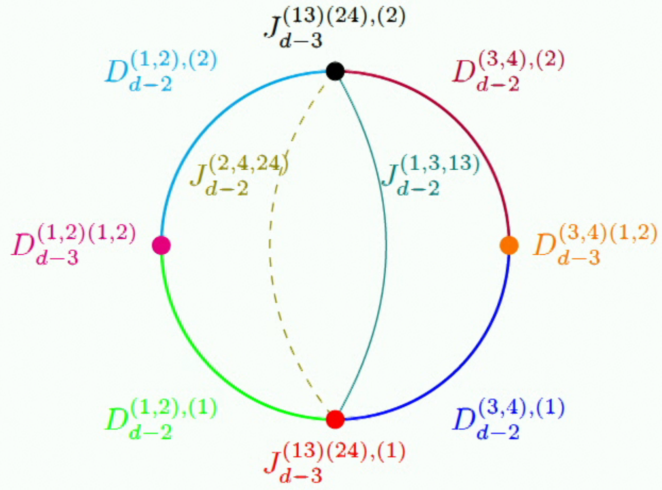
Fusion of 1-Morphisms

The 1-morphism $D_{d-2}^{(1,2)} \otimes D_{d-2}^{(3,4)}$ from $D_{d-1}^{(13)}$ to $D_{d-1}^{(24)}$ defined as a limit of the following combination of junctions:



Fusion of 2-Morphisms

Definition of the fusion $D_{d-3}^{(1,2)(1,2)} \otimes D_{d-3}^{(3,4)(1,2)}$ of 2-morphisms using codim 2 and 3 junctions:



Constructive approach in QFTs

Rather than developing the general formalism we will take a constructive approach, realizing such categories in down to earth QFTs:

We will determine the symmetry category from such a collection of topological defects in a physical QFT, including the fusion.

In this talk: non-invertible symmetries arise by gauging a 0-form outer automorphism.

Examples to keep in mind:

- Spin$(4N)$ Yang-Mills in any dim has a $\mathbb{Z}_2^{(0)}$ outer automorphism, gauging results in Pin$^+(4N)$
- Charge conjugation in $U(1)$ to $O(2)$ or $SU(N)$ to $\widetilde{SU}(N)$ Yang-Mills
- S_3-gauging of Spin(8)

2. Gauging Outer Automorphisms in Higher-Cats

We will focus on a sub-symmetry, which is a $(d-2)$-category $\mathcal{C}_{\mathrm{id},\mathfrak{T}}$, on $D_{d-1}^{(\mathrm{id})}$.

Let $G^{(0)}$ be a finite, but not necessarily abelian, 0-form symmetry group, which acts on $\mathcal{C}_{\mathrm{id},\mathfrak{T}}$, which will only have invertible symmetries.

Example.
3d $\mathrm{Spin}(4N)$ Yang-Mills, and the outer automorphism $G^{(0)} = \mathbb{Z}_2^{(0)}$ that exchanges the two factors in

$$\Gamma^{(1)} = \mathbb{Z}_2^{(S)} \times \mathbb{Z}_2^{(C)}$$

What is the symmetry category of $\mathfrak{T}/G$?

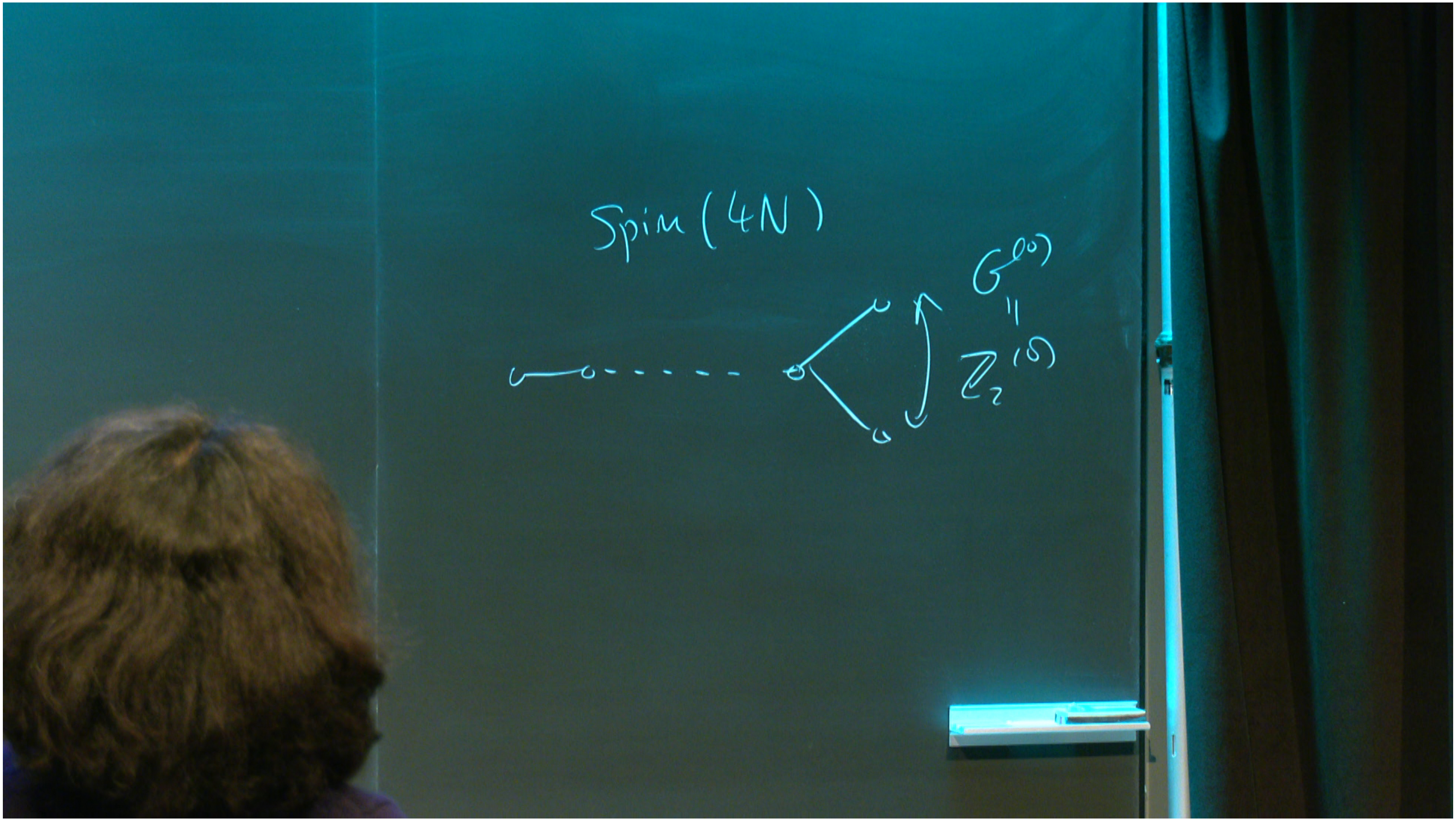
Conditions for Gaugeability

The 0-form symmetry G can be gauged if we can associate a so-called algebra in the symmetry category $\mathcal{C}_{\mathfrak{T}}$:

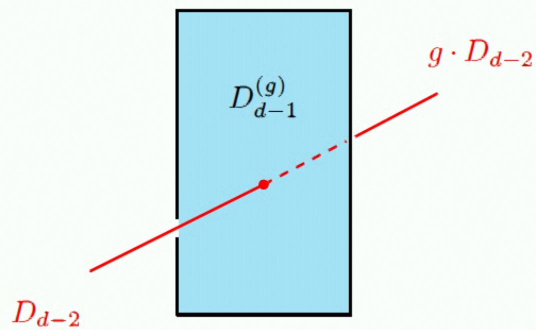
- Objects: $\bigoplus_g D_{d-1}^{(g)}$, $g \in G$ obey group law.
- 1-Morphisms: $\bigoplus_{g,g'} D_{d-2}^{(g,g')}$
- etc.

The condition for these to form an algebra in the symmetry category $\mathcal{C}_{\mathfrak{T}}$ is that they satisfy consistency conditions, e.g. associativity.

This ensures that correlation functions, with defects in algebra inserted do not depend on the triangulation of spacetime M_d, but only on the algebra and M_d. Allows gauging the algebra object.


Spin(4N)
$G^{(0)}$
$\parallel$
$\mathbb{Z}_2^{(0)}$

Action of G on Defects



Example (cont.). Outer automorphism of Spin$(4N)$ acts on the topological defects generating the 1-form symmetry by $D_1^{(S)} \leftrightarrow D_1^{(C)}$

Gauging Outer Automorphisms

Non-invertible symmetries can arise by gauging outer automorphisms of QFTs in 3d, 4d, 5d, 6d.

To determine the fusion, we need to construct the symmetry category (objects and morphisms).

In 3d we are dealing with a standard 1-category and 0-form gauging was developed in [Barkeshli, Bonderson, M. Cheng, Z. Wang][Teo, Hughes, Fradkin].

We generalize this to any dimension and propose a method to compute gauging of finite, (non-)abelian 0-form symmetries [Bhwardwaj, Bottini, SSN, Tiwari].

3. Gauging 0-Form Symmetries in 3d

In 3d the 0-form symmetry G is implemented by D_2^g defects, and the symmetry category $\mathcal{C}_{\text{id},\mathfrak{T}}$ is comprised only of topological lines and point operators.

Objects of $\mathcal{C}_{\text{id},\mathfrak{T}/G}$:

1. Topological lines $D_1^{(R)}$, R= irrep of G.
   They form a subcategory $\text{Rep}(G)$
2. Invariants lines of $\mathcal{C}_{\text{id},\mathfrak{T}}$, labeled by orbits O of G:
   $$D_1^{(O)} = \bigoplus_{i\in O} D_1^{(i)}$$
3. Mixture of both: consider $D_1^{(O)}$, and let G_O be the stabilizer group of any object $D_1^{(i)}$ with $i \in O$. $D_1^{(0)}$ can be dressed by rep R_O of G_O:
   $$D_1^{(O,R_O)}$$
   Obviously: $D_1^{(R)} = D_1^{(O_{\text{id}},R_{\text{id}})}$ the identity orbit, i.e. $G_{O_{\text{id}}} = G$.

Fusion of Lines in $\mathfrak{T}/G$

1. Fusion in $\text{Rep}(G)$:

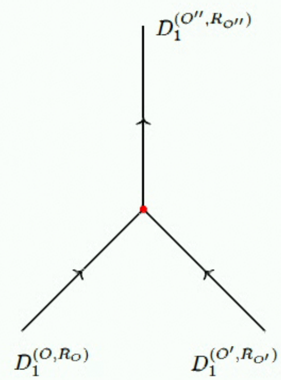
$$D_1^{(R)} \otimes D_1^{(R')} = D_1^{(RR')}\,, \qquad R, R' \in \widehat{G}$$

2. Mixed fusion

$$D_1^{(O,R_O)} \otimes D_1^{(S)} = D_1^{(O,R_O S_O)}$$

where S_O is the image of S under $\widehat{G} \to \widehat{G}_O$.

3. Fusion is determined from the morphism space:



Then

$$D_1^{(O,R_O)} \otimes D_1^{(O',R'_{O'})} \supset \dim\left(V^{(O'',R_{O''})}_{(O,R_O),(O',R_{O'})}\right) \times D_1^{(O'',R''_{O''})}$$

Example (cont.)

3d Spin$(4N)$ Yang-Mills, and the outer automorphism $G^{(0)} = \mathbb{Z}_2^{(0)}$ that exchanges the two factors in

$$\Gamma^{(1)} = \mathbb{Z}_2^{(S)} \times \mathbb{Z}_2^{(C)}$$

Objects:

$$\mathcal{C}^{\text{ob}}_{\text{Spin}(4N)} = \left\{ D_1^{(\text{id})}, D_1^{(S)}, D_1^{(C)}, D_1^{(V)} \right\}$$

V is the diagonal of S and C.

$G^{(0)} = \mathbb{Z}_2^{(0)}$ acts as the outer automorphism and gauging results in $\text{Pin}^+(4N)$

$$D_1^{(S)} \longleftrightarrow D_1^{(C)}, \qquad D_1^{(\text{id})} \text{ and } D_1^{(V)} \text{ are invariant}$$

Example (cont.)

3d Spin$(4N)$ Yang-Mills, and the outer automorphism $G^{(0)} = \mathbb{Z}_2^{(0)}$ that exchanges the two factors in

$$\Gamma^{(1)} = \mathbb{Z}_2^{(S)} \times \mathbb{Z}_2^{(C)}$$

Objects:

$$\mathcal{C}^{\text{ob}}_{\text{Spin}(4N)} = \left\{ D_1^{(\text{id})}, D_1^{(S)}, D_1^{(C)}, D_1^{(V)} \right\}$$

V is the diagonal of S and C.

$G^{(0)} = \mathbb{Z}_2^{(0)}$ acts as the outer automorphism and gauging results in Pin$^+(4N)$

$$D_1^{(S)} \longleftrightarrow D_1^{(C)}, \qquad D_1^{(\text{id})} \text{ and } D_1^{(V)} \text{ are invariant}$$

Example (cont.)

Objects in $\text{Pin}^+(4N)$:

1. The topological lines that generate $\text{Rep}(\mathbb{Z}_2)$: $D_1^{(\text{id})}$ and $D_1^{(-)}$.
2. Orbits, i.e. invariants:

$$D_1^{(SC)} := \left(D_1^{(S)} \oplus D_1^{(C)}\right)_{\mathcal{C}_{\text{Spin}}}$$

   to be a simple object. Furthermore we have $D_1^{(\text{id})}$ and $D_1^{(V)}$.

3. Combination of invariants and $D_1^{(-)}$:

$$D_1^{(V_-)}$$

Thus

$$\mathcal{C}^{\text{ob}}_{\text{Pin}^+(4N)} = \left\{D_1^{(\text{id})}, D_1^{(-)}, D_1^{(SC)}, D_1^{(V)}, D_1^{(V_-)}\right\}$$

Example (cont.) Fusion in $\mathcal{C}_{\mathrm{Pin}^+(4N)}$

$D_1^{(\mathrm{id})}$ and $D_1^{(-)}$ obey invertible fusion rules for $\mathbb{Z}_2$.

Furthermore

$$D_1^{(SC)} \otimes D_1^{(-)} = D_1^{(SC)}$$

$$D_1^{(V)} \otimes D_1^{(-)} = D_1^{(V-)}$$

$$D_1^{(V-)} \otimes D_1^{(-)} = D_1^{(V)}$$

The interesting fusion is $D_1^{(SC)}$ with itself and with $D_1^{(V\pm)}$.

Example (cont.) Fusion in $\mathcal{C}_{\mathrm{Pin}^+(4N)}$

It is useful to compute the fusion in the original $\mathcal{C}_{\mathrm{Spin}(4N)}$:

$$\left(D_1^{(SC)} \otimes D_1^{(V)}\right)_{\mathcal{C}_{\mathrm{Spin}(4N)}} = \left((D_1^{(S)} \oplus D_1^{(C)}) \otimes D_1^{(V)}\right) = \left(D_1^{(C)} \oplus D_1^{(S)}\right) = \left(D_1^{(SC)}\right)_{\mathcal{C}_{\mathrm{Spin}(4N)}}$$

To see whether $D_1^{(SC)}$ is also present in the gauged category we need to determine the $\mathbb{Z}_2$ transformation properties of the morphisms

$$D_1^{(SC)} \otimes D_2^{(V)} \to D_1^{(SC)}$$

There are two morphisms:

$$D_0^{(S\otimes C,V)}: \qquad D_1^{(S)} \otimes D_1^{(C)} \to D_1^{(V)}$$
$$D_0^{(C\otimes S,V)}: \qquad D_1^{(C)} \otimes D_1^{(S)} \to D_1^{(V)}$$

These are exchanged under $\mathbb{Z}_2^{(0)}$ and thus the morphism space is 2d splitting as $1_+ \oplus 1_-$. Since both $\mathbb{Z}_2$ representations are present:

$$D_1^{(SC)} \otimes D_1^{(V)} = D_1^{(SC)}$$
$$D_1^{(SC)} \otimes D_1^{(V_-)} = D_1^{(SC)}$$

Example (cont.) Fusion in $\mathcal{C}_{\text{Pin}^+(4N)}$

$$\left(D_1^{(SC)} \otimes D_1^{(SC)}\right)_{\mathcal{C}_{\text{Spin}(4N)}} = \left(2D_1^{(\text{id})} \oplus 2D_1^{(V)}\right)_{\mathcal{C}_{\text{Spin}(4N)}}$$

Again determining the morphism spaces and their $\mathbb{Z}_2$ representation decomposition we find

$$D_1^{(SC)} \otimes D_1^{(SC)} = D_1^{(\text{id})} \oplus D_1^{(-)} \oplus D_1^{(V)} \oplus D_1^{(V_-)}$$

In fact this category is of Tambara-Yamagami type for $\mathbb{Z}_2 \times \mathbb{Z}_2$ and has associators identifying it with

$$\mathcal{C}_{\text{Pin}^+(4N)} = \text{Rep}(D_8)$$

Note 1: a similar analysis for $\text{Pin}^+(4N+2)$ yields also $\text{Rep}(D_8)$, however the two differ by braiding [wip, Bhardwaj, Bottini, SSN, Tiwari]

Note 2: We can gauge non-abelian symmetries e.g. $\text{Spin}(8)$ with outer $S_3^{(0)}$ automorphism gauged.

4. Gauging Outer Automorphisms in 4d

In $d > 3$ the symmetry category after gauging is not a 1-category but a higher, $(d-2)$-category $\mathcal{C}_{\mathrm{id},\mathfrak{T}/G}$.

1. Again the objects are $D^{(i)}_{d-2}$: G-invariant combinations
2. p-morphisms for $p = 1, \cdots d-4$: take G-invariant combinations
3. $(d-3)$-morphisms: topological lines, which again have contributions from the dual symmetry to G and invariant lines
   (a) $D_1^{(R)}$ forming $\mathrm{Rep}(G)$
   (b) $D_1^{(O)}$ associated to G-orbits
   (c) $D_1^{(O,R_O)}$
4. $(d-2)$-morphisms: topological local operators, are again computed as before (decomposed into irreps)

Non-Invertible Symmetries in 4d

Lets consider the 4d Spin$(4N)$ Yang-Mills and outer-automorphism gauging to Pin$^+(4N)$. The objects are now

$$\mathcal{C}^{\text{ob}}_{\text{Spin}(4N)} = \left\{ D_2^{(\text{id})}, D_2^{(S)}, D_2^{(C)}, D_2^{(V)} \right\}$$

There are no 1-morphisms between two distinct such simple objects, but each object has a 1-endomorphism:

$$\mathcal{C}^{\text{1-endo}}_{\text{Spin}(4N)} = \left\{ D_1^{(\text{id})}, D_1^{(S)}, D_1^{(C)}, D_1^{(V)} \right\}$$

These satisfy group law

$$D_p^{(i)} \otimes D_p^{(j)} = D_p^{(ij)}$$

The $G = \mathbb{Z}_2$ outer automorphism acts by

$$D_i^{(S)} \longleftrightarrow D_i^{(C)}$$

4d $\mathrm{Pin}^+(4N)$ Symmetry Category

Gauging results in the following **objects**:

$$\mathcal{C}^{\mathrm{ob}}_{\mathrm{Pin}^+(4N)} = \left\{ D_2^{(\mathrm{id})}, D_2^{(SC)}, D_2^{(V)} \right\},$$

with the non-invertible surface defect fusion

$$D_2^{(SC)} \otimes D_2^{(SC)} = D_2^{(\mathrm{id})} \oplus D_2^{(V)}$$

1-morphisms:
There were no non-trivial 1-morphisms in the $\mathrm{Spin}(4N)$ category so that the only new 1-endomorphisms of the invariant surface defects are

$$\mathcal{C}^{\text{1-endo}}_{\mathrm{Pin}^+(4N)} = \left\{ D_1^{(\mathrm{id})}, D_1^{(-)}, D_1^{(SC)}, D_1^{(V)}, D_1^{(V_-)} \right\}$$

with the non-invertible fusion

$$D_1^{(SC)} \otimes D_1^{(SC)} = D_1^{(\mathrm{id})} \oplus D_1^{(-)} \oplus D_1^{(V)} \oplus D_1^{(V_-)}$$

4d $\mathrm{Pin}^+(4N)$ Symmetry Category

Gauging results in the following **objects**:

$$\mathcal{C}^{\text{ob}}_{\mathrm{Pin}^+(4N)} = \left\{ D_2^{(\text{id})}, D_2^{(SC)}, D_2^{(V)} \right\},$$

with the non-invertible surface defect fusion

$$D_2^{(SC)} \otimes D_2^{(SC)} = D_2^{(\text{id})} \oplus D_2^{(V)}$$

1-morphisms:
There were no non-trivial 1-morphisms in the $\mathrm{Spin}(4N)$ category so that the only new 1-endomorphisms of the invariant surface defects are

$$\mathcal{C}^{\text{1-endo}}_{\mathrm{Pin}^+(4N)} = \left\{ D_1^{(\text{id})}, D_1^{(-)}, D_1^{(SC)}, D_1^{(V)}, D_1^{(V_-)} \right\}$$

with the non-invertible fusion

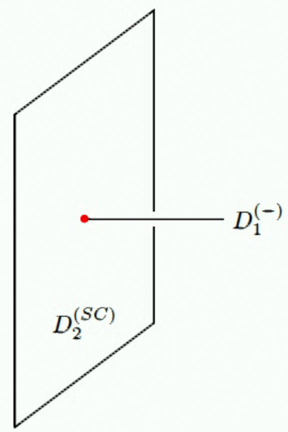
$$D_1^{(SC)} \otimes D_1^{(SC)} = D_1^{(\text{id})} \oplus D_1^{(-)} \oplus D_1^{(V)} \oplus D_1^{(V_-)}$$

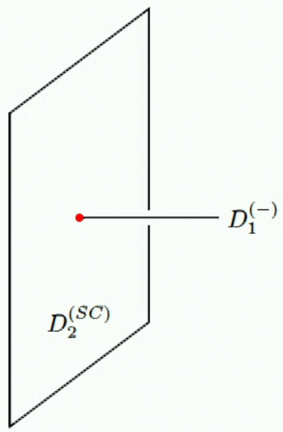
Hang on:

$$D_1^{(SC)} \otimes D_1^{(SC)} = D_1^{(\mathrm{id})} \oplus D_1^{(-)} \oplus D_1^{(V)} \oplus D_1^{(V_-)}$$

The RHS has line-defects which do not live on the $D_2^{(SC)}$ surface. So what exactly does this fusion rule mean?

E.g. $D_1^{(-)} \subset D_1^{(SC)} \otimes D_1^{(SC)}$ means that there is a non-zero local operator at the intersection of the genuine line $D_1^{(-)}$ and the genuine surface $D_2^{(SC)}$ (since $D_1^{(SC)}$ is the identity line on the surface $D_2^{(SC)}$).





One could be tempted to interpret this is a fusion "surface ×surface = line".

However in terms of the higher-category perspective this is a fusion of two non-genuine identity lines ($D_1^{(SC)}$) living on a non-identity surface $D_2^{(SC)}$ fusing to give rise to a genuine non-identity line $D_1^{(-)}$.

I.e. we retain standard fusion rule: objects to objects, morphisms to morphisms.

Gauging Non-Abelian Finite Groups: 4d

Spin$(8) \rtimes S_3$ in 4d.

Gauging in two-steps: $\mathbb{Z}_3 \rtimes \mathbb{Z}_2$.

The $\mathbb{Z}_3$ acts by $S \to C \to V \to S$. The invariant surface defect is then

$$D_2^{(SCV)} = \left(D_2^{(S)} \oplus D_2^{(C)} \oplus D_2^{(V)} \right)_{\mathcal{C}_{\mathrm{Spin}(8)}}$$

Then the fusion of 4d Spin$(8) \rtimes S_3$ gauge theory is ("Fibonacci-like")

$$D_2^{(SCV)} \otimes D_2^{(SCV)} = D_2^{(\mathrm{id})} \oplus D_2^{(SCV)}$$

The 1-endomorphisms are $\mathcal{C}^{\text{1-endo}}_{\mathbb{Z}_3} = \{D_1^{(\text{id})}, D_1^{(\omega)}, D_1^{(\omega^2)}, D_1^{(SCV)}\}$ where $D_1^{(\omega^k)}$ are the dual lines to the $\mathbb{Z}_3^{(0)}$, with $\omega = e^{2\pi i/3}$.

For S_3 we act with $\mathbb{Z}_2$: $D_1^{(\omega)} \leftrightarrow D_1^{(\omega^2)}$

$$\mathcal{C}^{\text{1-endo}}_{S_3} = \left\{D_1^{(\text{id})}, D_1^{(-)}, D_1^{(\omega\omega^2)}, D_1^{(SCV)}, D_1^{(SCV_-)}\right\}$$

The fusion of 1-endomorphisms is:

$$\begin{aligned}
D_1^{(-)} \otimes D_1^{(\omega\omega^2)} &= D_1^{(\omega\omega^2)} \\
D_1^{(-)} \otimes D_1^{(SCV)} &= D_1^{(SCV_-)} \\
D_1^{(-)} \otimes D_1^{(SCV_-)} &= D_1^{(SCV)} \\
D_1^{(\omega\omega^2)} \otimes D_1^{(SCV)} &= D_1^{(SCV)} \oplus D_1^{(SCV_-)} \\
D_1^{(\omega\omega^2)} \otimes D_1^{(SCV_-)} &= D_1^{(SCV)} \oplus D_1^{(SCV_-)} \\
D_1^{(\omega\omega^2)} \otimes D_1^{(\omega\omega^2)} &= D_1^{(\text{id})} \oplus D_1^{(-)} \oplus D_1^{(\omega\omega^2)} \\
D_1^{(SCV)} \otimes D_1^{(SCV)} &= D_1^{(\text{id})} \oplus D_1^{(\omega\omega^2)} \oplus D_1^{(SCV)} \oplus D_1^{(SCV_-)} \\
D_1^{(SCV_-)} \otimes D_1^{(SCV_-)} &= D_1^{(\text{id})} \oplus D_1^{(\omega\omega^2)} \oplus D_1^{(SCV)} \oplus D_1^{(SCV_-)} \\
D_1^{(SCV)} \otimes D_1^{(SCV_-)} &= D_1^{(-)} \oplus D_1^{(\omega\omega^2)} \oplus D_1^{(SCV)} \oplus D_1^{(SCV_-)} .
\end{aligned}$$

5. Global Fusion

So far we have discussed local fusion: i.e. the fusion without taking into account global aspects of the spacetime manifold M_d. We can lift the local fusion to global fusions.

In a nutshell: the global fusion of p-dimensional topological defects $D_p^{(1)} \otimes D_p^{(2)}$ is obtained by gauging the symmetry localized on the defect $D_p^{(1)} \otimes D_p^{(2)}$.

E.g. for 4d $\mathrm{Pin}^+(4N)$ the fusion of surfaces

$$D_2^{(SC)} \otimes D_2^{(SC)} = D_2^{(\mathrm{id})} \oplus D_2^{(V)}$$

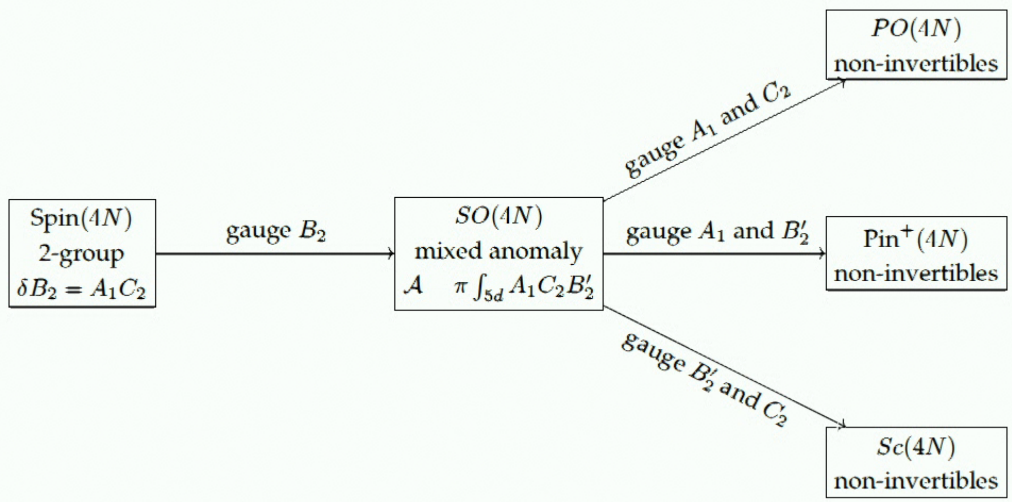
becomes on a 2-manifold M_2

$$D_2^{(SC)}(M_2) \otimes D_2^{(SC)}(M_2) = \frac{D_2^{(\mathrm{id})}}{\mathbb{Z}_2}(M_2) \oplus \frac{D_2^{(V)}}{\mathbb{Z}_2}(M_2)$$

where $\frac{D_2^{(i)}}{\mathbb{Z}_2}(M_2)$ for $i \in \{\mathrm{id}, V\}$ denotes the surface defect obtained by gauging the $\mathbb{Z}_2$ 0-form symmetry of $D_2^{(i)}$ wrapped along M_2. These are precisely condensation defects – see [Gaiotto, Johnson-Freyd][Choi, Cordova, Hsin, Lam Shao][Rumpedakis, Seifnashri, Shao]

For abelian G we can compare to the complementary approach by [Kaidi, Ohmori, Zheng] (and a slightly different approach [Choi, Cordova, Lam, Shao]), who derived non-invertible symmetries by gauging mixed anomalies.

Indeed, the $\mathrm{Pin}^+(4N)$ non-invertible symmetries can be obtained also by following the KOZ approach:



Non-Invertibles from Mixed Anomalies

E.g. using the [Kaidi, Ohmori, Zheng] method for $\text{Pin}^+(4N)$ the fusion is

$$\mathcal{N}(M_2;C_2)\times\mathcal{N}(M_2;C_2)=\frac{1}{|H^0(M_2,\mathbb{Z}_2)|}(1+T(M_2))\left(\sum_{M_1\in H_1(M_2,\mathbb{Z}_2)}L(M_1)\right)$$

$$\mathcal{N}(M_2;C_2)\times T(M_2)=\mathcal{N}(M_2;C_2)$$

$$\mathcal{N}(M_2;C_2)\times L(M_1)=\mathcal{N}(M_2;C_2)\,.$$

$T(M_2)=e^{i\pi\oint_{M_2}b_2'}$ is the defect generating the $\mathbb{Z}_2^{(1)}$ and $L(M_1)=e^{i\pi\oint_{M_1}a_1}$ is the defect generating the $\mathbb{Z}_2^{(2)}$ 2-form symmetry dual to $\mathbb{Z}_2^{(0)}$.

$$D_2^{(SC)}\longleftrightarrow\mathcal{N}(M_2;C_2)$$

$$D_2^{(V)}\longleftrightarrow T(M_2)$$

$$D_1^{(-)}\longleftrightarrow L(M_1)$$

Non-Invertible 3-categorical symmetry in 5d SYM

5d $\mathcal{N}=2$ Spin$(4N)$ super Yang-Mills has a 3-categorical symmetry

$$\left\{D_i^{(\text{id})}, D_i^{(S)}, D_i^{(C)}, D_i^{(V)}\right\}, \qquad i=3,2,1$$

Gauging $\mathbb{Z}_2^{(0)}$ outer automorphism we get simple objects

$$\mathcal{C}^i_{\text{Pin}^+(4N)} = \left\{D_i^{(\text{id})}, D_i^{(SC)}, D_i^{(V)}\right\}$$

$i=3$: objects; $i=2$: 1-endomorphisms. These have fusion

$$D_i^{(SC)} \otimes D_i^{(SC)} = D_i^{(\text{id})} \oplus D_i^{(V)}$$

The 2-endomorphisms are

$$\mathcal{C}^{\text{2-endo}}_{\text{Pin}^+(4N)} = \left\{D_1^{(\text{id})}, D_1^{(-)}, D_1^{(SC)}, D_1^{(V)}, D_1^{(V_-)}\right\}$$

with fusion of TY type

$$D_1^{(SC)} \otimes D_1^{(SC)} = D_1^{(\text{id})} \oplus D_1^{(-)} \oplus D_1^{(V)} \oplus D_1^{(V_-)}$$

The non-trivial global fusion is

$$D_3^{(SC)}(\Sigma_3) \otimes D_3^{(SC)}(\Sigma_3) = \frac{D_3^{(\text{id})}}{\mathbb{Z}_2^{(1)}}(\Sigma_3) \oplus \frac{D_3^{(V)}}{\mathbb{Z}_2^{(1)}}(\Sigma_3)$$

where

$$\frac{D_3^{(\text{id})}}{\mathbb{Z}_2^{(1)}} : \quad \mathbb{Z}_2^{(1)} \quad \text{generated by } D_1^{(\text{id})}, D_1^{(-)}$$

$$\frac{D_3^{(V)}}{\mathbb{Z}_2^{(1)}} : \quad \mathbb{Z}_2^{(1)} \quad \text{generated by } D_1^{(V)}, D_1^{(V_-)}$$

Comments and Conclusions

We only started to dip our toes into higher-categorical symmetries in QFTs, proposing a working definition for symmetries of 3d and 4d QFTs.

Things I did not talk about but are in our paper [Bhardwaj, Bottini, SSN, Tiwari]:

1. Non-invertible symmetries for disconnected gauge groups $O(2)$ (consistent with [Heidenreich, McNamara, Monteiro, Reece, Rudelius, Valenzuela])and $\widetilde{SU}(N)$ (gauging charge conjugation)
2. Allows gauging non-abelian finite symmetries
3. Non-invertible 3-categorical symmetries in absolute 6d $(2,0)$ theories, e.g.

$$\left[SO(2n) \times SO(2n)\right] \rtimes \mathbb{Z}_2$$

4. Non-invertible symmetries in 5d SCFTs

Outlook

Non-invertible symmetries are ubiquitous in QFTs.

1. Develop a mathematically sound framework for higher fusion categories (higher meaning ≥ 2)
2. Learn to gauge higher-form/non-invertible symmetries in higher-categories; 't Hooft anomalies
3. Physical implications of these symmetries: see Shu-Heng's talk. Clearly much to explore here.
4. Is this the most general "symmetry structure" for QFTs?
5. String theory realization of non-invertibles (e.g. using SymTFT)

Thank you!