Title: Higher S-matrices and higher modular categories

Speakers: David Reutter

Collection: Global Categorical Symmetries

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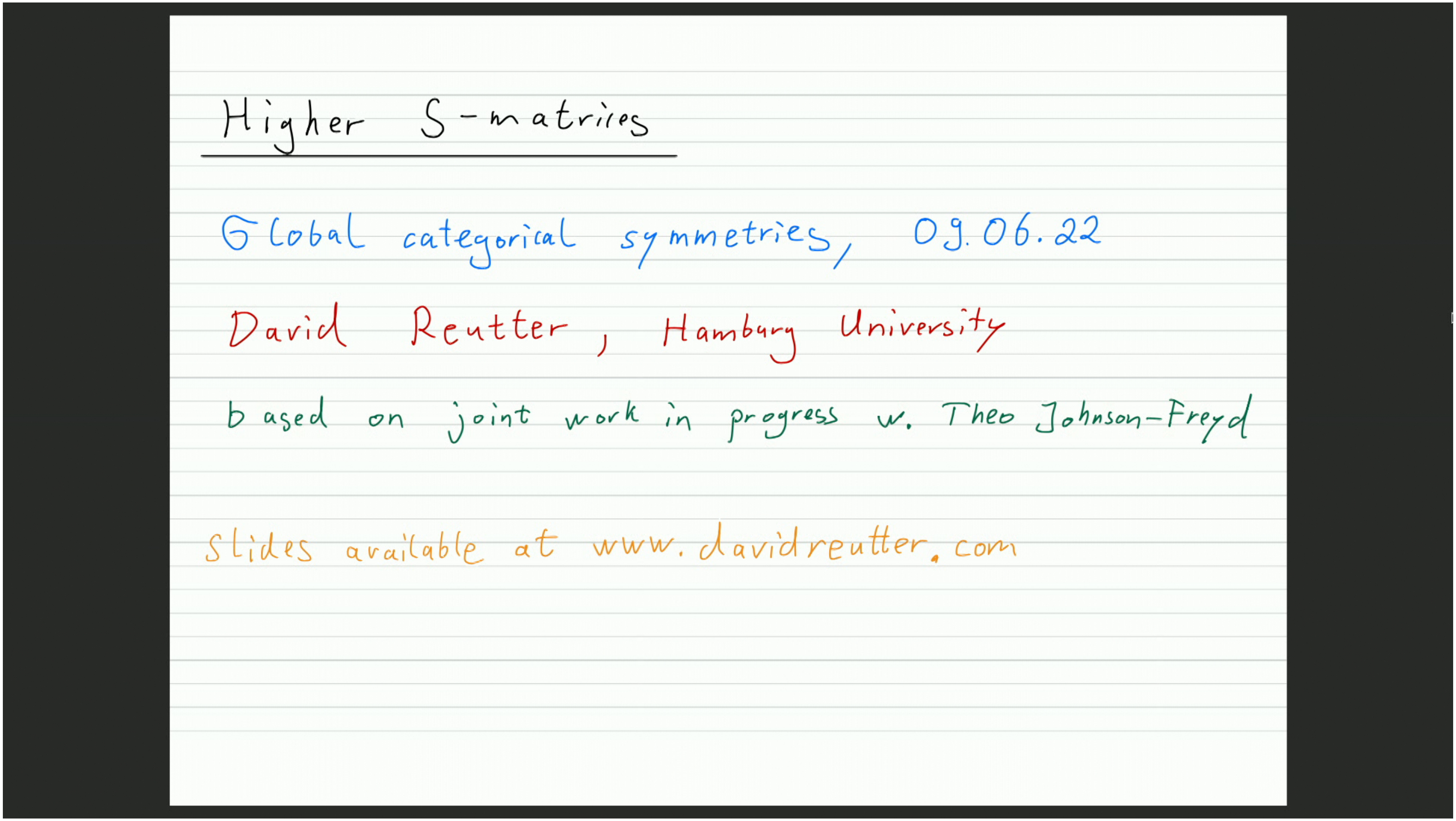
Abstract: "Every braided fusion category has a `framed S-matrix pairing' which records the braiding between simple objects. Non-degeneracy/Morita invertibility of the category (aka `modularity' in the oriented case) is equivalent to non-degeneracy of this pairing. I will define higher-dimensional versions of S-matrices which pair morphisms of complementary dimension in higher semisimple categories and sketch a proof that these pairings are non-degenerate if and only if the higher category is. Along the way, I will introduce higher semisimple categories and higher fusion categories and interpret these results in terms of the associated anomalous topological quantum field theories.

This is based on joint work in progress with Theo Johnson-Freyd."



Higher S-matrices

Global categorical symmetries, 09.06.22

David Reutter, Hamburg University

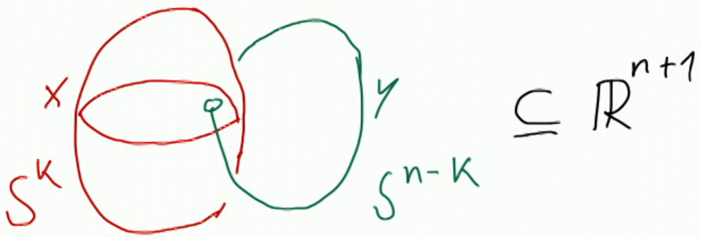
based on joint work in progress w. Theo Johnson-Freyd

Slides available at www.davidreutter.com

The story on one slide

In a semisimple anomalous $(n+1)$-dim. TQFT* the pairing

$$\left\{\begin{matrix}\text{operators of}\\ \text{dim } k\end{matrix}\right\} \times \left\{\begin{matrix}\text{operators of}\\ \text{dim } n-k\end{matrix}\right\} \longrightarrow \mathbb{C}$$



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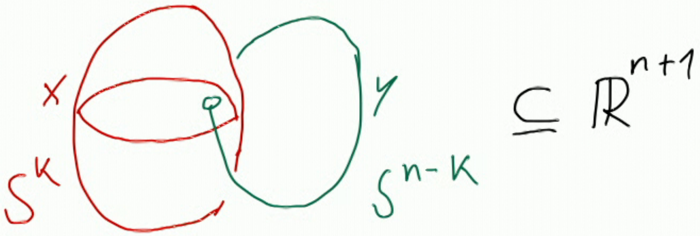

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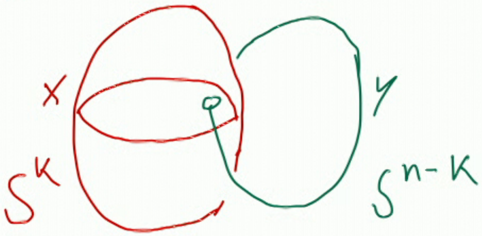

$$\subseteq \mathbb{R}^{n+1}$$

is non-degenerate.

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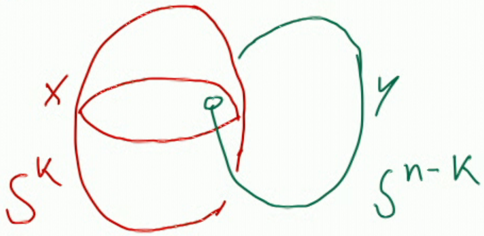

$\subseteq \mathbb{R}^{n+1}$

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Moreover, this completely characterizes anomalous TQFTs among relative (n+2)/(n+1)-d TQFTs.

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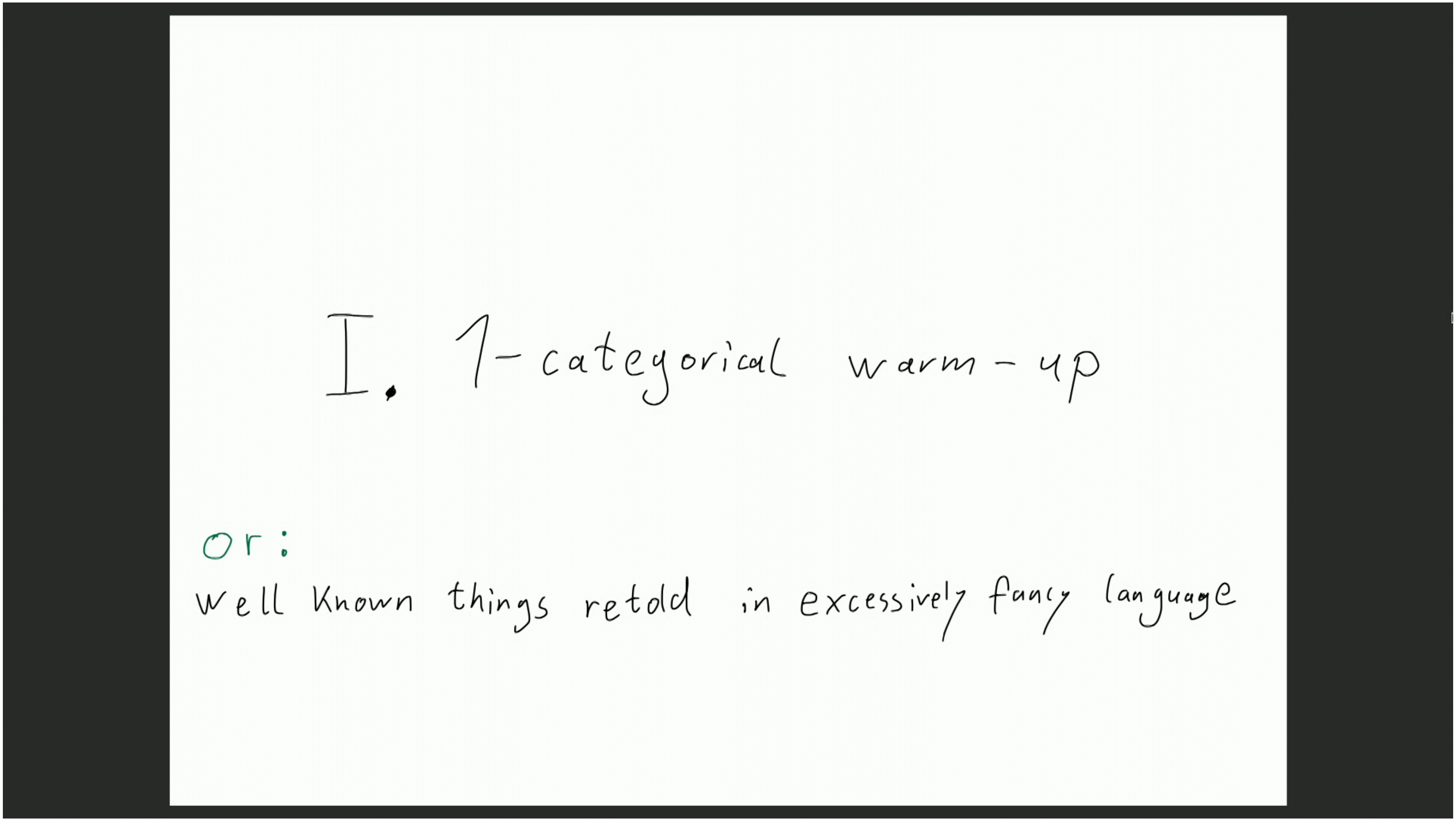
I. 1-categorical warm-up



I. 1-categorical warm-up

or:

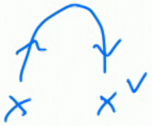
well Known things retold in excessively fancy language

1-categorical warm-up: string diagrams

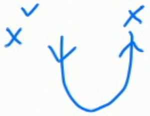
all objects have left & right duals.

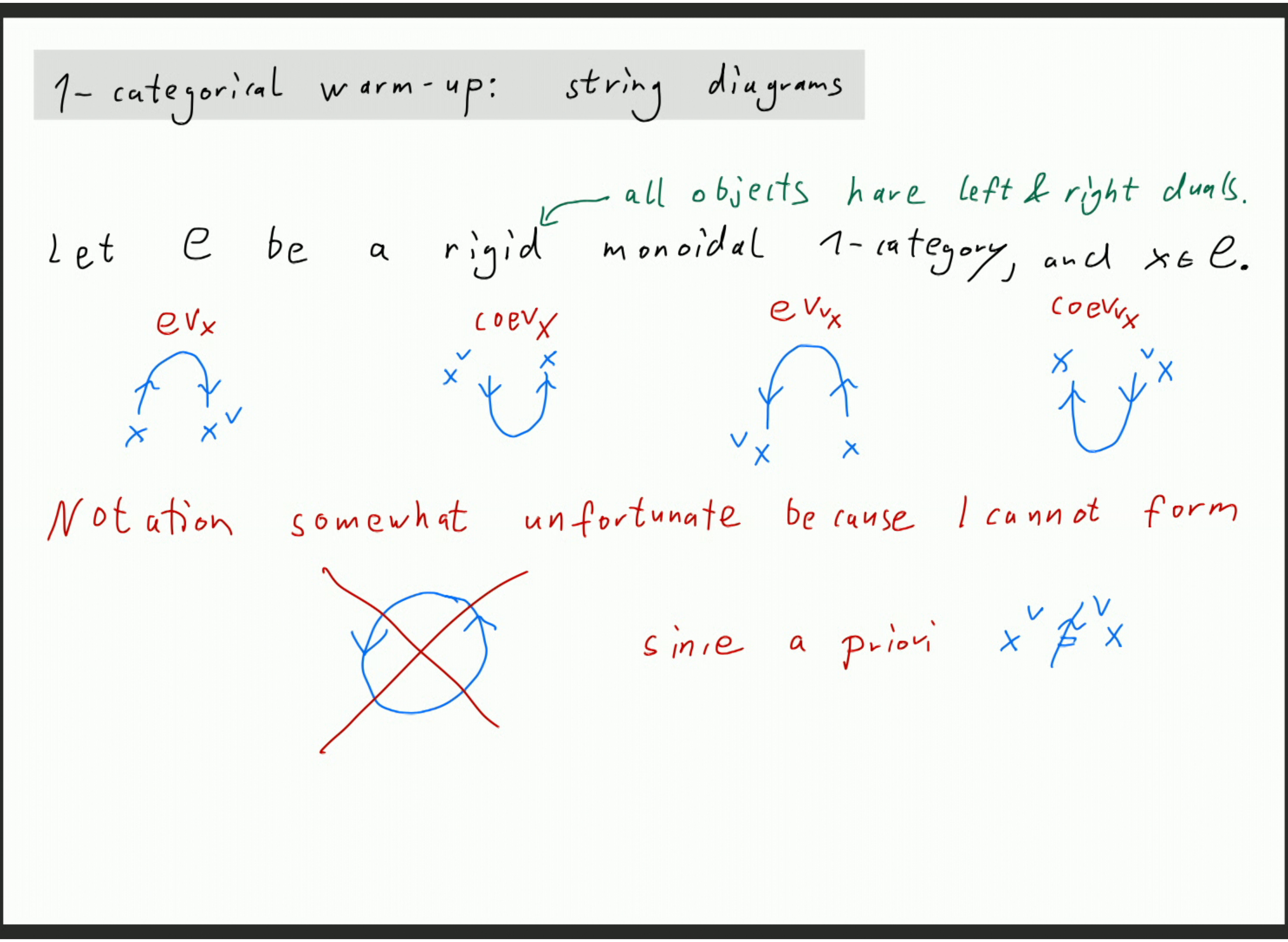
Let $\mathcal{C}$ be a rigid monoidal 1-category, and $x \in \mathcal{C}$.

ev_x



coev_x





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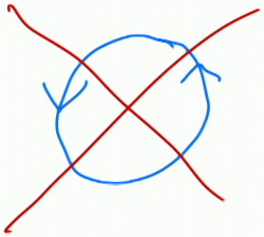

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⇒ Need to record how often string has been twisted around.

1-categorial warm-up: 1d-tangle hypothesis

Let $\mathcal{C}$ be a rigid monoidal 1-category and $x \in \mathcal{C}$. Then, x can be evaluated at an embedded 1-mfld $S \hookrightarrow \mathbb{R}^2$ with the following additional framing data:

- tangential framing $\tau : \underset{\text{tang. bundle}}{T_S} \simeq \underset{\text{triv. bundle}}{\underline{\mathbb{R}}}$
- normal framing $\nu : \underset{\text{normal bundle}}{N_S} \simeq \underline{\mathbb{R}}$
- a homotopy filling the triangle

$$\begin{tikzcd}
T_S \oplus N_S \arrow[rr, "\tau \oplus \nu"] \arrow[d, phantom, "\wr"] & \arrow[d, phantom, "\eta \ \cong"] & \underline{\mathbb{R}}^2 \\
T\mathbb{R}^2|_S \arrow[rr, "\text{blackboard framing}"'] & & \underline{\mathbb{R}}^2
\end{tikzcd}$$

[Baez-Dolan, Lurie, ...]

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such an η does not exist for $S = S^1 \hookrightarrow \mathbb{R}^2$

Notation: $\int_{S,\tau,\nu,\eta} x = \int_S x \in \mathrm{End}_{\mathcal{C}}(\mathbb{I})$

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Same data lets you draw defects on any framed 2-manifold M

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[Baez–Dolan, Lurie, ...]

1-categorical warm-up: extra fractional dualizability

$$\int_{S^0} X = X \otimes X^\vee$$

$$\int_{D^1} X = \mathrm{ev}_X : X X^\vee \xrightarrow{\curvearrowright} I$$

1-categorical warm-up: extra fractional dualizability

Now assume: $\mathcal{C}$ is fusion

semisimple, $|\pi_0 \mathcal{C}| < \infty$, I simple

useful notation

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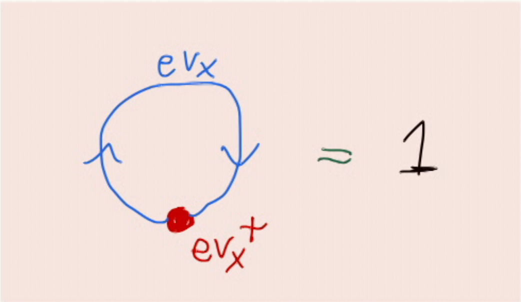

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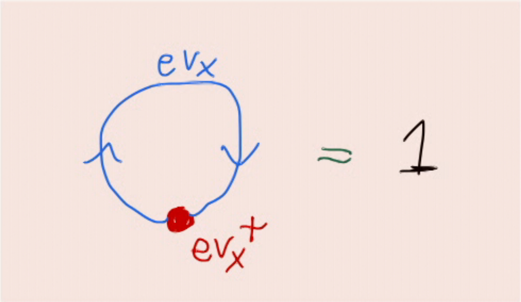

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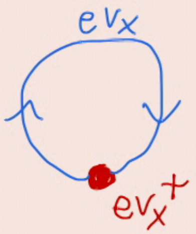
Think: value of X on $S^1_b := \partial D^2$ with "bounding 2-framing"



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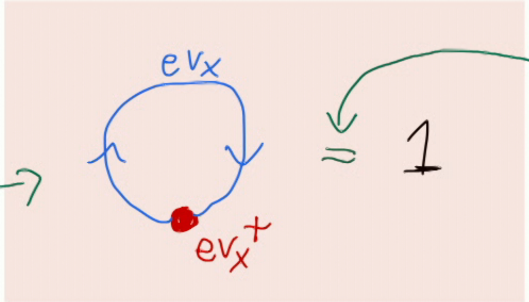


Think: value of X on D^2

Leads to a canonical iso ${}^\vee X \cong X^\vee$. But: usually not monoidal!

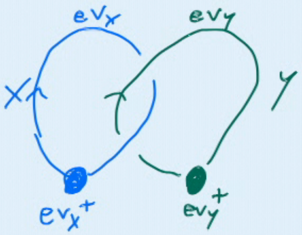
A monoidal such iso is a pivotal structure.

1-categorical warm-up: framed S-matrices

Let $\mathcal{C}$ be a braided fusion category.

Def [Etingof?] The framed S-matrix of $\mathcal{C}$ is the $\pi_0 \mathcal{C} \times \pi_0 \mathcal{C}$ matrix

$$\tilde{S}_{x,y} :=$$



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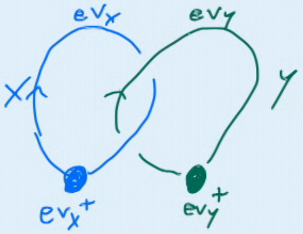

(N.B.: If $\mathcal{C}$ is ribbon/oriented:

$$\underset{\nwarrow}{S_{X,Y}} = \underset{\nearrow}{\dim(X)}\ \underset{\nearrow}{\dim(Y)}\ \tilde{S}_{X,Y}$$

depend on ribbon structure)

TQFT interpretation of $\mathcal{C}$: framed 4d $\overset{\text{Crane-Yetter/Walker-Wang}}{\text{bulk theory}}$ + 3d $\overset{\text{Reshetikhin-Turaev}}{\text{boundary theory}}$

(valued in a certain Morita 4-category)

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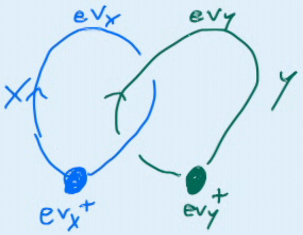

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Crane-Yetter/Walker-Wang

Reshetikhin-Turaev

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(valued in a certain Morita 4-category)

Theorem [Etingof-Drinfeld-Gelaki-Nikshych-Ostrik, Freed-Teleman, Shimizu, Brochier-Jordan-Safronov-Snyder, ...].

$\tilde{S}$ invertible $\Leftrightarrow$ $Z_2(\mathcal{C})$ is trivial $\Leftrightarrow$ 4d bulk theory is invertible

Müger center: subcategory of transparent objects

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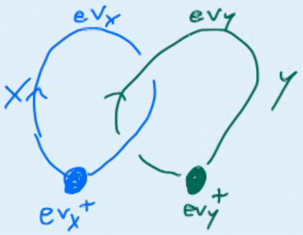
$\tilde{S}_{X,Y} :=$ 

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Modular categories := ribbon/oriented + one of these equiv. conditions.

n-categorical version: higher fusion categories

Slogan: $$\frac{\text{multifusion } (n+1)\text{-cat.}}{\text{multifusion } n\text{-cat.}} = \frac{\text{multifusion } 1\text{-cat.}}{\text{f.d. semisimple algebra}}$$

fusion := multifusion + I simple

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Def [Douglas-R. for n=2]

A multifusion n-category is a $\mathbb{C}$-linear monoidal n-cat. $\mathcal{C}$, s.t.

- it has direct sums and all n-idempotents split

  (n-cat. version of idempotent [n>2: Gaiotto-Johnson-Freyd] physical interpretation: all operators/defects which can be condensed from operators in $\mathcal{C}$ are already in $\mathcal{C}$)
- Locally multifusion: $\forall X \in \mathcal{C}$, $\mathrm{End}(X)$ is a multifusion $(n-1)$-category
- rigid: all objects have duals.
- finite: $|\pi_0 \mathcal{C}| < \infty$ (defined on next slide!)

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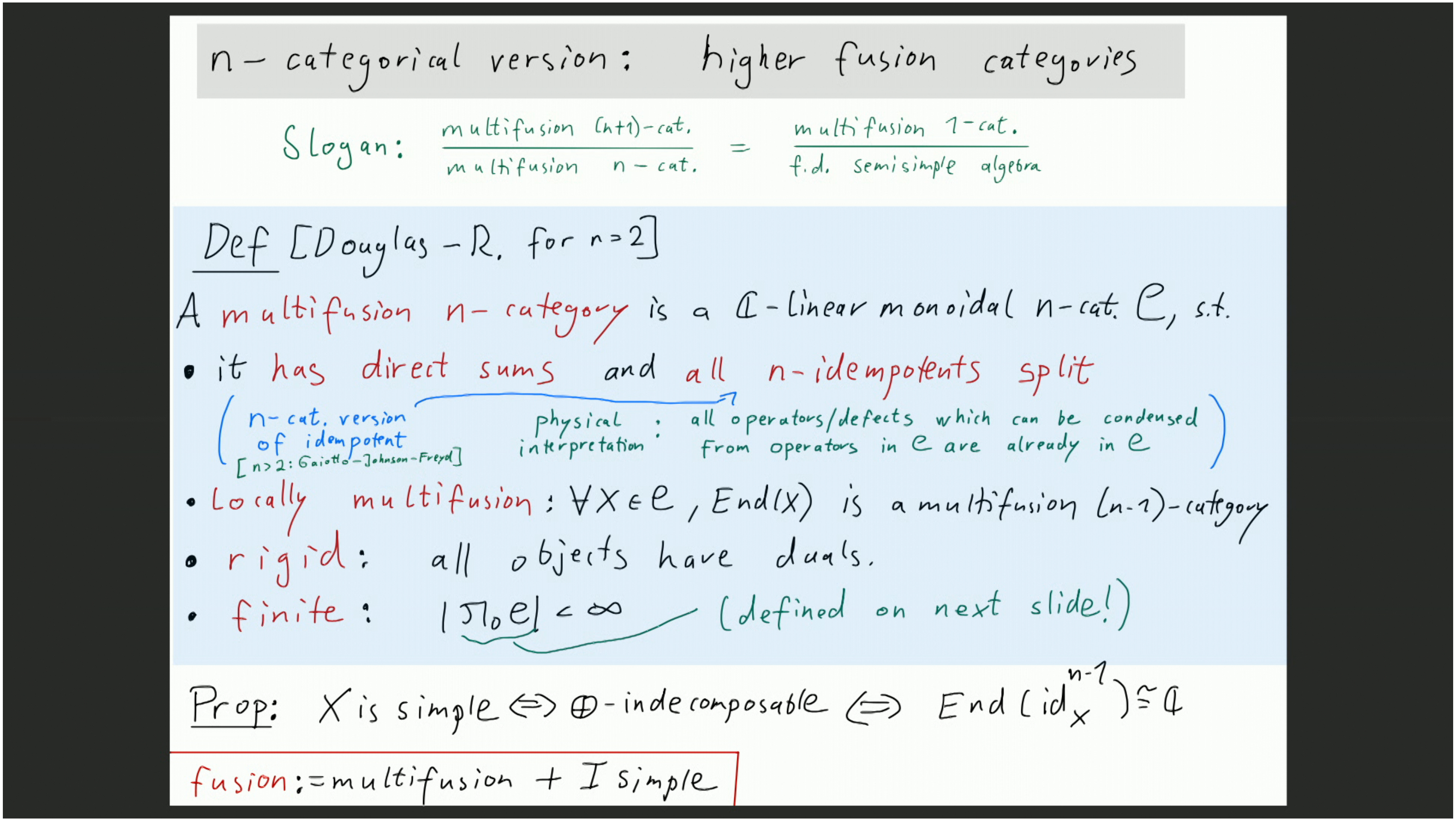

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Prop: X is simple $\Leftrightarrow$ $\oplus$-indecomposable $\Leftrightarrow$ $\mathrm{End}(\mathrm{id}_X^{n-1}) \cong \mathbb{C}$

fusion := multifusion + I simple

n-categorical version: higher fusion categories

<u>Def</u>: Simples X, Y are Schur connected if $\underbrace{\exists\, X \xrightarrow{\neq 0} Y}$

$n=1$: connected $\Leftrightarrow$ iso $n \geq 2$ connected $\overset{\Leftarrow}{\not\Rightarrow}$ iso

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Higher-cat. Schur's lemma

This is an equivalence relation.

Schur components

$\pi_0 \mathcal{C} := \text{simples} / \text{Schur connectivity} \quad \left(= \text{simples} / \text{condensation} \right)$

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Slogan: fusion cats generalize finite groups $(\mathrm{Vec}^{\omega}[G])$

fusion n-cats $\rightsquigarrow$ finite n-groups $(n\mathrm{Vec}^{\omega}[G])$

$(:=$ loop spaces w. $|\pi_k| < \infty \; \forall k$, $\pi_{\geq n} = 0)$

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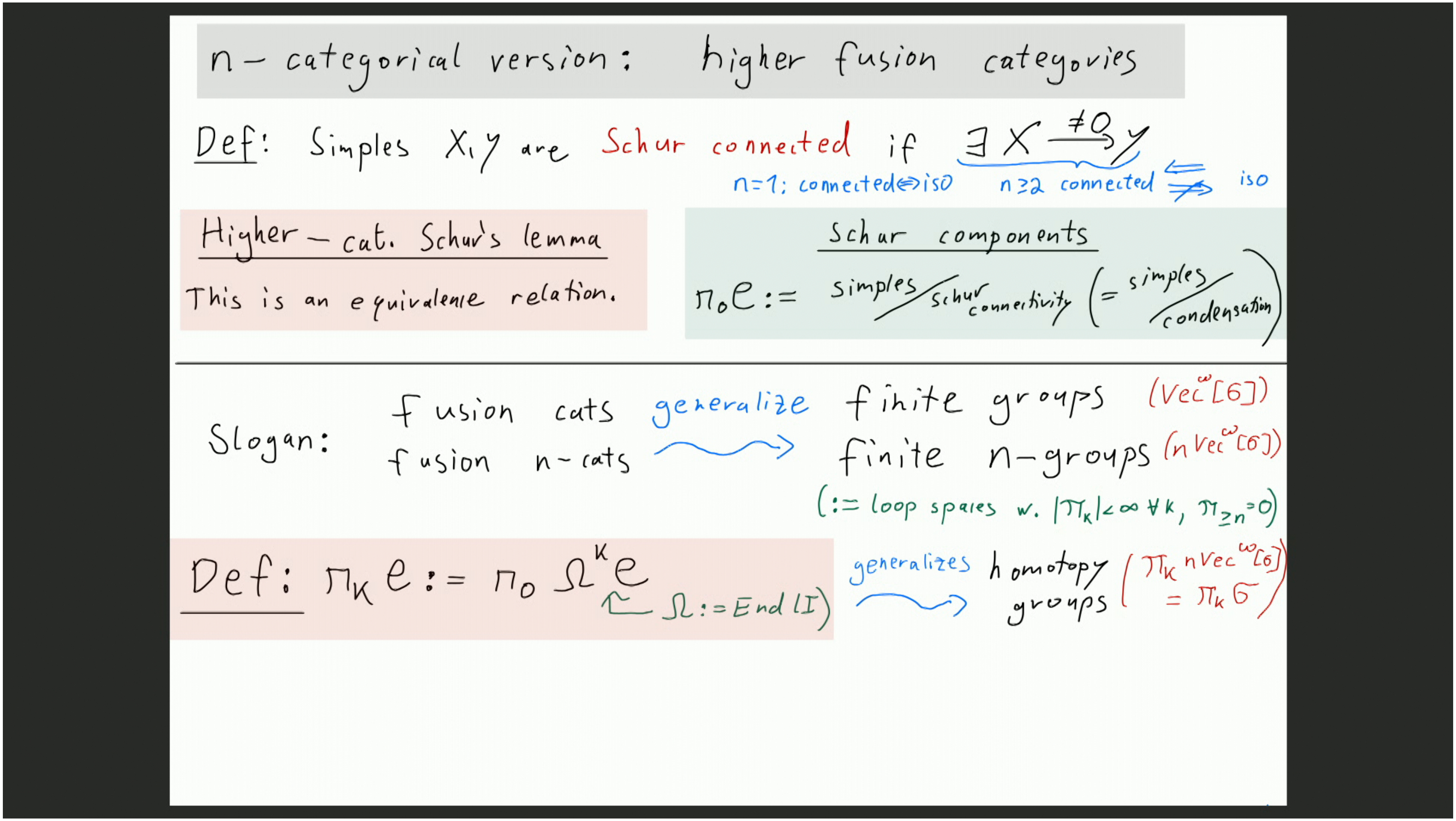
Def: $\pi_k \mathcal{C} := \pi_0 \Omega^k \mathcal{C}$ $\quad (\Omega := \mathrm{End}(I))$



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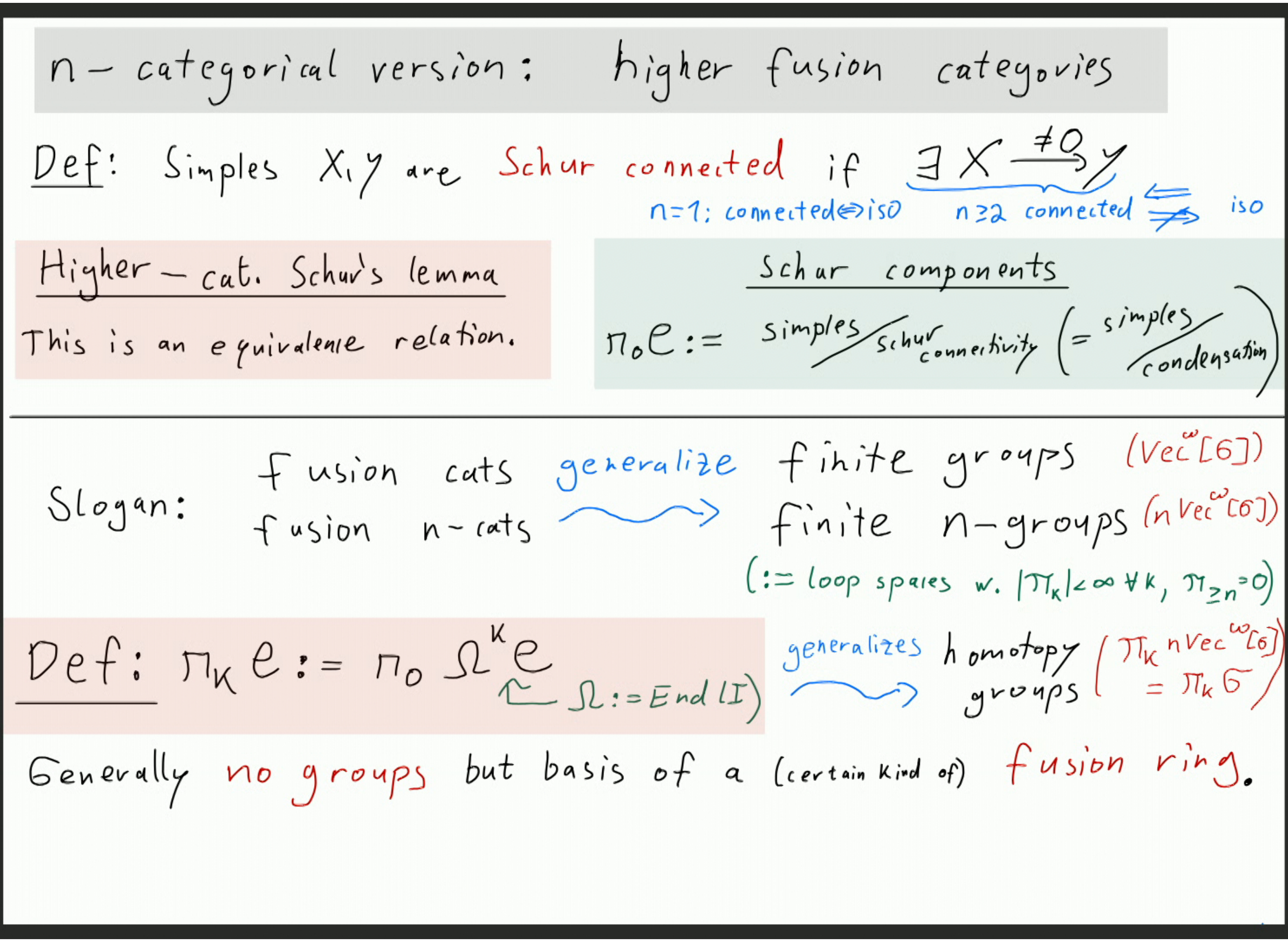
generalizes $\rightsquigarrow$ homotopy groups $\left(\begin{array}{l}\pi_k n\mathrm{Vec}^{\omega}[G] \\ = \pi_k G\end{array}\right)$



n – categorical version: higher fusion categories

Def: Simples X, Y are Schur connected if $\underbrace{\exists\, X \xrightarrow{\neq 0} Y}$

n=1: connected ⇔ iso n ≥ 2 connected $\overset{\Leftarrow}{\not\Rightarrow}$ iso

Higher – cat. Schur's lemma

This is an equivalence relation.

Schur components

$\pi_0 \mathcal{C} := \text{simples} / \text{Schur connectivity} \left(= \text{simples} / \text{condensation} \right)$

Slogan: fusion cats generalize finite groups $(\mathrm{Vec}^{\omega}[G])$

fusion n-cats $\rightsquigarrow$ finite n-groups $(n\mathrm{Vec}^{\omega}[G])$

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Generally no groups but basis of a (certain kind of) fusion ring.

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Aside: $\pi_k \mathcal{C}$ can be a group, even for non-invertible symmetry $\mathcal{C}$, namely $\Leftrightarrow \forall X\ \ X \otimes X^*$ is condensable from I (e.g. happened in Justin Kaidi's and Shu-Heng Shao's talk)

n-categorical version : tangle hypothesis

Let $\mathcal{C}$ be a monoidal n-category. A fully dualizable (n-k)-morphism $X \in \Omega^{n-k}\mathcal{C}$ can be evaluated at an embedded k-dimensional manifold $M^k \hookrightarrow \mathbb{R}^{n+1}$ with framing data:

- tangential framing $\tau : \underset{\text{tang. bundle}}{T_M} \simeq \underset{\text{triv. bundle}}{\underline{\mathbb{R}}^k}$
- normal framing $\nu : \underset{\text{normal bundle}}{N_M} \simeq \underline{\mathbb{R}}^{n+1-k}$
- a homotopy filling the triangle

$$\begin{tikzcd}
T_M \oplus N_M \arrow[rrd, bend left, "\tau\oplus\nu"] \arrow[d, phantom, "\wr"] & & \\
 & \eta \;\; \wr\wr & \underline{\mathbb{R}}^{n+1} \\
T_{\mathbb{R}^{n+1}}\big|_M \arrow[rru, "\text{blackboard framing}"'] & &
\end{tikzcd}$$

Notation: $\int_{M,\tau,\nu,\eta} X = \int_M X$

[Baez-Dolan, Lurie, ...]

n-categorical version: extra fractional dualizability

Let $\mathcal{C}$ be a fusion n-category and $x \in \Omega^{n-k}\mathcal{C}$

$$\Omega^{n-1}\mathcal{C} \ni \int_{S_b^{k-1}} x$$

sphere w. k-framing induced by $S_b^{k-1} = \partial D^k$

evaluation morphism $\int_{D^k} x : \int_{S_b^{k-1}} x \longrightarrow I$

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Let $\mathcal{C}$ be a fusion n-category and $x \in \Omega^{n-k}\mathcal{C}$

$$x \in \Omega^{n-k}\mathcal{C} \text{ simple} \Longleftrightarrow \underbrace{\Omega^{n-1}\mathcal{C}}_{\text{fusion 1-category}} \ni \int_{S_b^{k-1}} x \cong \mathbb{1} \oplus \dots$$

appears with multiplicity 1

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Conclude: Can place x on S_b^k:

(k+1)-framing: one more than dualizability allows

This needs semisimplicity!

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S_b^{k-1} ← sphere w. k-framing induced by $S_b^{k-1} = \partial D^k$

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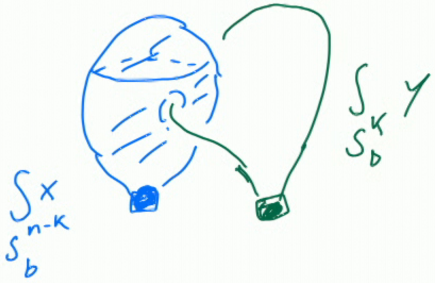
n-categorical version: framed S-matrix

For $x \in \Omega^k \mathcal{C}$, $y \in \Omega^{n-k} \mathcal{C}$ simple, $0 \le k \le n$ define:

$\tilde{S}^k_{x,y} :=$



$\subseteq \mathbb{R}^{n+1}$

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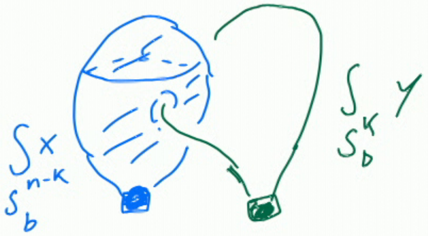

$$\in \Omega^n \mathcal{C} \cong \mathbb{C}$$

Thm: $\tilde{S}^k$ only depends on components $\tilde{S}^k : \pi_k \mathcal{C} \times \pi_{n-k} \mathcal{C} \to \mathbb{C}$

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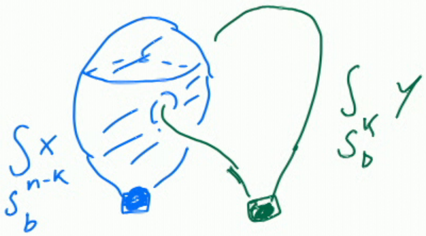

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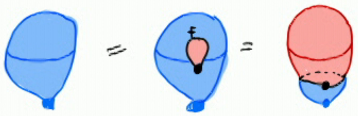
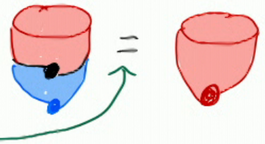
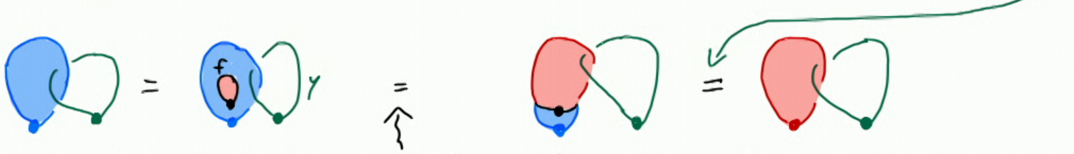


(2) Uniqueness of splittings ⇒



(3)



f and y don't link since $\dim f + \dim y = (n-k-1) + k < n$

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$\tilde{S}$ generalizes Whitehead bracket $\pi_k \Omega X \times \pi_{n-k} \Omega X \to \pi_n \Omega X$

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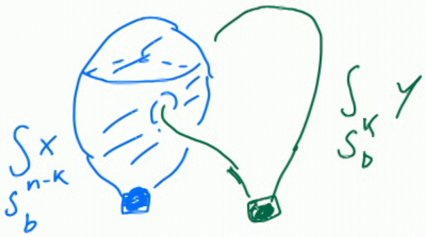

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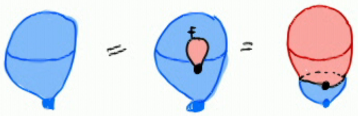

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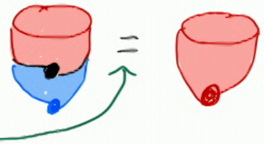

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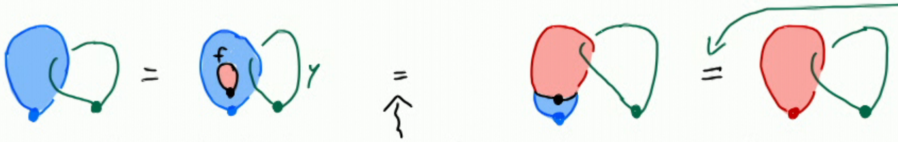

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S-matrices and invertibility

Morita class of $\mathcal{C}$

$\mathcal{C}$ itself

(Conjecture: Automatic over char. zero field)

(separable) fusion n-cat. $\mathcal{C} \rightsquigarrow$ framed $(n+2)d$ TQFT with $(n+1)d$ boundary cond.

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<u>Theorem</u> (in progress w. Johnson-Freyd)

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Drinfeld center

(ie. bulk is invertible, ie. $\mathcal{C}$ defines an anomalous $(n+1)$d TQFT)

[proves a conjecture of Kong-Wen '14 in condensed matter theory.]

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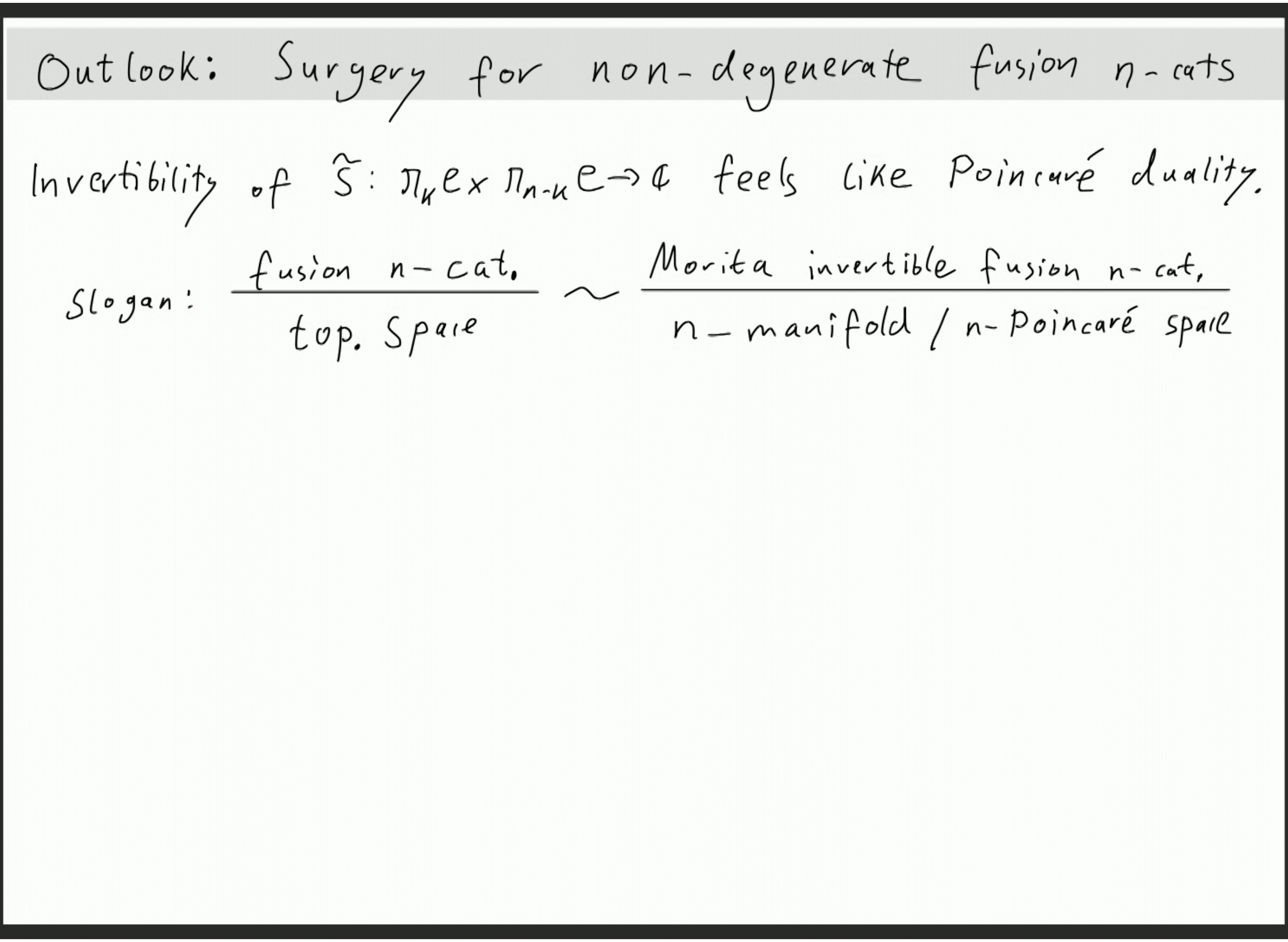
Exm: anomalous s.s. TQFT without local operators $(\pi_n \mathcal{C} = *)$

[JF'20] $\Rightarrow$ all codim. 1 operators condensable from vacuum $(\pi_0 \mathcal{C} = *)$

Math version: $\mathcal{C}$ invertible fusion n-cat $\Rightarrow \mathcal{C} \simeq \mathrm{Mod}$(braided fus. $(n-1)$-cat)

$\mathcal{C}$ invertible fusion 1-cat $\Rightarrow \mathcal{C} \simeq \mathrm{Vect}$



Outlook: Surgery for non-degenerate fusion n-cats

Invertibility of $\tilde{S}: \pi_k \mathcal{C} \times \pi_{n-k} \mathcal{C} \to \mathbb{C}$ feels like Poincaré duality.

Slogan: $$\frac{\text{fusion } n\text{-cat.}}{\text{top. space}} \sim \frac{\text{Morita invertible fusion } n\text{-cat.}}{n\text{-manifold / } n\text{-Poincaré space}}$$

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$\Rightarrow$ Nondegeneracy of $\tilde{S}$ is a first step towards a classification of Morita invertible higher fusion categories via surgery:

- ⇝ Kill high π_k's via "ungauging"
- ⇝ Poincare duality: have also killed π_{n-k}
- ⇝ possible S-matrix obstruction in middle dimension

Work in progress w. Johnson-Freyd...

Thank you for your attention!

Some references on higher fusion categories:

Douglas - R., Fusion 2-categories and a state-sum invariant for 4-manifolds, arXiv: 1812.11933

Johnson-Freyd — Gaiotto, Condensation in higher categories, arXiv: 1905.09566

Johnson-Freyd — Yu, Fusion 2-categories with no line operators, arXiv: 2010.07950

Décoppet, Weak fusion 2-categories, arXiv: 2103.15150

Johnson-Freyd — R., Minimal nondegenerate extensions, arXiv: 2105.15167

and the work of Kong, Lan, Wen, Zhang, Zheng, ... in condensed matter.