Title: Knot categorification from homological mirror symmetry

Speakers: Mina Aganagic

Collection: Global Categorical Symmetries

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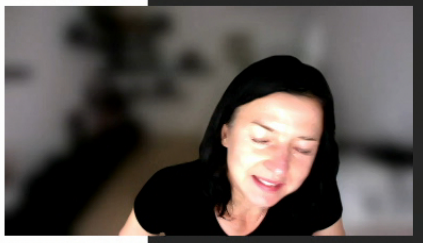
Abstract: "Khovanov showed in '99 that the Jones polynomial arises as the Euler characteristic of a homology theory. The knot categorification problem is to find a general construction of knot homology groups and to explain their meaning: what are they homologies of?

Homological mirror symmetry, formulated by Kontsevich in '94, naturally produces hosts of homological invariants. Sometimes, it can be made manifest, and then its striking mathematical power comes to fore. Typically though, it leads to invariants which have no particular interest outside of the problem at hand.

I will explain that there is a vast new family of mirror pairs of manifolds for which homological mirror symmetry does lead to interesting invariants, and solves the knot categorification problem. "



Knot categorification from homological mirror symmetry

Mina Aganagic

UC Berkeley

Based on: M. A., 2004.14518

M.A., 2105.06039

as well as M.A, M. Rapcak, E. LePage, to appear

M.A, I. Danilenko, P. Zhao, M. McBreen, V. Shende, in progress



In '98 Khovanov showed one can associate to every link

a complex of bi-graded vector spaces

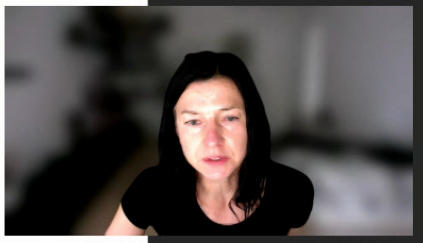
$$C^{*,j}(K) = \dots C^{i-1,j}(K) \xrightarrow{\partial^{i-1}} C^{i,j}(K) \xrightarrow{\partial^{i}} \dots$$

such that its homology groups

$$\mathcal{H}^{i,j}(K) = \ker \partial^i / \mathrm{im}\, \partial^{i-1}$$

categorify the Jones polynomial

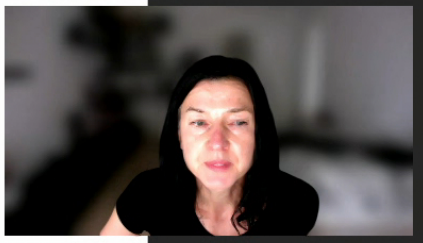
$$J_K(\mathfrak{q}) = \sum_{i,j} (-1)^i \mathfrak{q}^{j/2} \dim_{\mathbb{Z}} \mathcal{H}^{i,j}(K).$$



Khovanov's construction is part of

categorification program

pioneered by Crane and Frenkel.



A simple toy model of categorification comes from a Riemannian manifold M whose Euler characteristic

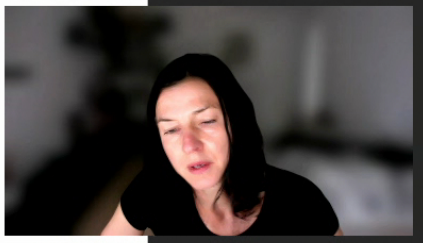
$$\chi(M) = \sum_{k \in \mathbb{Z}} (-1)^k \dim_{\mathbb{Z}} \mathcal{H}^k(M)$$

is categorified by the cohomology groups

$$\mathcal{H}^k(M) = \ker d_k / \mathrm{im}\, d_{k-1}$$

of the de Rahm complex

$$C^* = \dots C^{k-1} \xrightarrow{d_{k-1}} C^k \xrightarrow{d_k} \dots .$$



The action of the Q supercharge

$$Q = \sum_k d_k$$

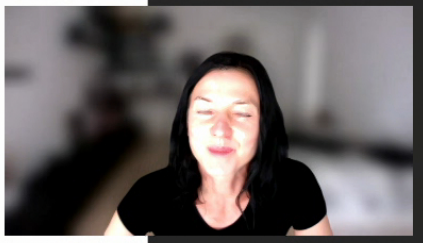
on the complex

$$C^* = \dots C^{k-1} \xrightarrow{d_{k-1}} C^k \xrightarrow{d_k} \dots .$$

is generated by instantons.

Q defines a differential as it squares to zero,

$$Q^2 = 0$$



In 2013 Webster showed there is an

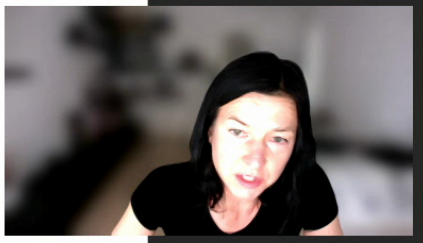
abstract algebraic framework

for categorification of quantum link invariants for arbitrary

$${}^{L}\mathfrak{g}$$

based on KLRW algebras,

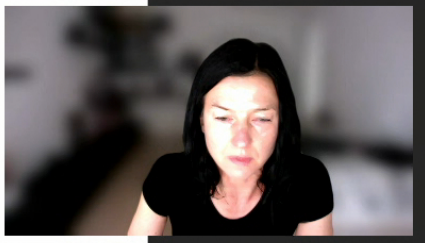
generalizing algebras of Khovanov and Lauda, and of Rouquier.



Khovanov's categorification of the Jones polynomial

it is explicit and easily calculable.

The theory Webster formulated was intractable.



Unlike in

our toy example of categorification of the Euler characteristic

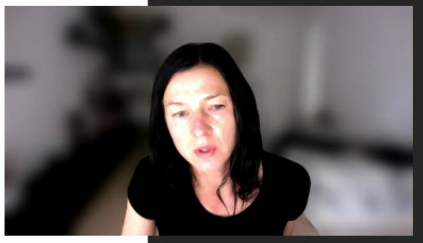
of a Riemannian manifold

$$\chi(M) = \mathrm{Tr}(-1)^F e^{-\beta H}$$

Khovanov's construction and its generalizations

do not come from either geometry,

or physics in any unified way.



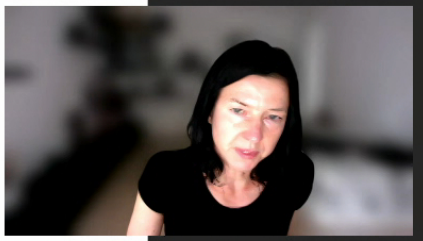
I will explain that

Khovanov's construction also has an origin in physics,

one that generalizes to

all gauge groups,

just like the Jones polynomial does.



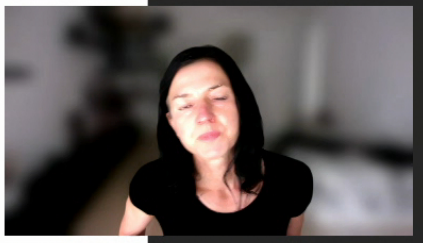
Homological mirror symmetry relates

categories of boundary conditions in

two dimensional sigma models.

It naturally gives rise to hosts of homological invariants.

Most of the time though, the invariants it gives rise to are not of any particular interest outside of the problem at hand, e.g. if one studies the quintic.

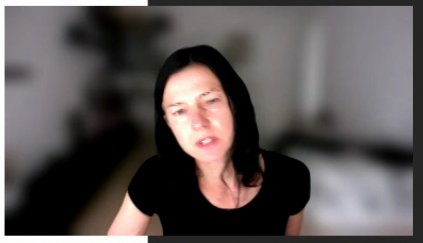


We will discover here a vast new family,

of mirror pairs of manifolds,

where homological mirror symmetry does lead to interesting new invariants.

It solves the knot categorification problem.

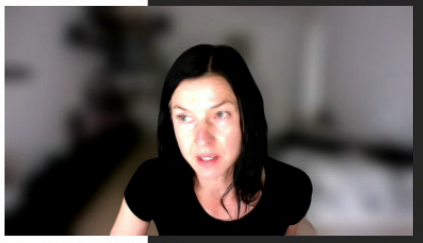


Chern-Simons theory,

which Witten showed leads to quantum link invariants,

is a striking example of an interacting quantum field theory

which is solvable exactly.

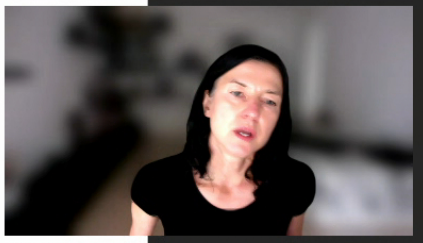


Its solvability turns out to carry over to the homological world.

The sigma models that arise are extremely special,

and their very special features

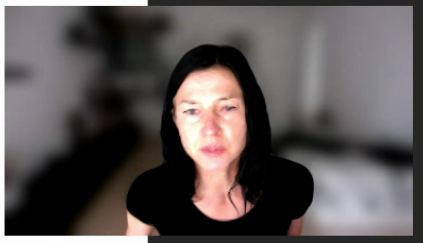
make the resulting theory explicitly solvable.



Our sigma models will describe the theory on

two dimensional defects

of the six dimensional (0,2) theory.



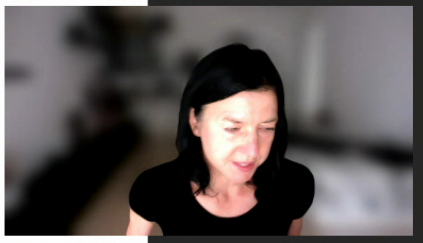
The relation to representation theory may be anticipated

by the fact that possible (0,2) theories in six dimensions

are labeled by a choice of a simply laced Lie algebra

$$L_{\mathfrak{g}}$$

and the codimension four defects are labeled by its weights.



The fact that six dimensional (0,2) theory is central to the problem

was conjectured by Ooguri and Vafa in 1999,

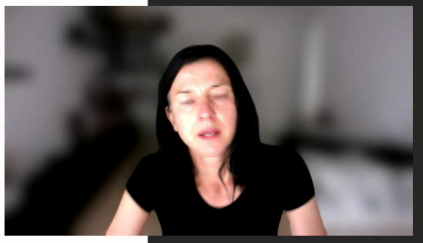
and Gukov, Schwarz and Vafa in 2004.

A six, or five dimensional approach to the problem

is being developed by Witten.

We will describe the same physics from the perspective

of the two-dimensional theories on the defects.

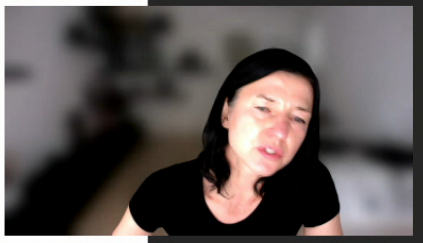


In '88, Witten showed

that the Jones polynomial comes from

Chern-Simons theory with gauge group based on the Lie algebra

$$^{L}\mathfrak{g} = \mathfrak{su}_2$$



In the same paper, he showed that

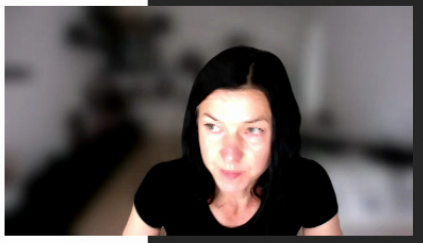
underlying Chern-Simons based on Lie algebra

$L\mathfrak{g}$ at level κ

is a two-dimensional conformal field theory with

affine Lie algebra symmetry

$\widehat{L\mathfrak{g}}_\kappa$



The Hilbert space of Chern-Simons theory

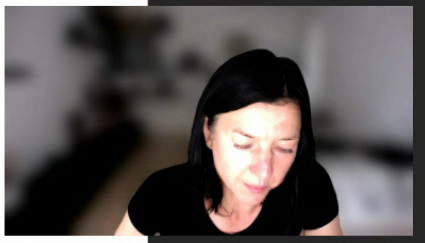
is the space of conformal blocks of

$$\widehat{L\mathfrak{g}}_\kappa$$

on $\mathcal{A}$ where Wilson lines puncturing the Riemann surface

translate to insertions of vertex operators.





To get invariants of knots in $\mathbb{R}^3$ or S^3

one typically starts with a Riemann surface

$\mathcal{A}$

which is a complex plane with punctures,





It is equivalent, but for our purpose better, to take

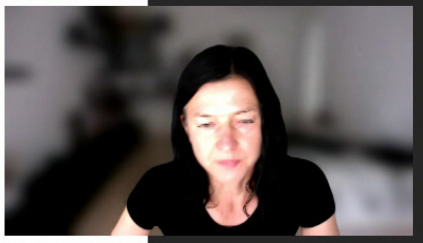
$\mathcal{A}$



to be a punctured infinite cylinder.

This way, the theory will be able to describe links in

$\mathbb{R}^2 \times S^1$ and not just $\mathbb{R}^3$



Rather than characterizing conformal blocks

$$\langle \lambda |\ \Phi_{V_1}(a_1) \cdots \Phi_{V_m}(a_m)\ | \lambda' \rangle$$

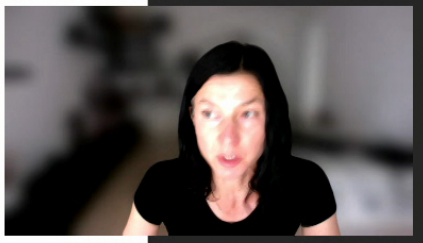
in terms of vertex operators and sewing,



$\mathcal{A}$

one can describe them as solutions of the

equation discovered by Knizhnik and Zamolodchikov in '84.

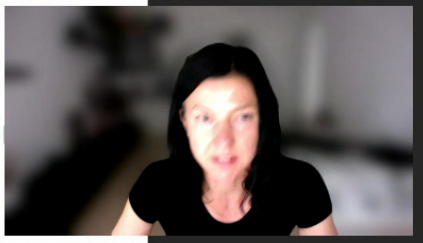


We will need its trigonometric version,

$$\kappa\, a_\ell \frac{\partial}{\partial a_\ell}\, \mathcal{V} = \sum_{j\neq\ell} r_{\ell i}(a_\ell/a_j)\, \mathcal{V}.$$

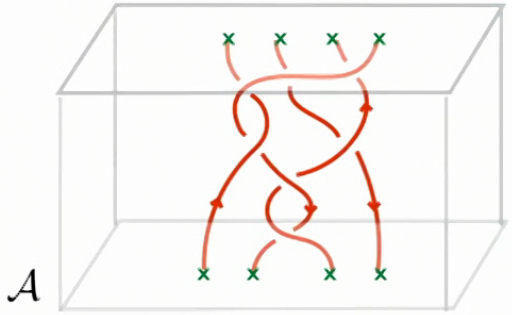
since the Riemann surface is an infinite cylinder.





By varying the positions of punctures on $\mathcal{A}$ as a function of "ti

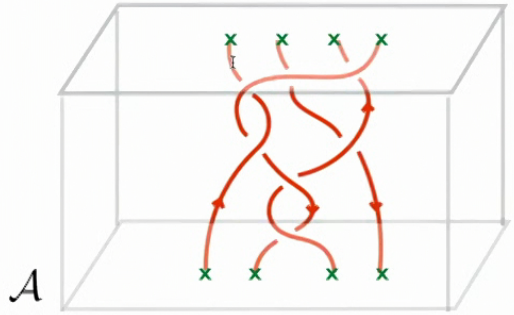
we get a colored braid and a monodromy matrix



of the KZ equation,

along a path in the parameter space corresponding to the braid.

The monodromy matrix



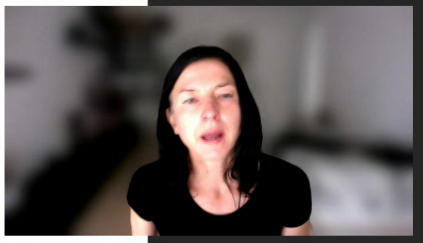
is given in terms of R-matrices of the quantum group

$$U_{\mathfrak{q}}({}^{L}\mathfrak{g})$$

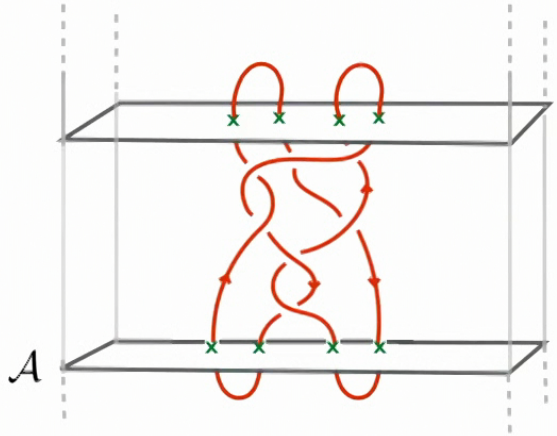
corresponding to ${}^{L}\mathfrak{g}$

This was observed first in works of Tsuchia and Kanyie,

and proven by Drinfeld and by Kazhdan and Lustig in '89.

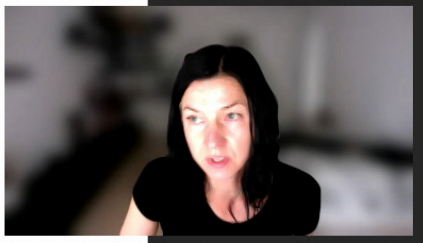


Any link invariant can be represented as a matrix element



of a braiding matrix between conformal blocks

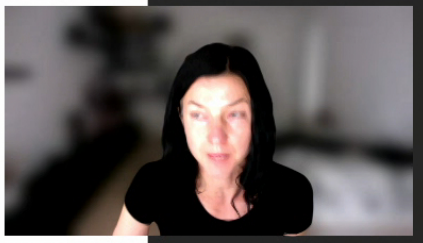
corresponding to cups and caps.



Thus, both braiding and fusion in conformal field theory



play an important role in the story.



To categorify quantum knot invariants,

one would like to associate

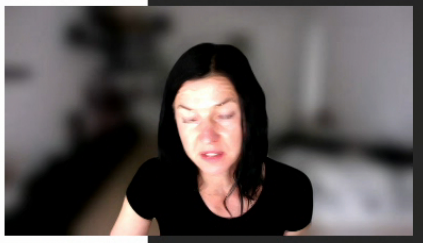
to the space conformal blocks one obtains at a fixed time slice



a bi-graded category.

In addition to the usual fermion number grading

the category should have an additional grading associated to q



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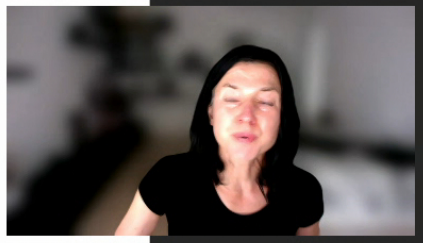
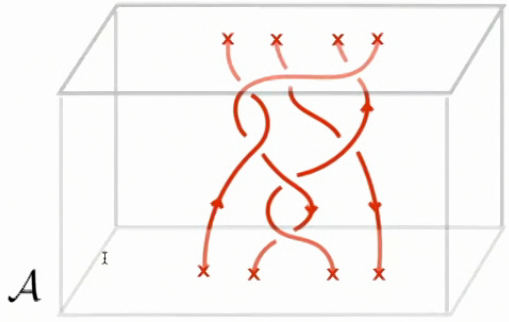
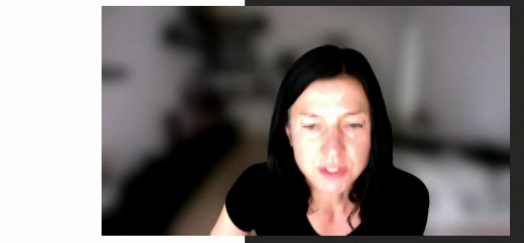


To braids,

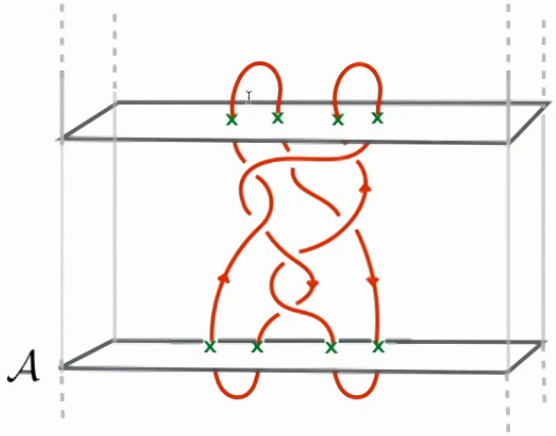


one would like to associate functors between the categories

corresponding to the top and the bottom.



To links one would like to associate

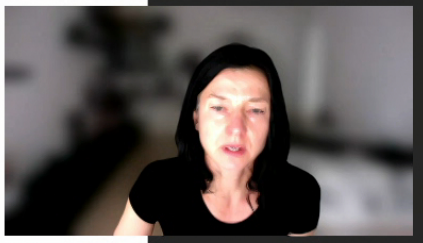


a vector space

whose elements are morphisms

between the objects of the categories the top and the bottom,

up to the action of the braiding functor.

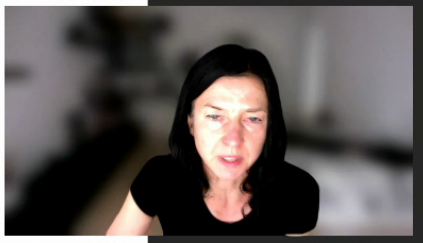


Moreover,

one would like to do that in the way that

recovers the quantum knot invariants upon

de-categorification.

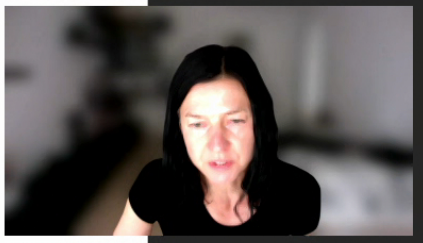


One typically proceeds by coming up with a category,

and then one has to work to prove

that de-categorification gives

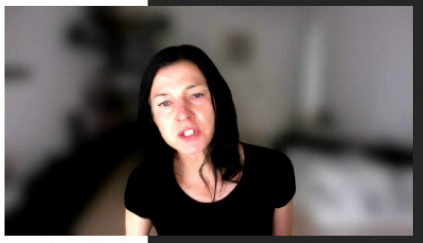
the quantum knot invariants one set out to categorify.



The geometric solutions to the knot categorification problem

I will describe

share the virtue of both that the second step is automatic.



The Knizhnik-Zamolodchikov equation

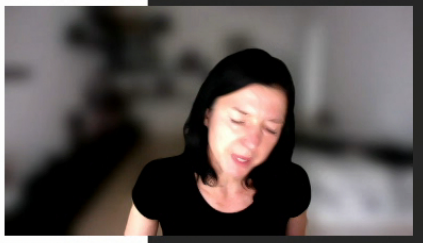
and its solutions,

have a geometric realization,

in fact a pair of them.

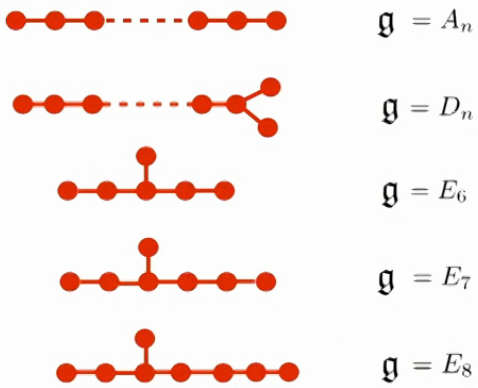
This serves as the starting point for categorification

of quantum link invariants.



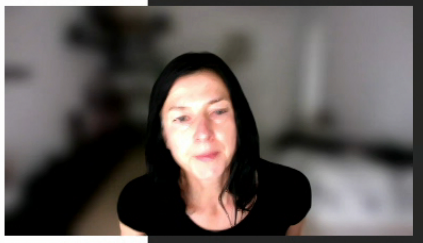
We will specialize ${}^L\mathfrak{g}$ to be a simply laced Lie algebra

so ${}^L\mathfrak{g} = \mathfrak{g}$ are one of the following types:



The generalization to non-simply laced Lie algebras

involves an extra step, which we will not have time for.

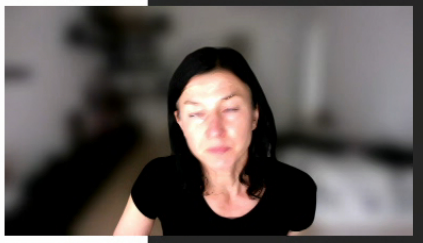


The Knizhnik-Zamolodchikov equation

$$\kappa\,\partial_i \mathcal{V}_\alpha - (r_i)^\beta_\alpha\ \mathcal{V}_\beta = 0$$

which plays a central role in knot theory,

has a geometric counterpart.

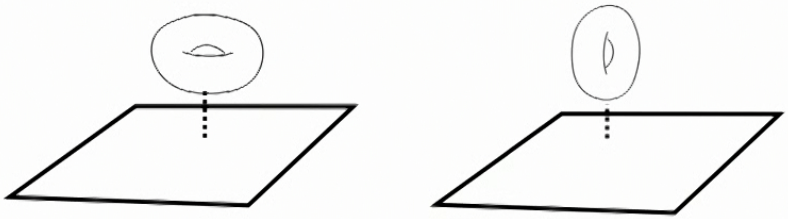


Mirror symmetry,

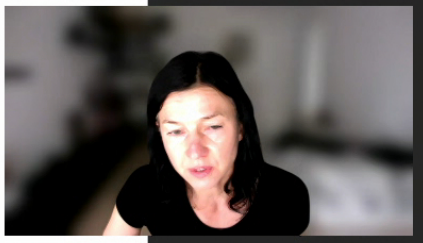
relates pairs of Calabi-Yau manifolds

$\mathcal{X}$ and $\mathcal{Y}$

which are fibrations by a pair of "dual tori"



over a common base.

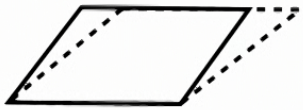


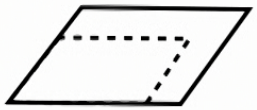
Mirror symmetry exchanges

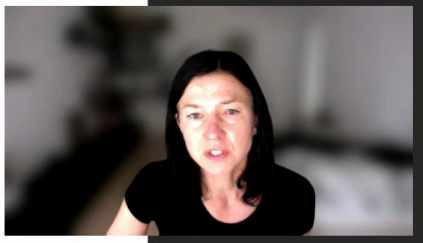
$\mathcal{X}$ and $\mathcal{Y}$

while exchanging variations of

complex structure "B-type" and symplectic or Kahler structures "A-type"







The "quantum differential equation"

$$\partial_i \mathcal{V}_\alpha - (C_i)^\beta_\alpha \, \mathcal{V}_\beta = 0.$$

is a linear differential equation for a vector valued function

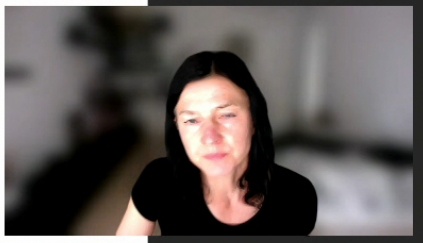
over the moduli space of either

the symplectic (A-type)

$\mathcal{X}$

or complex structures (B-type) on its mirror

$\mathcal{Y}$



The name

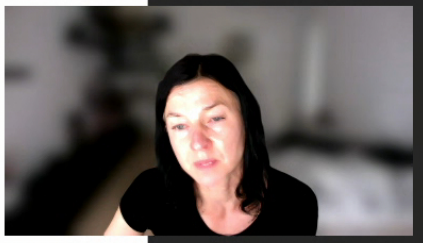
"quantum differential equation"

comes from symplectic (A-type) geometry where the coefficients in

$$\partial_i \mathcal{V}_\alpha - (C_i)^\beta_\alpha \mathcal{V}_\beta = 0.$$

are computed by "quantum multiplication" with a class in

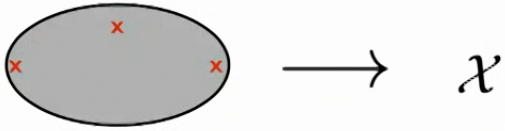
$$C_i \in H^2(\mathcal{X})$$

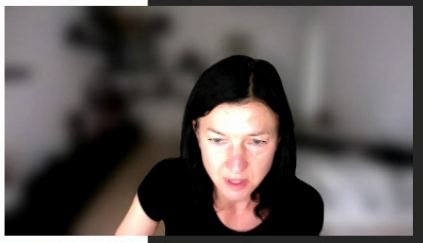


Quantum product on $H^*(\mathcal{X})$

$$\langle \gamma_i \star \gamma_j, \gamma_k \rangle = \sum_{d \geq 0, d \in H_2(\mathcal{X})} (\gamma_i, \gamma_j, \gamma_k)_d \, a^d$$

is defined by Gromov-Witten counts of rational curves





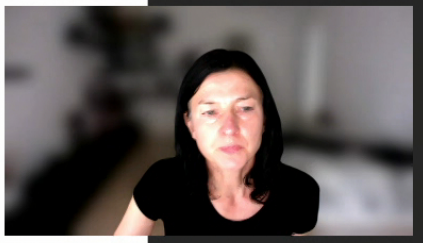
The first, $d = 0$ term of quantum multiplication

$$\langle \gamma_i \star \gamma_j, \gamma_k \rangle = \sum_{d \geq 0, d \in H_2(\mathcal{X})} (\gamma_i, \gamma_j, \gamma_k)_d \, a^d$$

is the classical product on $H^*(\mathcal{X})$:

$$(\gamma_i, \gamma_j, \gamma_k)_0 = \int_{\mathcal{X}} \gamma_i \wedge \gamma_j \wedge \gamma_k$$

subsequent $d > 0$ terms are quantum corrections.

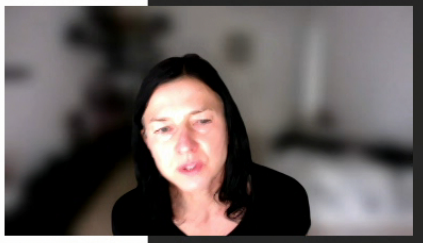


Both the equation,

$$\partial_i \mathcal{V}_\alpha - (C_i)^\beta_\alpha \, \mathcal{V}_\beta = 0.$$

and its monodromy problem,

featured prominently starting with the very first papers on mirror symmetry.



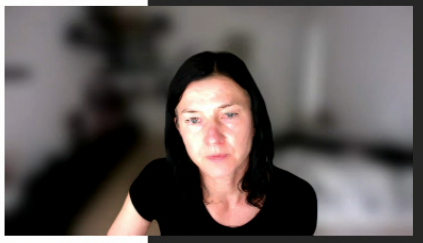
The solutions to the quantum differential equation

$$\partial_i \mathcal{V}_\alpha - (C_i)^\beta_\alpha \, \mathcal{V}_\beta = 0.$$

live in a finite dimensional vector space

associated to the manifold

which is spanned by the K-theory classes of its branes.



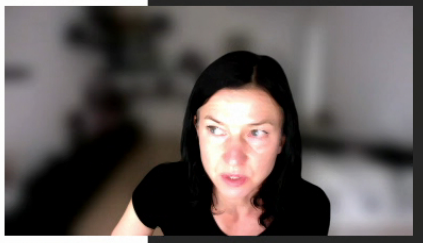
From perspective of $\mathcal{X}$ the solutions are obtained



by counting holomorphic maps of all degrees

from a domain curve D, which is best thought of an infinite cigar,

with an S^1 boundary at infinity.



We get a specific solution of the equation

$$\partial_i \mathcal{V}_\alpha - (C_i)^\beta_\alpha \, \mathcal{V}_\beta = 0.$$

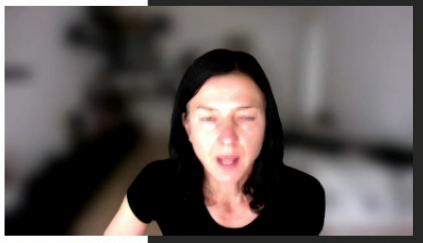
by choosing a B-type brane

as the boundary condition at infinity,



The solution depends on the brane only through its K-theory class,

and not the details of its shape.



The Knizhnik-Zamolodchikov equation

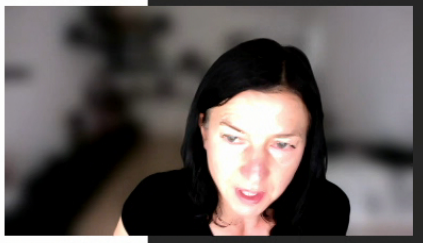
$$\kappa\,\partial_i \mathcal{V}_\alpha - (r_i)^\beta_\alpha\,\mathcal{V}_\beta = 0$$

not only has the same flavor

as the quantum differential equation:

$$\partial_i \mathcal{V}_\alpha - (C_i)^\beta_\alpha\,\mathcal{V}_\beta = 0,$$

under certain conditions, they coincide.



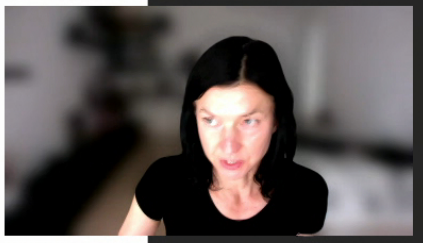
On the knot theory side, we want to take

the Riemann surface to be a punctured infinite cylinder,



rather than a complex plane with punctures.





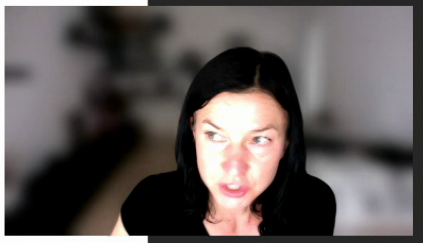
This enriches the theory, allowing it to describe

invariants of knots in

$$M_3 = \mathbb{R}^2 \times S^1$$

and not only in

$$M_3 = \mathbb{R}^3$$



The Chern-Simons gauge group

^{L}G

(whose Lie algebra is $^{L}\mathfrak{g}$) is related to

G

by Langlands, or

electric-magnetic type duality.



In Chern-Simons theory,

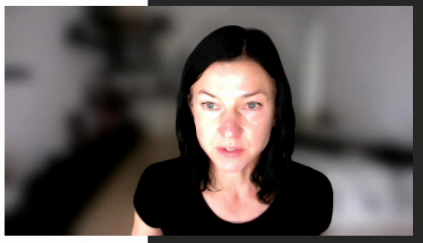
view the knots in three dimensional space



V_i

as paths of heavy particles electrically charged under

${}^L G$



In the geometric description, the same heavy particles



V_i

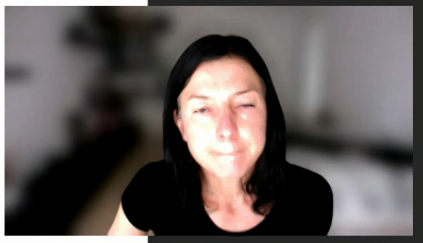
appear as Dirac monopoles of the Langlands dual group

G

This magnetic description is what is needed

to understand categorification,

as anticipated in works of Witten.



The manifold

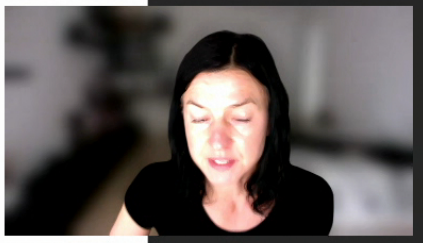
$\mathcal{X}$

has played an important role in mathematics before.

In geometric Langlands correspondence

it is known as a

"transversal slice to affine Grassmanian of G ."

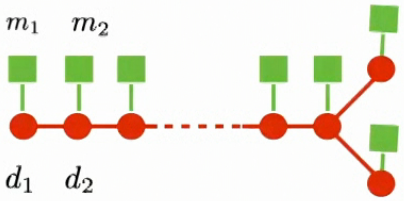


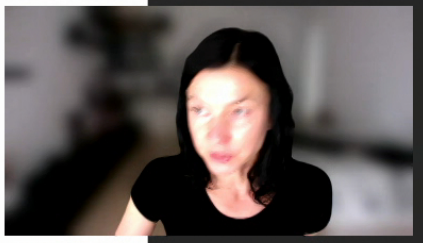
To physicists,

$\mathcal{X}$

is the Coulomb branch of a certain three dimensional quiver gauge theory

$\mathfrak{g} = D_n$





The monopole moduli space

$\mathcal{X}$

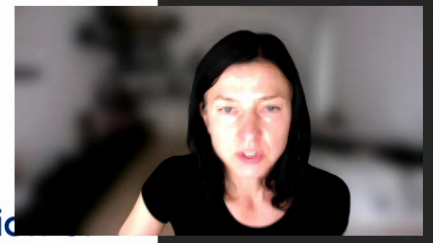
is parameterized in part by positions of some number of smooth

't Hooft-Polyakov type monopoles on

$\mathbb{R}^3$

whereas positions of singular, Dirac-type monopoles are fixed,

and determine the metric on $\mathcal{X}$

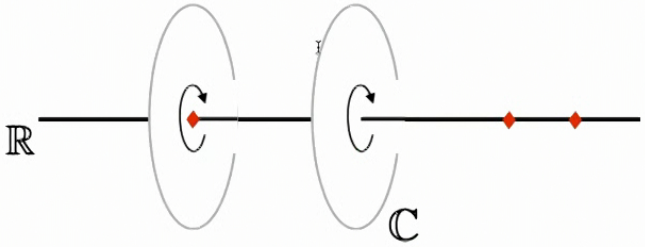


To get the KZ equation to coincide with the quantum differential equatio

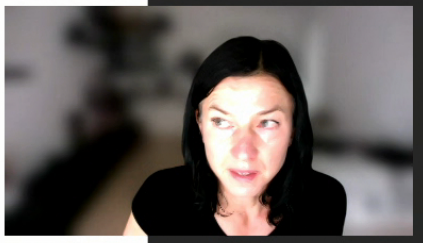
$\mathcal{X}$

we want to place all the singular, Dirac-type monopoles at the origin of

a complex plane in $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$



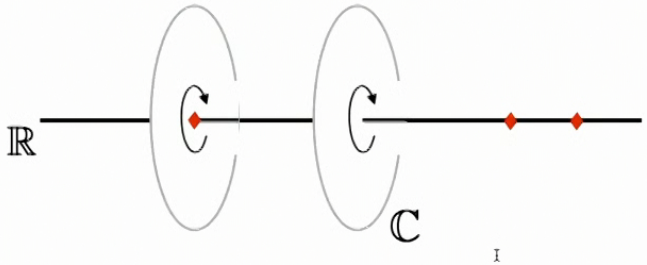
Then, rotations of this plane are an isometry of $\mathcal{X}$



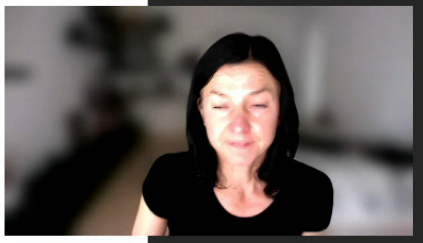
The parameter

q

of knot theory will be related to keeping track of charges of states



under this symmetry.

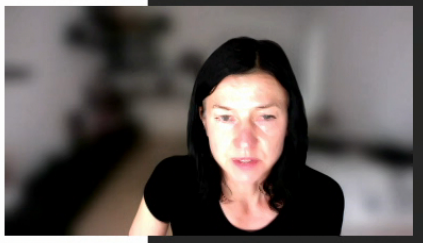


The fact that quantum group invariants

become interesting only for

$$q \neq 1$$

has a geometric counterpart.



The monopole moduli space

$$\mathcal{X}$$

has more symmetries than a typical Calabi-Yau,

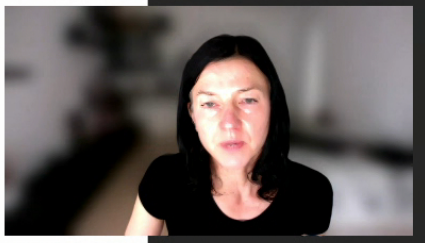
so the quantum multiplication

$$\langle \gamma_i \star \gamma_j, \gamma_k \rangle = \sum_{d \geq 0, d \in H_2(\mathcal{X})} (\gamma_i, \gamma_j, \gamma_k)_d \; a^d$$

differs from classical only as long as

$$\mathfrak{q} \neq 1$$

($\mathcal{X}$ is "hyper-Kahler" as opposed to just Kahler)



The Knizhnik-Zamolodchikov equation

coincides with the quantum differential equation of

$\mathcal{X}$

provided we work equivariantly with respect to a larger torus of symmetries

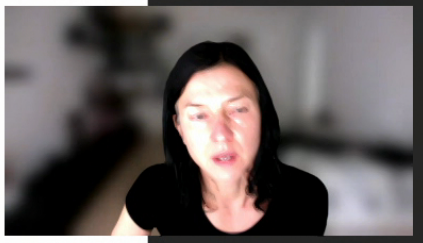
$$\mathrm{T} = \Lambda \times \mathbb{C}_q^\times$$

where Λ -action preserves the holomorphic symplectic form,

and encodes holonomies of the Chern-Simons gauge fields

around the S^1 in





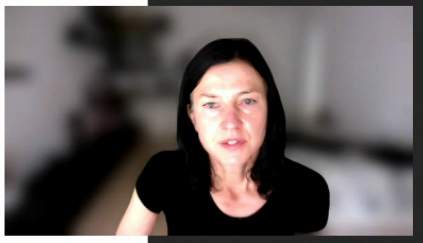
The fact that trigonometric Knizhnik-Zamolodchikov equation of

$\widehat{L\mathfrak{g}}$

has a geometric interpretation as the quantum differential equation of

$\mathcal{X}$

is a recent theorem by Ivan Danilenko.



It follows that which solution of the KZ equation we get

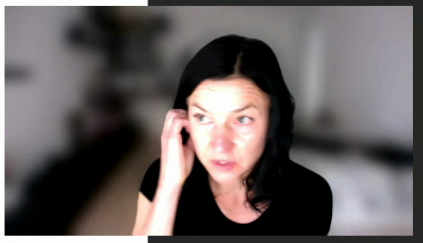
$$\partial_i \mathcal{V}_\alpha - (C_i)^\beta_\alpha \, \mathcal{V}_\beta = 0.$$

is determined by the choice of a B-type brane on

$$\mathcal{X}$$

as the boundary condition at infinity.





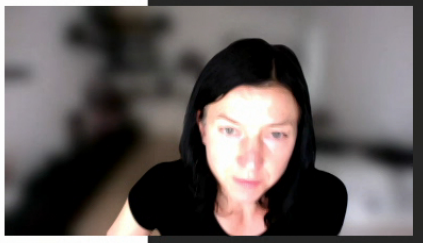
Boundary conditions form a category, and

the category of B-type branes

working equivariantly with respect to T is known as its

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

the derived category of $\mathrm{T}-$ equivariant coherent sheaves.

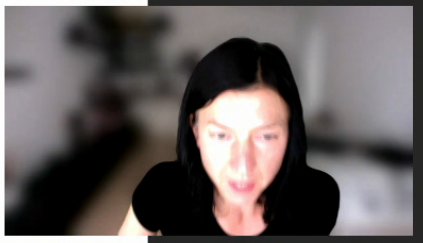


The positions of punctures on the Riemann surface

$\mathcal{A}$ 

coincide with the A-type, or Kahler moduli of

$\mathcal{X}$

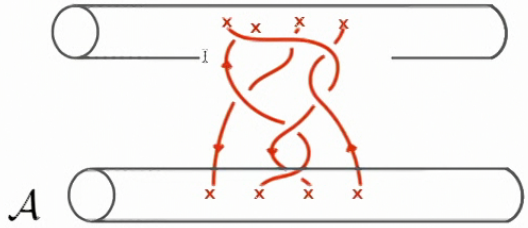


It follows that a braid

B

has a geometric interpretation

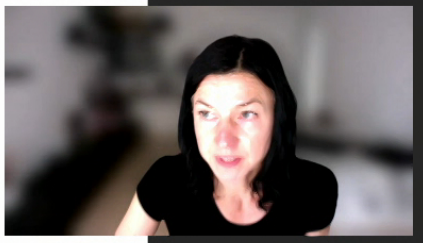
as a path in (complexified) Kahler moduli



since these moduli of

$\mathcal{X}$

are the relative positions of punctures on $\mathcal{A}$



A central expectation in mirror symmetry,

is the fact that

monodromy of the quantum differential equation

$\mathfrak{B}$

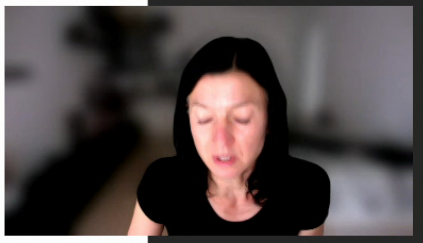


is categorified by a functor acting on the category of branes,

$$\mathscr{B} : \mathscr{D}_{\mathcal{X}} \to \mathscr{D}_{\mathcal{X}'}$$

which "transports" the category along the path B

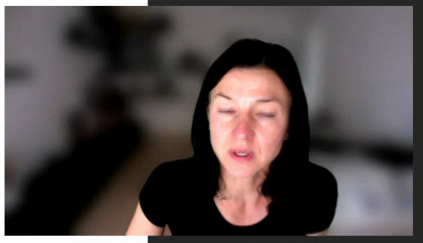
and which is an equivalence.



In settings relevant to us

the proof was given by Bezrukavnikov and Okounkov,

using quantization in characteristic p.



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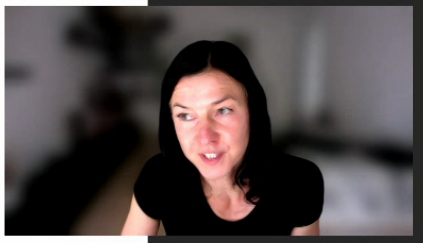
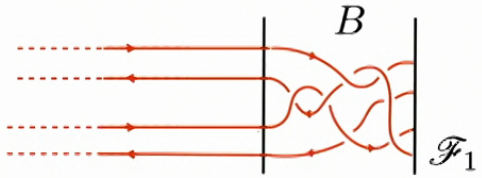
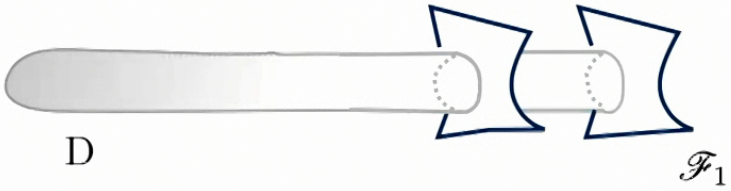


Physically, braid group



acts, in the sigma model on the cigar,

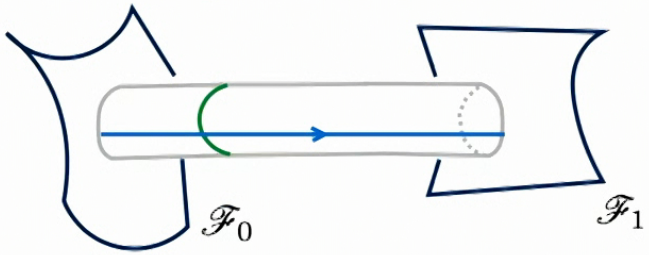
by letting the moduli of the theory vary near the boundary at infinity.



The direction along the (Euclidian) cigar

coincides with the "time" along the braid.

It follows sigma model on the annulus



with moduli that vary according to the braid computes

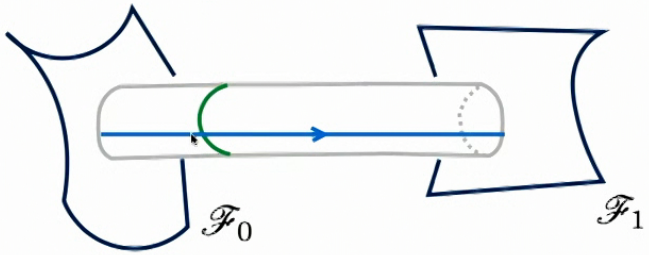
the matrix element of the monodromy

$$\mathfrak{B} \in U_{\mathfrak{q}}({}^{L}\mathfrak{g})$$

between a pair of branes

For this, we view the time to run along the annulus.

It follows sigma model on the annulus



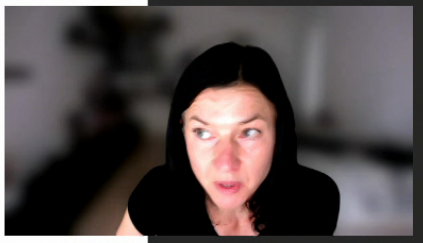
with moduli that vary according to the braid computes

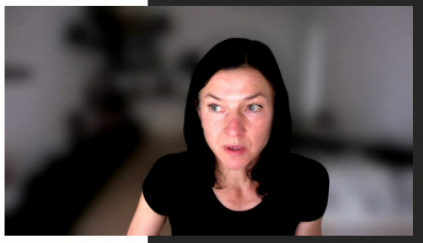
the matrix element of the monodromy

$$\mathfrak{B} \in U_{\mathfrak{q}}({}^{L}\mathfrak{g})$$

between a pair of branes

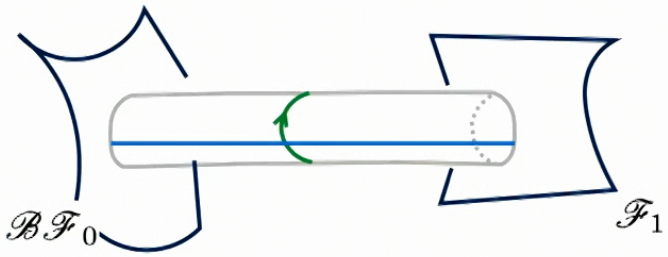
For this, we view the time to run along the annulus.





Sigma model on the same Euclidian annulus

with "time" viewed as running around the S^1 ,

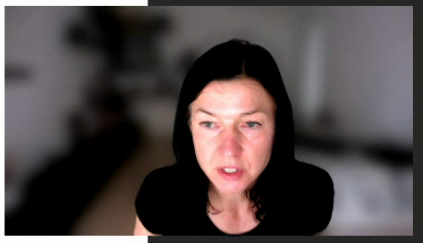


computes the supertrace

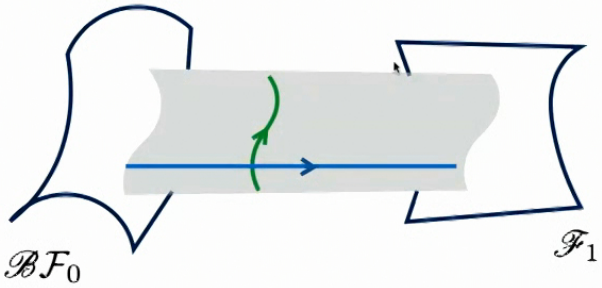
$$\mathrm{Tr}(-1)^F e^{-\beta H}$$

which is the Euler characteristic

of the supercharge Q preserved by the two branes.



The cohomology of the supercharge Q



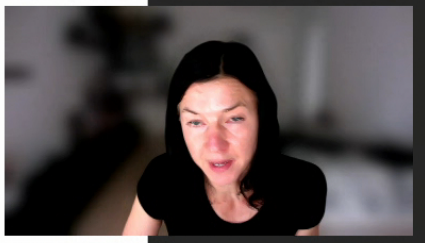
is the basic ingredient in the category of branes

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

a graded vector space,

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathscr{B}\mathcal{F}_0, \mathcal{F}_1)$$

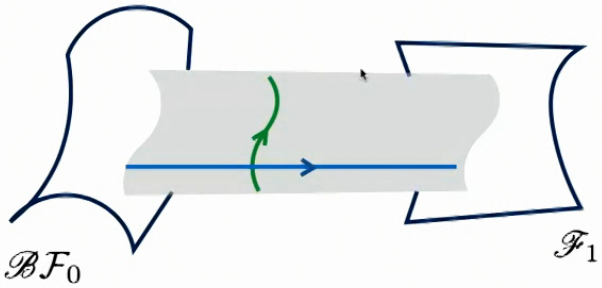
the space of morphisms between a pair of branes.



So, for any pair of branes, we get a the graded vector space,

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathscr{B}\mathcal{F}_0, \mathcal{F}_1)$$

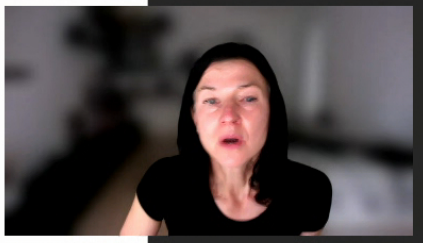
the space of morphisms between a pair of branes,



whose Euler characteristic

$$\chi(\mathscr{B}\mathcal{F}_0, \mathcal{F}_1)$$

is the braiding matrix element.



So far, we understood that

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

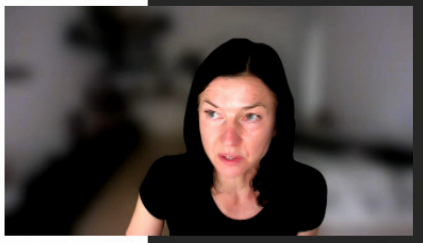
manifestly categorifies



the matrix elements of

$$\mathfrak{B} \in U_{\mathfrak{q}}({}^{L}\mathfrak{g})$$

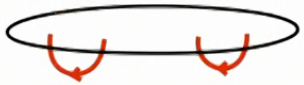
between pairs of conformal blocks.



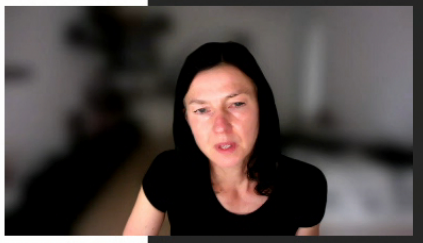
To extend this to link invariants, we need objects of

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

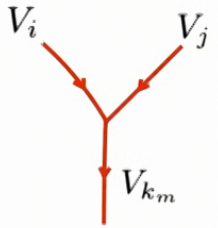
which lead to conformal blocks



in which pairs of vertex operators fuse to trivial representation.



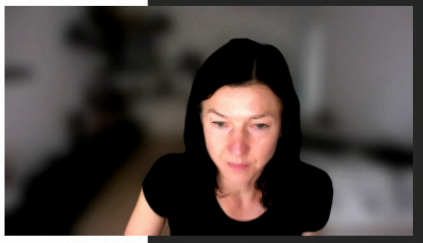
For this, one needs to understand



the geometric interpretation of fusion in terms of

$\mathcal{X}$

and its category of B-type branes.



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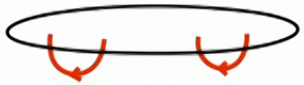

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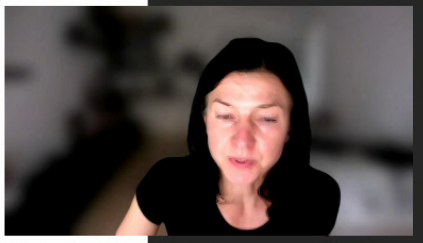

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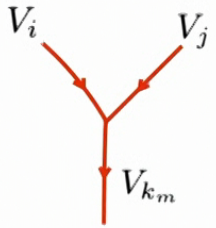

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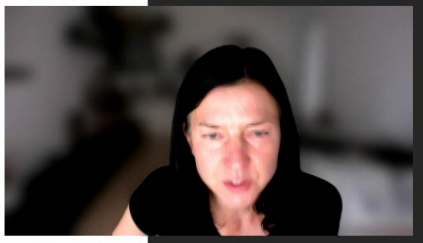


As one brings a pair of singular monopoles close together



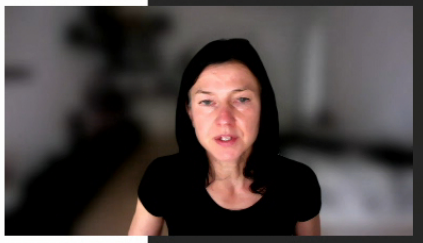
our manifold

$\mathcal{X}$

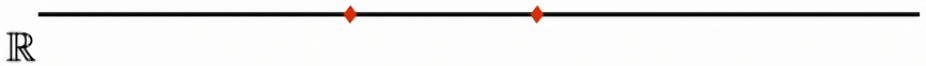
develops a singularity where a collection of cycles vanishes,

as the distance between the monopoles

controls their size.

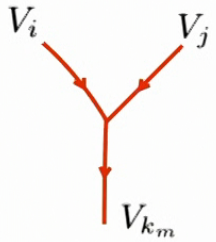


The singularities occur due to monopole "bubbling"

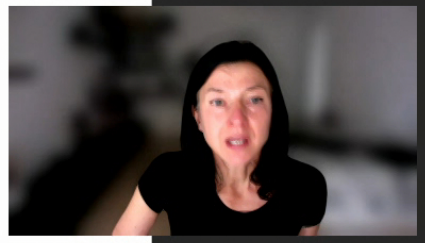


When a pair of singular monopoles come together,

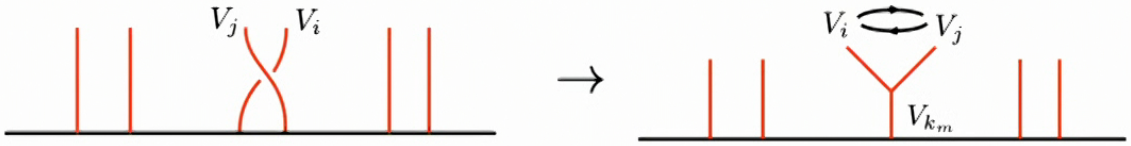
a number of smooth monopoles can bubble off of them and disappear.



to leave behind a single singular monopole of lower charge.



In conformal field theory fusion diagonalizes braiding.



The analogue of this in the category of branes is

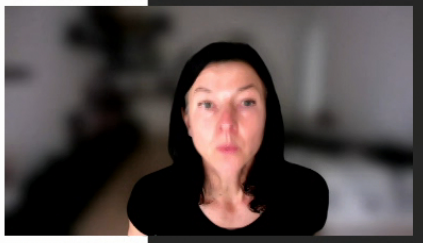
turns out to be existence of a "perverse filtration"

envisioned by Chuang and Rouquier

in abstract terms,

as the right structure in which to describe

the action of braiding on derived categories.



One learns from this that cups and caps



come from branes on

$$\mathcal{X}$$

with simple geometric meaning.

These branes are structure sheaves of vanishing cycles in $\check{\mathcal{X}}$

$$\mathcal{U} = \mathcal{O}_U$$

known as "minuscule Grassmanians"

$$U = G/P_1 \times \ldots \times G/P_d$$

The cycles shrink to a point as punctures come together in pairs.



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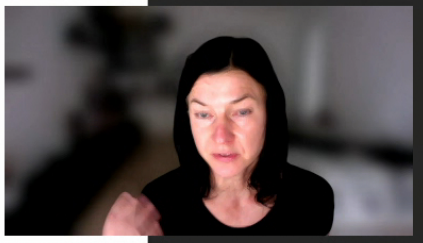


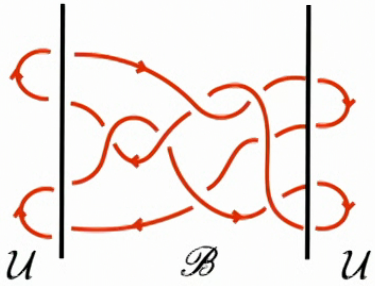
Using very special properties of perverse filtrations

and these vanishing cycle branes,

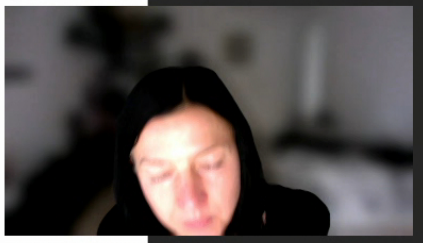
it is easy to prove that, not only do the homology groups

$$Hom^{*,*}_{\mathscr{D}_X}(\mathscr{B}\mathcal{U},\mathcal{U})$$

manifestly categorify the corresponding $U_q({}^L\mathfrak{g})$ link invariants,



they are themselves link invariants.



The approach by

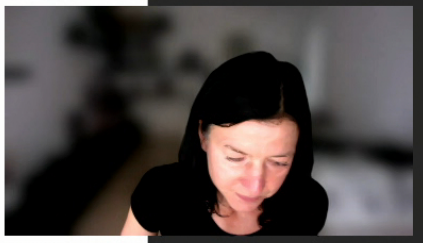
$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

explains the origin of homological link invariants

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathscr{B}\mathcal{U}, \mathcal{U})$$

in geometry and physics,

in a manner that works uniformly for all groups.



Recently, Ben Webster proved that homological link invariants

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathscr{B}\mathcal{U},\mathcal{U})$$

that come from B-type branes on

$$\mathcal{X}$$

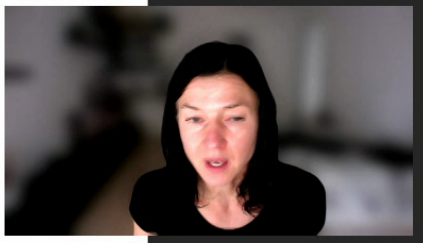
are equivalent to algebraic invariants he defined in '13,

using a (cylindrical version) of the KLRW algebra

$$\mathscr{A}$$

and moreover generalize them to links in

$$\mathbb{R}^2 \times S^1$$

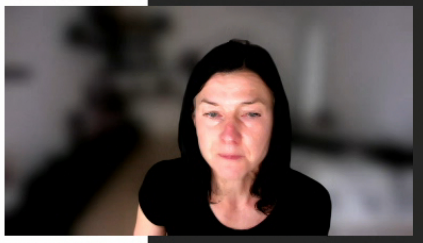


As stated, the approach by

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

is not much more explicit than that by the KLRW algebra.

Neither are really amenable to calculations.

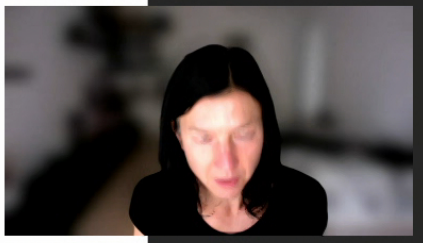


To solve the theory

we will make use of homological mirror symmetry,

or more precisely,

an equivariant version of it.



Kontsevich conjectured in 1994, that the way to understand mirror symmetry

is as an equivalence

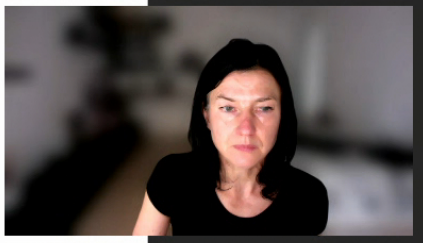
of categories of boundary conditions of two dimensional

sigma models on a pair of Calabi-Yau manifolds

$\mathcal{X}$ and $\mathcal{Y}$

one of which comes from complex,

and the other from symplectic geometry.



The category of branes coming from complex geometry is

(derived) category of coherent sheaves

whose objects are

B-branes

supported on complex submanifolds of

X



The category of branes coming from symplectic geometry is

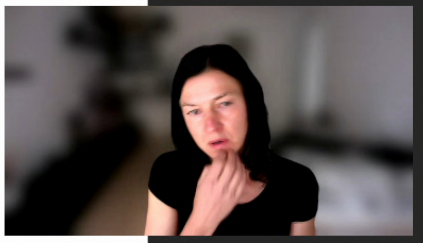
derived Fukaya category

whose objects are

A-branes

supported on real, or Lagrangian submanifolds of

$\mathcal{Y}$



Kontsevich's homological mirror symmetry

is a conjecture that

category of B-branes

on

$\mathcal{X}$

(the derived category of coherent sheaves)

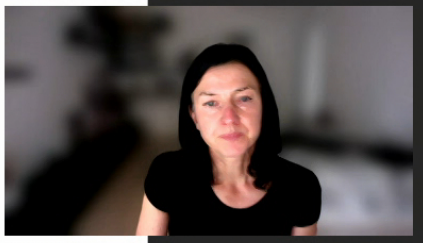
category of A-branes

on

$\mathcal{Y}$

(derived Fukaya category)

are equivalent.



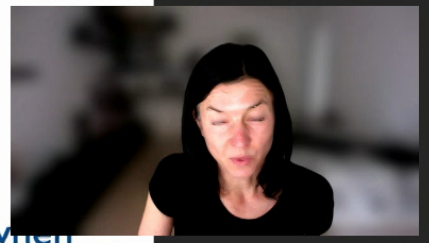
In the very best instances,

one learns how to make homological mirror symmetry manifest,

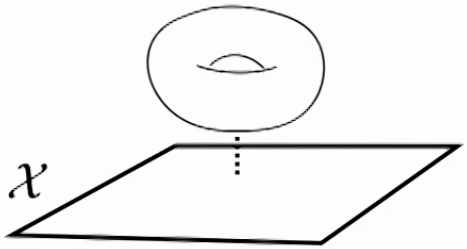
and both theories based on

$\mathcal{X}$ and on $\mathcal{Y}$

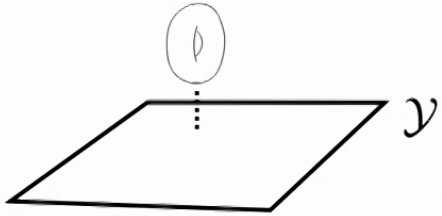
solvable exactly.



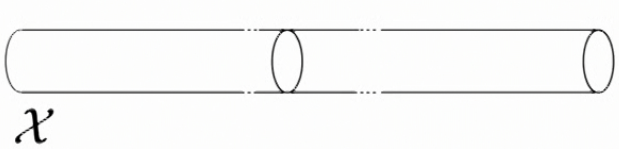
One of the very simplest examples of homological mirror symmetry is when

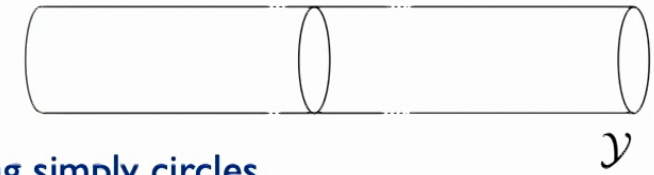


and



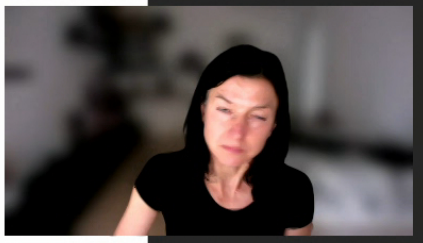
are taken to be simply a pair of infinite cylinders,





their torus fibers being simply circles,

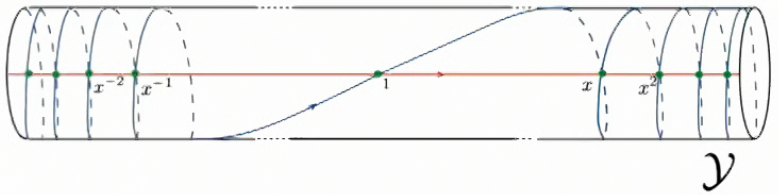
the common base a real line.



Categories of branes on the two sides turn out to be

each generated by a single brane:

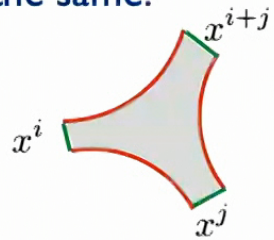




While the branes look different,

their algebras of open strings are the same.

$$\mathscr{A} = \mathbb{C}[x, x^{-1}]$$



The algebra is simply the algebra of functions on a complex cylinder

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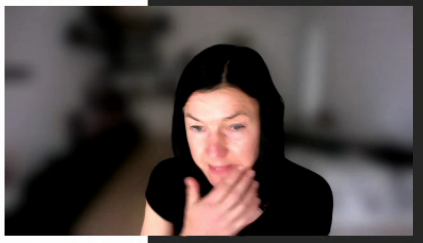
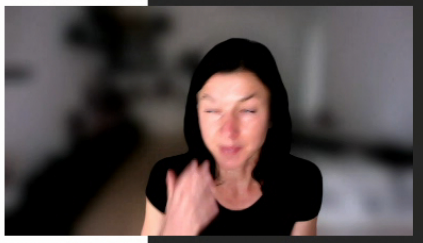


This simple example is the model for

how one hopes to understand homological mirror symmetry in all cases.



Ordinary, non-equivariant mirror of

$\mathcal{X}$

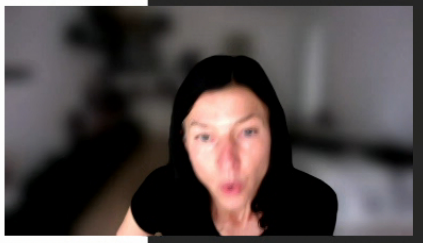
is a hyper-Kahler manifold

$\mathcal{Y}$

which is, to a first approximation,

given by a hyper-Kahler rotation of $\mathcal{X}$

($\mathcal{Y}$ is also a moduli space of G monopoles, just on $\mathbb{R}^2 \times S^1$)



The ordinary homological mirror symmetry,

$$\mathcal{X} \quad \overset{\text{mirror}}{\longleftrightarrow} \quad \mathcal{Y}$$

should be a consequence of a pair of equivalences

$$\mathcal{D}_{\mathcal{X}} \cong \mathcal{D}_{\mathcal{A}} \cong \mathcal{D}_{\mathcal{Y}}$$

where $\mathcal{D}_{\mathcal{Y}}$ is an appropriate version of

the derived Fukaya category of $\mathcal{Y}$



Webster's proof of equivalence of categorification

of $U_{\mathfrak{q}}({}^{L}\mathfrak{g})$ link invariants

via B-type branes on

$\mathscr{X}$

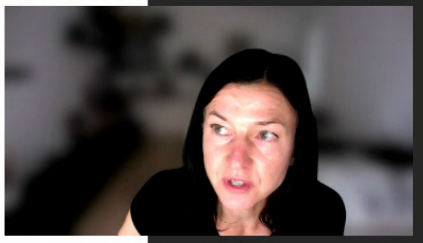
and via KLRW algebra

$\mathscr{A}$

is really the first of the two equivalences in homological mirror symmetry:

$$\mathscr{D}_{\mathscr{X}} \cong \mathscr{D}_{\mathscr{A}} \cong \mathscr{D}_{\mathscr{Y}}$$

where is the derived category of modules of the algebra $\mathscr{A}$

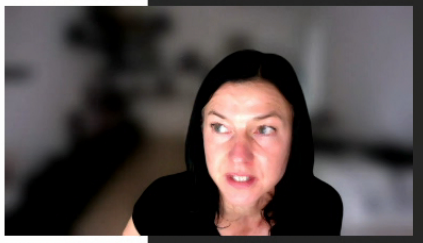


We will not describe

$\mathscr{D}_y$

directly, but rather we will make use of a further simplification,

available in the present setting.



Recall that

$\mathcal{X}$

is a moduli space of monopoles on

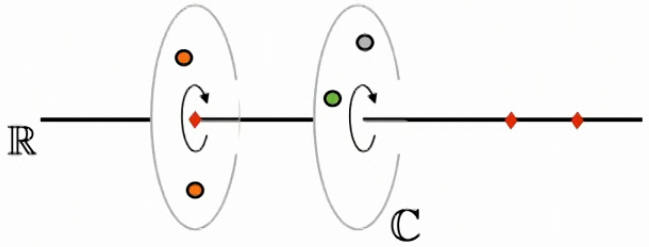
$\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$

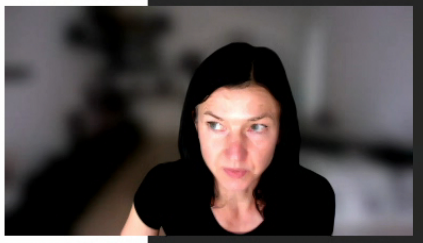
where a symmetry that

corresponds to rotations of

$\mathbb{C}$

plays a key role — this is how we got $\mathfrak{q}$ into the problem.





The simplification comes from the fact that,

working equivariantly with respect to this symmetry

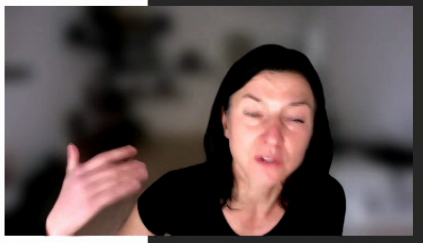
all the relevant information about the geometry

$\mathcal{X}$

must be contained in a "core" locus

X

preserved by it.

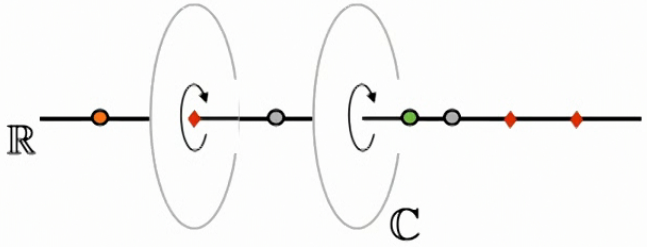


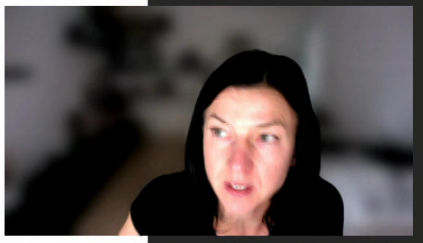
The core is a subspace

$$X \subset \mathcal{X}$$

of half the dimension, where all monopoles,

singular or not, are at the origin of $\mathbb{C}$ and at points in $\mathbb{R}$





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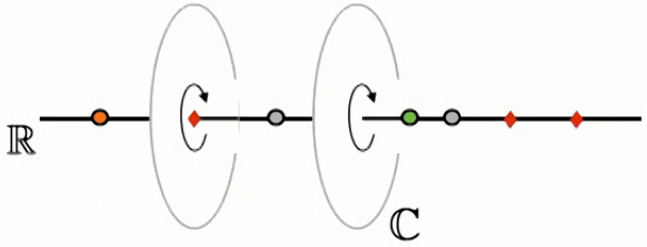
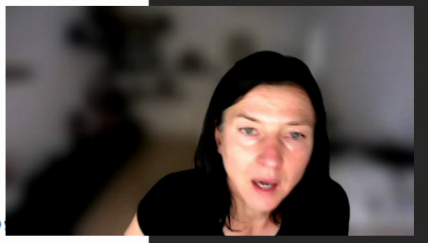
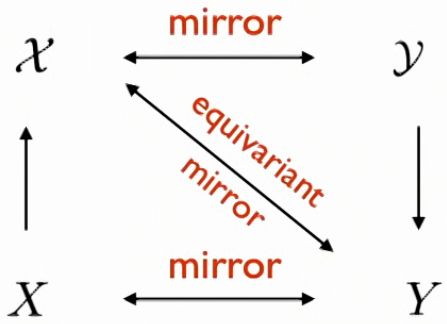
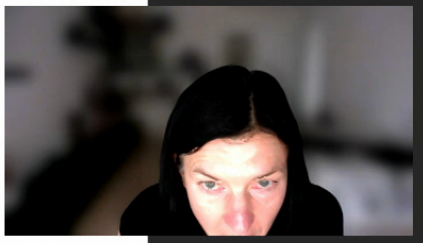




We will define the equivariant mirror of the "big" $\mathcal{X}$, call it "small" Y

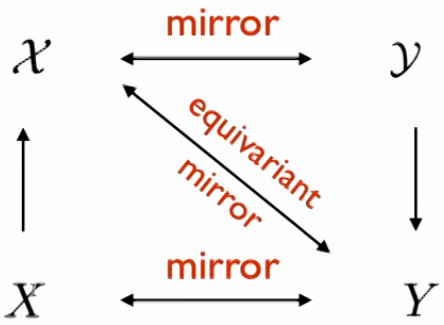
to be the ordinary mirror of the "small" X :



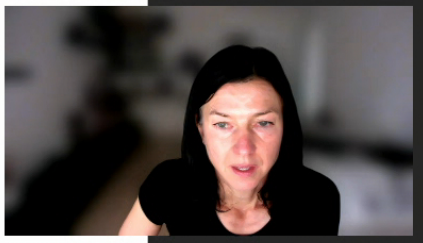


While X embeds into $\mathcal{X}$ as a

holomorphic Lagrangian submanifold of dimension $D = \dim_{\mathbb{C}} \mathcal{X}/2$



$\mathcal{Y}$ fibers over Y with holomorphic Lagrangian $(\mathbb{C}^\times)^D$ fibers

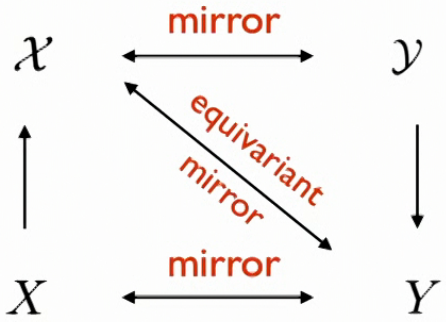


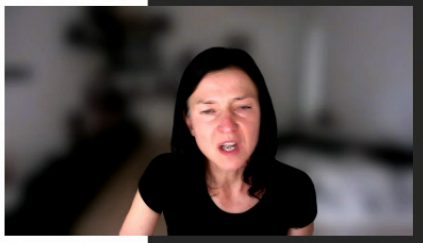
We will define the equivariant mirror of $\mathcal{X}$ which we will call

Y

to be the ordinary mirror of its core:

X



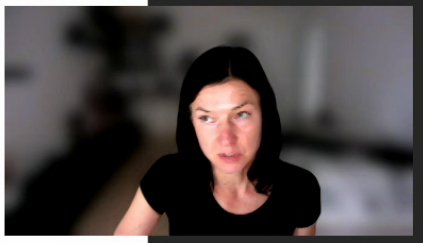


A model example to keep in mind is

$\mathcal{X}$ which is the resolution of an A_{m-1} surface singularity.

$\mathcal{X}$ is the moduli space of a single smooth $G = SU(2)/\mathbb{Z}_2$ monopole,

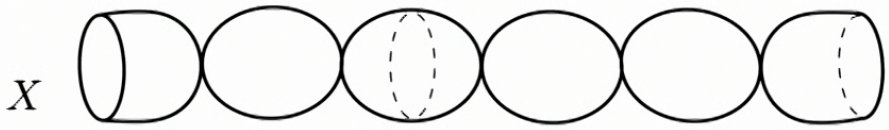
in presence of m singular ones.



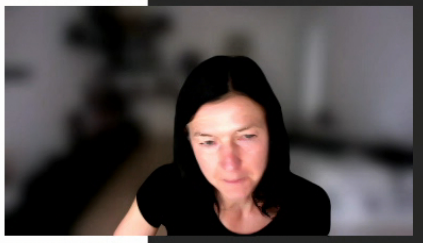
For

$\mathcal{X}$ which is the resolution of an A_{m-1} surface singularity,

its core X looks like:



It is a collection of $m-1$ $\mathbb{P}^1$'s with a pair of infinite discs attached.



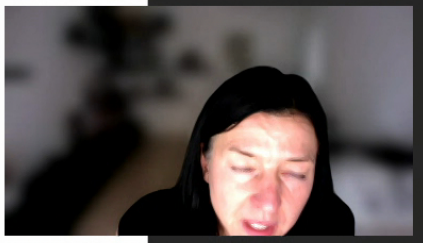
The ordinary mirror of $\mathcal{X}$ which is a resolution of

an A_{m-1} surface singularity,

is $\mathcal{Y}$ which is a complex structure deformation of an multiplicative

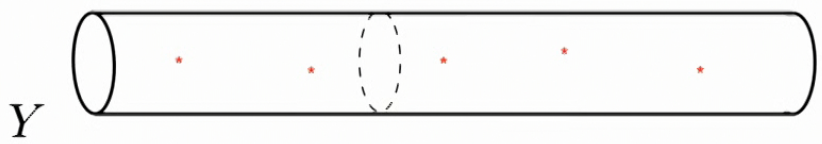
A_{m-1} surface singularity,

with a potential which we will not need.



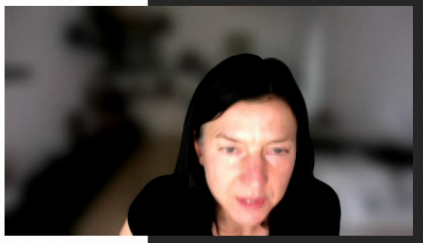
The multiplicative A_{m-1} surface $\mathcal{Y}$,

is a $\mathbb{C}^\times$-fibration over Y



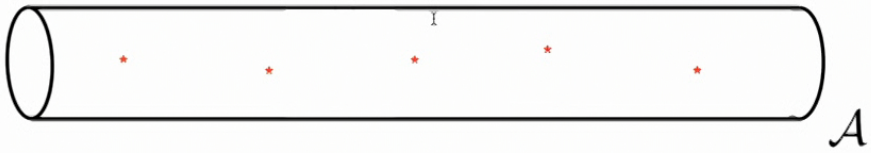
which is an infinite cylinder with m marked points in the interior.

At the marked points, the $\mathbb{C}^\times$ fibers of $\mathcal{Y}$ degenerate .



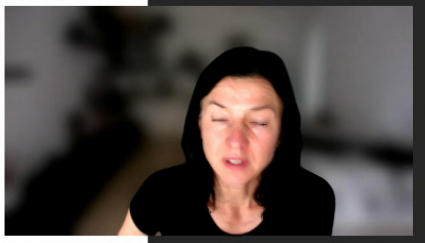
Y is a single copy of the Riemann surface

where the conformal blocks live:



The positions of punctures correspond to the marked points

where the $\mathbb{C}^\times$ fibration $\mathcal{Y} \to Y$ degenerates.



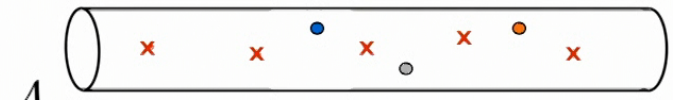
More generally, the equivariant mirror

Y

is the configuration space of points,

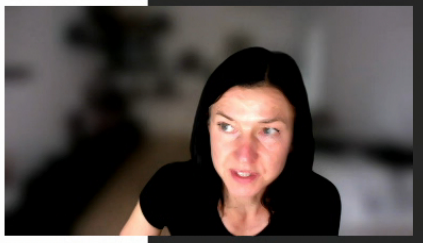
"colored" by simple roots of ${}^L\mathfrak{g}$ but otherwise indistinguishable,

one for each smooth monopole,



on the Riemann surface $\mathcal{A}$ where conformal blocks live,

(with some locus deleted and singularities resolved).



Including an equivariant $\mathrm{T}-$ action

on $\mathcal{X}$ and on X

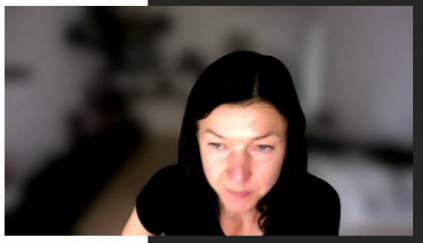
corresponds to adding to the sigma model

a specific potential,

$$W = \lambda_0 W^0 + \sum_{a=1}^{\mathrm{rk}} \lambda_a W^a$$

which is a multi-valued holomorphic function.

$$W^0 = \ln f^0 \quad \text{and} \quad W^a = \ln \prod_{\alpha} y_{a,\alpha}$$



From the mirror perspective, the conformal block of

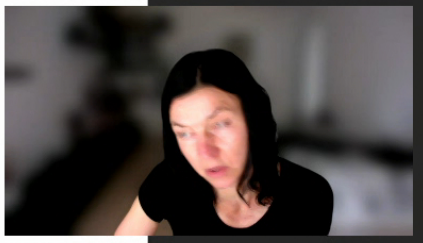
$$\widehat{L_{\mathfrak{g}}}$$

is the partition function of the B-twisted theory on D ,



with A-type boundary condition at infinity, corresponding to a

Lagrangian L in Y.



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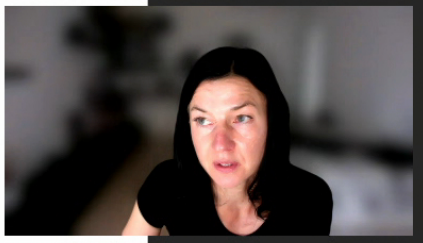


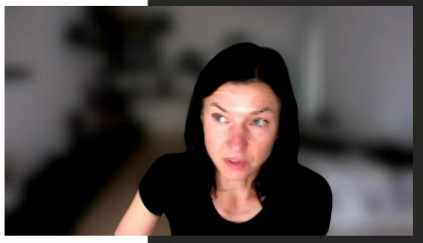
Such amplitudes have the following form

$$\mathcal{V}_\alpha[L] = \int_L \Phi_\alpha \, \Omega \, e^{-W}$$

where Ω is the top holomorphic form on Y ,

W is the Landau-Ginsburg potential,

and Φ 's are the chiral ring operators.



The integral

$$\mathcal{V}_\alpha[L] = \int_L \Phi_\alpha \, \Omega \, e^{-W}$$

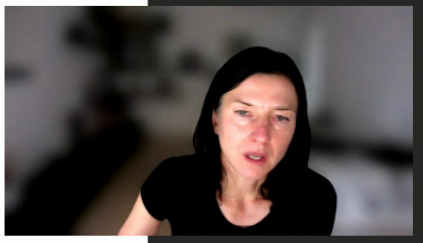
is known as the "free field representation" of conformal blocks of

$$\widehat{L\mathfrak{g}}$$

which goes back to work of Feigin and E.Frenkel in the '80's

and Schechtman and Varchenko.

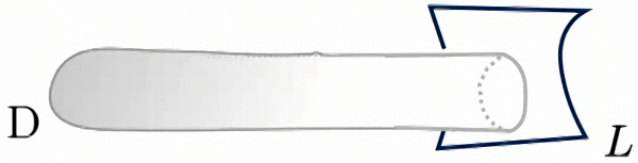
Mirror symmetry "explains" why such representation exists.



Corresponding to a solution of the

Knizhnik-Zamolodchikov equation

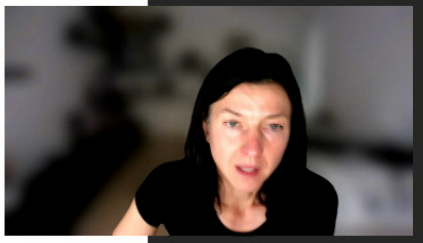
is an A-brane at the boundary of D at infinity,



The brane is an object of the category of A-branes

$$\mathscr{D}_Y$$

the "derived Fukaya-Seidel category" of Y with potential W .

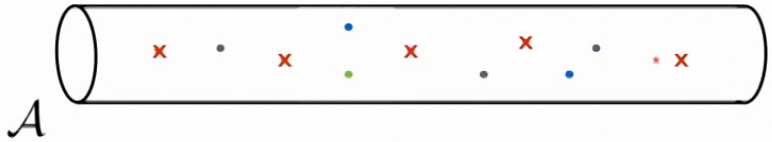


One can describe this category very explicitly

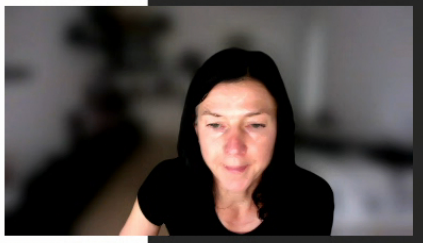
thanks to the fact

Y

a (close cousin of) configuration space of colored points



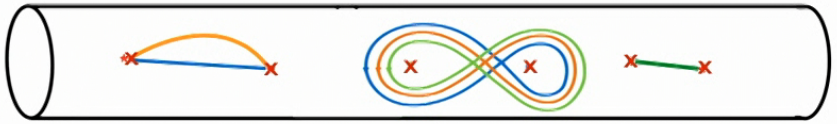
on the punctured Riemann surface.



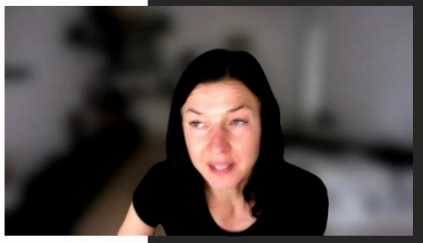
Objects of the category of A-branes on

Y

are all products of one dimensional curves



on the Riemann surface, colored by simple roots.

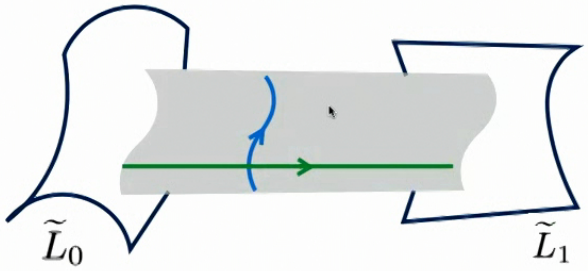


In any category of A-branes

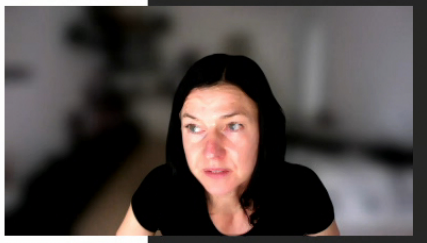
the spaces of morphisms between a pair of branes

$$Hom^{*,*}_{\mathscr{D}_Y}(\widetilde{L}_0, \widetilde{L}_1) = \mathrm{Ker}\, Q/\mathrm{Im}\, Q.$$

are defined by Floer theory,



which is modeled after Morse theory approach to supersymmetric quantum mechanics.



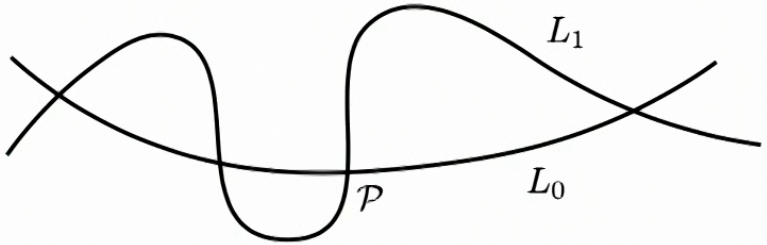
The role of the Morse complex from the beginning of the talk

is taken by the Floer complex, which is a vector space

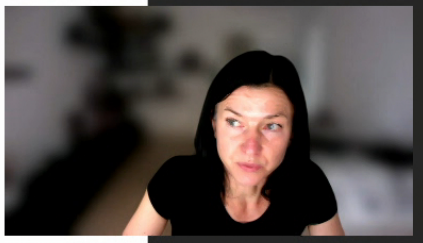
$$CF^{*,*}(L_0, L_1) = \bigoplus_{\mathcal{P} \in L_0 \cap L_1} \mathbb{C}\mathcal{P}.$$

spanned by the intersection points of the two Lagrangians,

and graded by the cohomological



and $\mathfrak{q}$ -degrees, the latter is thanks to the fact W is multi-valued.



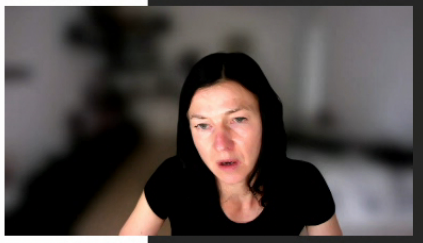
The action of the differential Q on this space

$$CF^{*,*}(L_0, L_1) = \bigoplus_{\mathcal{P} \in L_0 \cap L_1} \mathbb{C}\mathcal{P}.$$

of perturbative supersymmetric ground states

$$Q : CF^{*,*} \to CF^{*,*}$$

is, as always, generated by instantons.



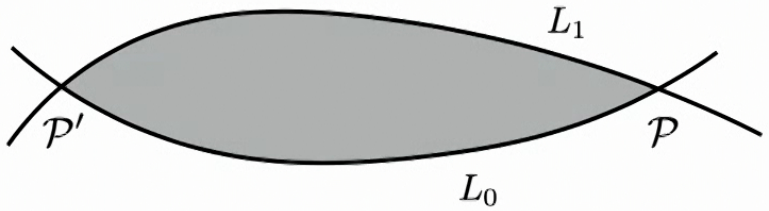
In Floer theory,

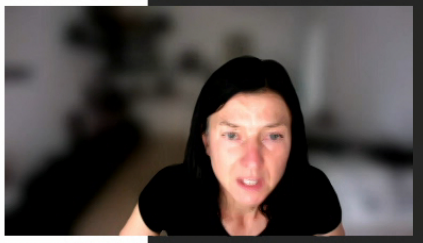
the coefficient of $\mathcal{P}'$ in $Q\mathcal{P}$

is obtained by counting holomorphic disk instantons in Y

interpolating from $\mathcal{P}$ to $\mathcal{P}'$, of Femion number one

and $\mathfrak{q}$-degree zero.

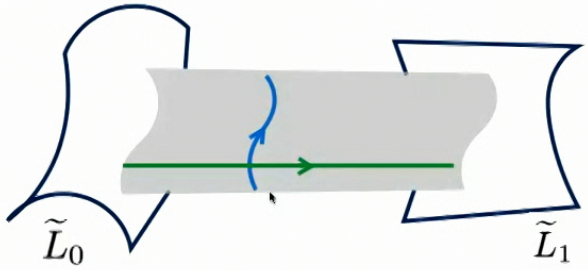


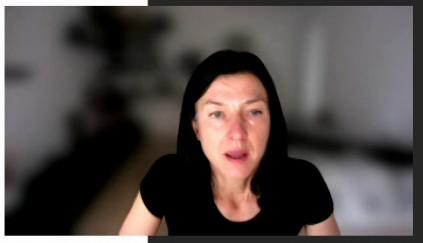


The cohomology of the resulting complex

$$Hom^{*,*}_{\mathscr{D}_Y}(\widetilde{L}_0, \widetilde{L}_1) = \mathrm{Ker}\, Q/\mathrm{Im}\, Q.$$

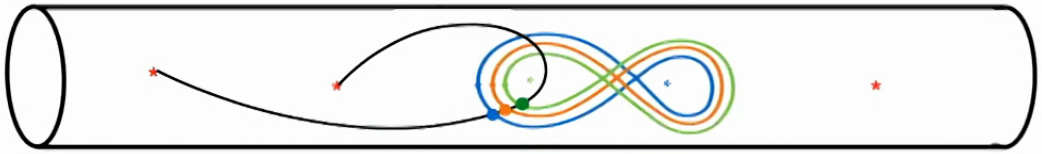
is the space of morphisms between the branes in $\mathscr{D}_Y$.





A vast simplification in the present case is that

just as the branes have a description in terms of the Riemann surface



so do their intersection points,

as well as the maps between them.

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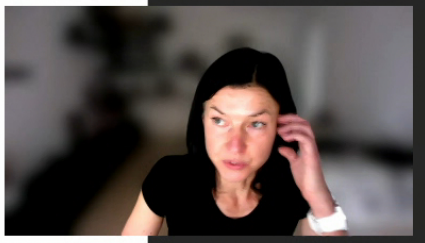


The theory which results is a generalization,

to arbitrary simply laced Lie algebra

$${}^{L}\mathfrak{g}$$

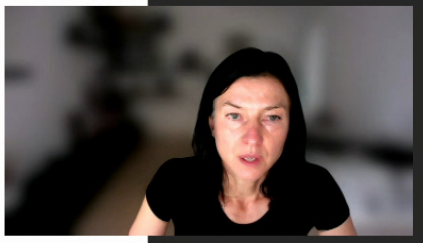
of "Heegard-Floer" theory.

The latter is associated to

$${}^{L}\mathfrak{g} = \mathfrak{gl}_{1|1}$$

and categorifies the Alexander polynomial.

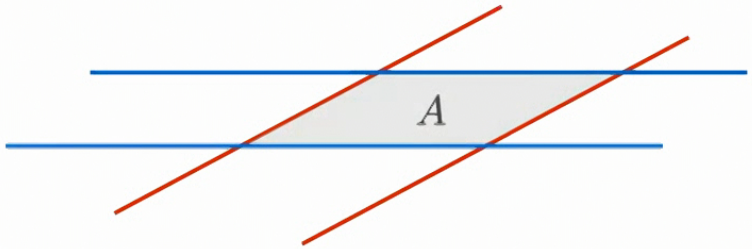
Heegard-Floer theory is phrased in the same, one dimensional, terms.



In the present case, every holomorphic map from a disk to Y,

$$y : \mathrm{D} \to Y$$

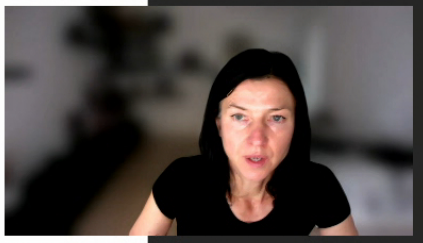
projects with non-negative multiplicities, to domains



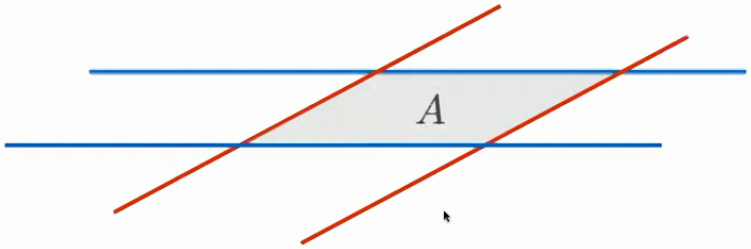
on the Riemann surface $\mathcal{A}$

with boundaries on the one dimensional Lagrangians

and vertices at their intersection points.



As in Heegard-Floer theory, one can read off

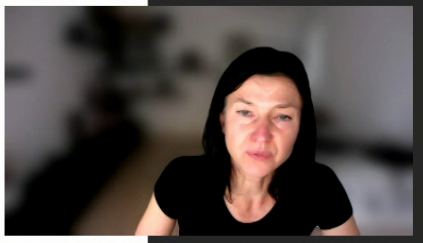


from the domains on the Riemann surface the cohomological,

and the equivariant degrees of the map to

$$y : \mathrm{D} \to Y$$

as well as degrees of all the intersection points.

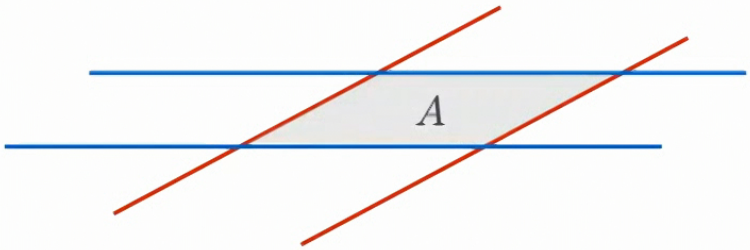


For example, for ${}^L\mathfrak{g} = \mathfrak{su}_2$ Maslov index of a disk is

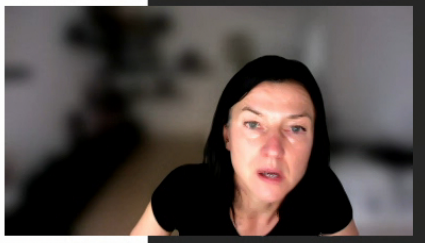
$$\mathrm{ind}[y] = 2e(A)$$

where

$$e(A) = \chi(A) - \#\mathrm{acute}/4 + \#\mathrm{obtuse}/4,$$



The above disk has cohomological degree zero.



Mirror symmetry

helps us understand exactly which questions we need to ask



to recover homological knot invariants from Y,

for an arbitrary simply laced Lie algebra

$L_{\mathfrak{g}}$



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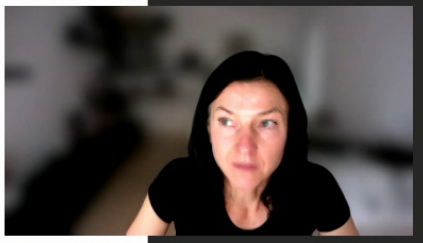


Every B-brane on $\mathcal{X}$ which is relevant to us

"comes from"

a B-brane on the core X

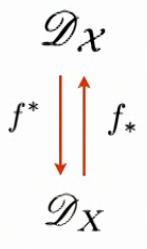
via a functor,

$$f_* : \mathscr{D}_X \to \mathscr{D}_{\mathcal{X}}$$

that interprets a brane F on X, an object of $\mathscr{D}_X$

as a brane $\mathcal{F} = f_* F$ on $\mathcal{X}$, an object of $\mathscr{D}_{\mathcal{X}}$

The functor f_* has an adjoint f^* that goes the other wa



Adjointness means that given any pair of branes on $\mathcal{X}$ that come from X

the Homs between them, computed upstairs, in $\mathscr{D}_{\mathcal{X}}$

$$\mathcal{F} = f_* F, \quad \mathcal{G} = f_* G$$

agrees with the Hom downstairs, in $\mathscr{D}_X$,

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}) = Hom^{*,*}_{\mathscr{D}_X}(f^* f_* F, G)$$

after replacing F with $f^* f_* F$

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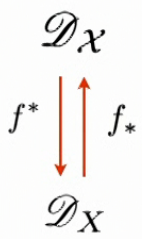

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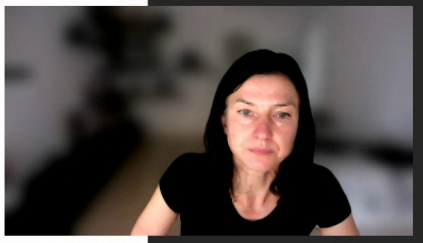


Equivariant homological mirror symmetry

relating

$\mathcal{X}$ and Y

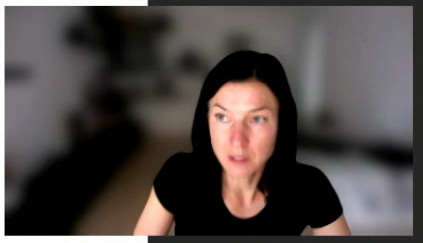
is not an equivalence of categories,

but a correspondence of branes and associated vector spaces,

which come from a pair of "adjoint functors",

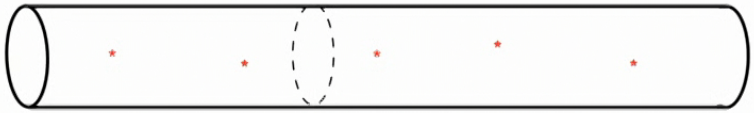
$$\mathscr{D}_{\mathcal{X}} \underset{h^*}{\overset{h_*}{\rightleftarrows}} \mathscr{D}_Y$$

which come from composing f_*, f^* with the downstairs mirror symmetry.



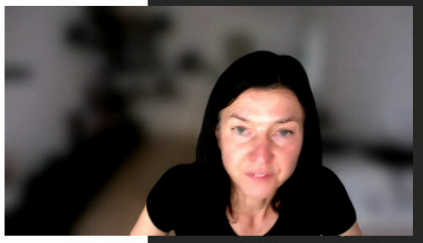
Recall our example, Y the equivariant mirror to

$\mathcal{X}$ which is the A_n surface.



Mirror to i-th vanishing $\mathbb{P}^1$ in X is the Lagrangian





The functor going up

$$\mathscr{D}_{\mathcal{Y}}$$

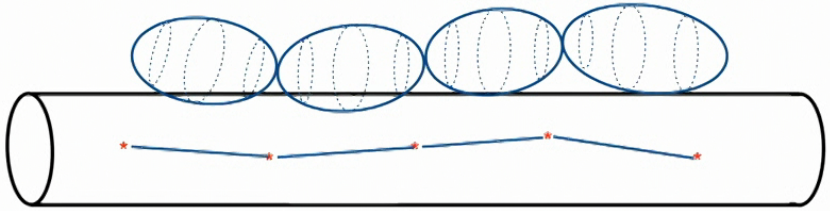
$$\uparrow k_*$$

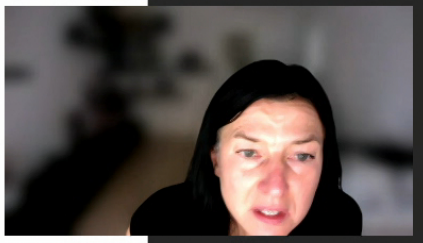
$$\mathscr{D}_{Y}$$

amounts to pairing the brane downstairs with the

$$(S^1)^D \subset (\mathbb{C}^\times)^D$$

fiber over it, which is how one gets this picture:





The functor going the other way

$$k^* : \mathscr{D}_{\mathcal{Y}} \to \mathscr{D}_Y$$

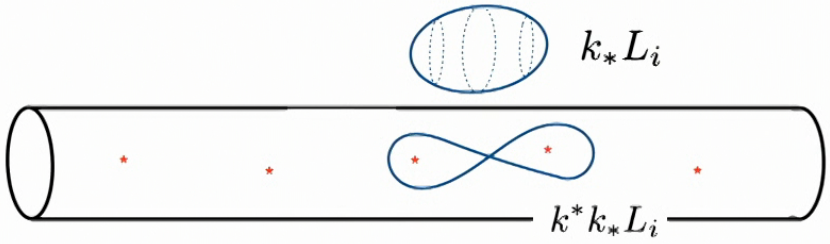
does not send the vanishing sphere $k_* L_i$ back to L_i :

$$k^* k_* L_i \neq L_i$$

Instead, either computing it either from mirror symmetry,

or its via its definition (coming from a Lagrangian correspondence),

one finds a figure eight Lagrangian



For any simply laced Lie algebra

$^{L}\mathfrak{g}$

branes which serve as "cups" and "caps" "upstairs" on

$\mathcal{X}$

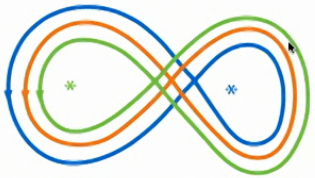
(associated to "minuscule Grassmanians")

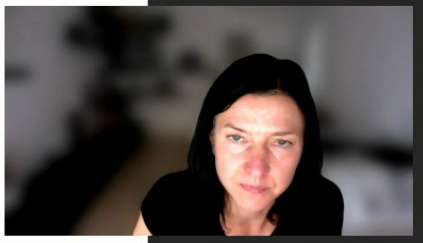
originate from h_* the vanishing cycle branes of the downstairs theory

which are generalized intervals



and project back down h^* as generalized figure eight branes.





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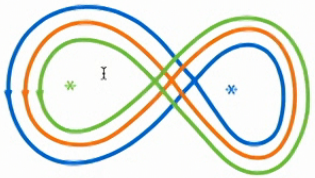


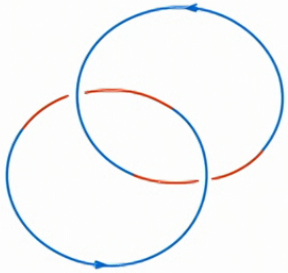


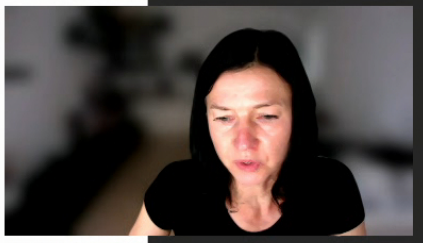
In the description based on

Y

both the Lagrangians and the action of braiding on them are geometric.

Start with a projection of a link to a the surface $\mathcal{A}$:





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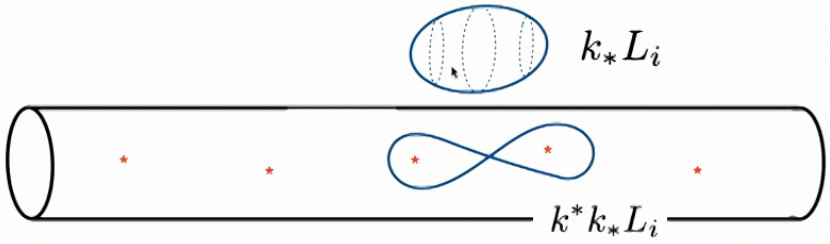
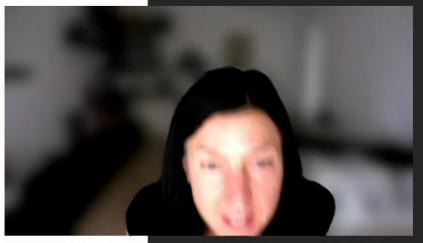
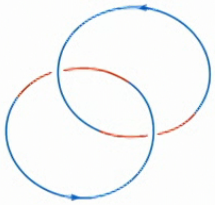




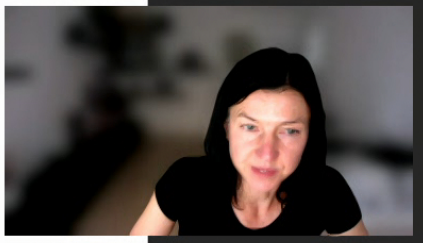
To translate it to a pair of A-branes by choosing a bicoloring,



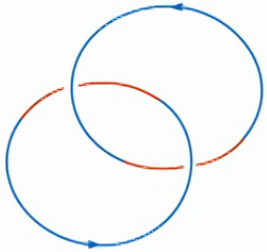
of every link component

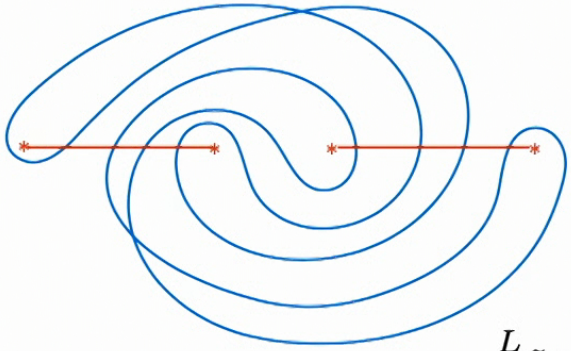
by an equal number of segments of each color,

such that red always underpasses the blue.

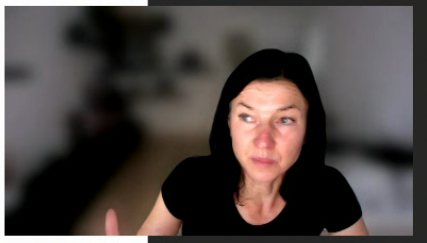


The mirror Lagrangians $I_{\mathcal{U}}$ and $\mathscr{B}E_{\mathcal{U}}$ are obtained by replacing all the red segments by the interval-type branes and the blue segments by figure eight-type branes:





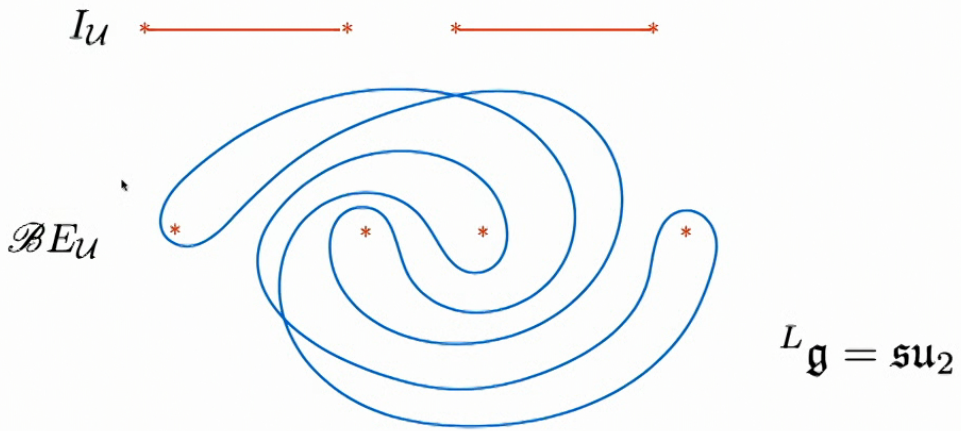
$L_{\mathfrak{g}} = \mathfrak{su}_2$



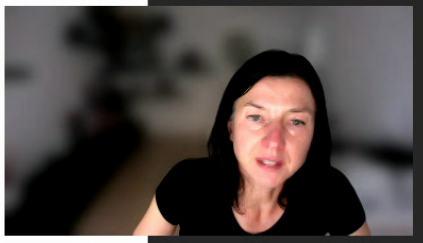
The homological link invariant is the vector space of morphisms

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$$

between the pair of branes.



graded by cohomological and $\mathfrak{q}$ -gradings .



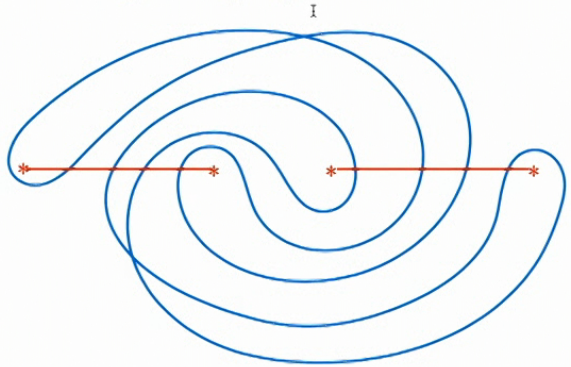
The fact that, for

$$^{L}\mathfrak{g} = \mathfrak{su}_2$$

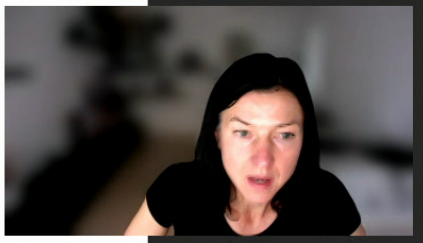
the graded count of the intersection points in

$$\chi(E, \mathscr{B}I) = \sum_{\mathcal{P} \in E \cap \mathscr{B}I} (-1)^{M(\mathcal{P})} \mathfrak{q}^{J(\mathcal{P})}$$

computes the Jones polynomial is a theorem by



Bigelow from the '90s.



The dimensions of the complex

$$CF^{*,*}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}) = \bigoplus_{\mathcal{P}\in\mathscr{B}E_{\mathcal{U}}\cap I_{\mathcal{U}}} \mathbb{C}\,\mathcal{P}$$

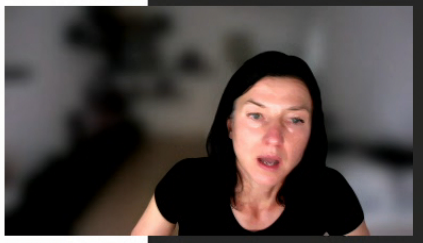
the vector space with the action of the differential

whose cohomology is the Link homology

grows polynomially with the number of crossings in our case,

which should be compared to exponential growth

in Khovanov's case.

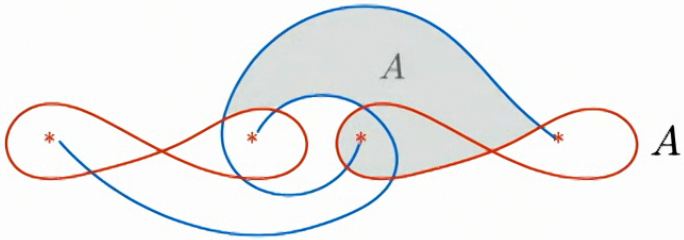


As in Heegard-Floer theory,

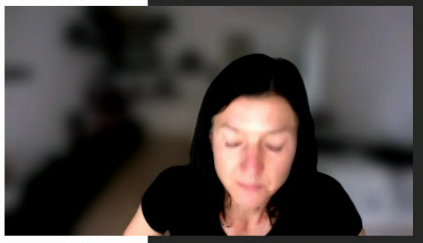
computing the action of the differential

can be translated to a sequence well defined, but hard

problems in complex analysis in one dimension.



Surprisingly, this problem can be solved.



One solves all the disk counting problems

at once,

by making the homological mirror symmetry that relates

"downstairs" mirror pair

$$\mathscr{D}_X \cong \mathscr{D}_A \overset{I}{\cong} \mathscr{D}_Y$$

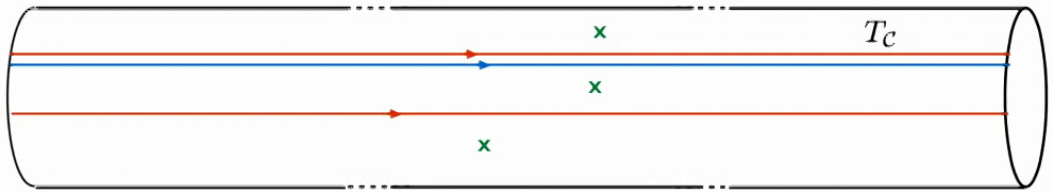
manifest.

The fact mirror symmetry can sum up curve counts is its basic property.

From perspective of

Y

the generating set of branes

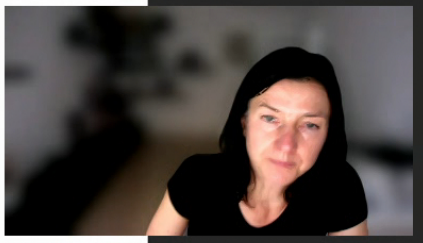


are products of real line Lagrangians,

$$T_C = T_{i_1} \times T_{i_2} \times \ldots \times T_{i_D}$$

colored by simple roots.

This is a simple generalization of very simplest example of mirror symmetry.

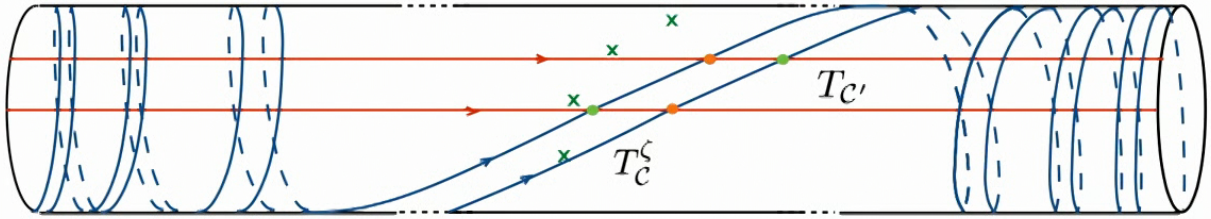


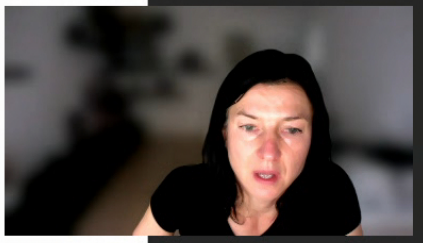
The associated "downstairs" algebra of open strings

$$A$$

is a smaller cousin of the "upstairs" KLRW algebra.

$$\mathscr{A}$$





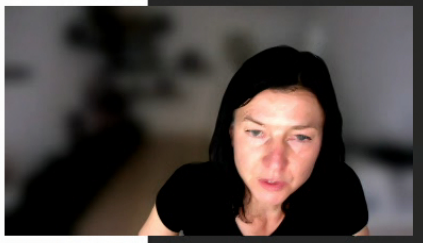
The algebraic description of link homology

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$$

based on the "small" algebra

$$A$$

has a simple geometric meaning.



By the virtue of the equivalence

$$\mathscr{D}_A \cong \mathscr{D}_Y$$

as any brane in $\mathscr{D}_Y$, the braided cap branes

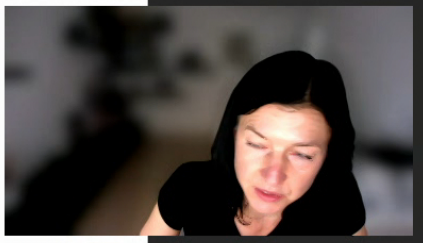
$$\mathscr{B}E_{\mathcal{U}} \in \mathscr{D}_Y$$

have a description as a complex ,

$$\mathscr{B}E_{\mathcal{U}} \cong \ldots \xrightarrow{e_1} \mathscr{B}E_1(T) \xrightarrow{e_0} \mathscr{B}E_0(T)$$

every term of which is a direct sum of thimble $T_{\mathcal{C}}$ branes.

The maps in the complex are the gluing prescriptions.



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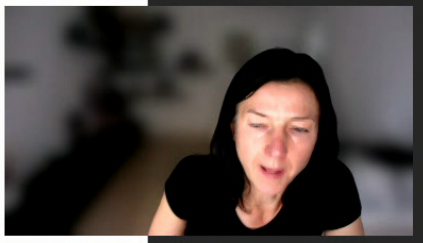
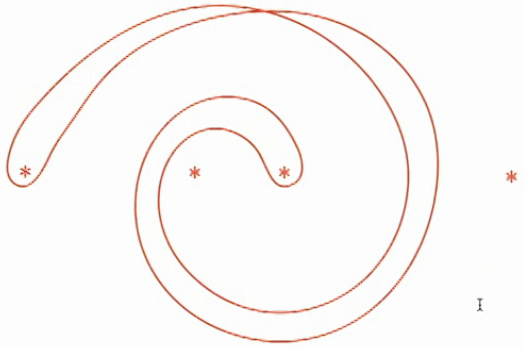


The complex

$$\mathscr{B}E_{\mathcal{U}} \cong \ldots \xrightarrow{e_1} \mathscr{B}E_1(T) \xrightarrow{e_0} \mathscr{B}E_0(T)$$

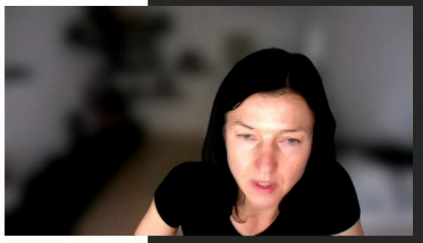
describing braided figure eight branes



delivers the homology of links obtained by capping the braid

without any further work,

encoding all necessary disc counts.



For any simple Lie algebra ${}^{L}\mathfrak{g}$

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$$

the link homology can be read off directly from the complex

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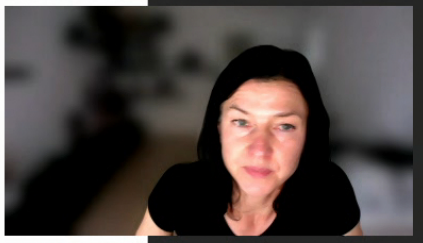
describing the brane.



This is thanks to the fact that the vanishing cycle branes

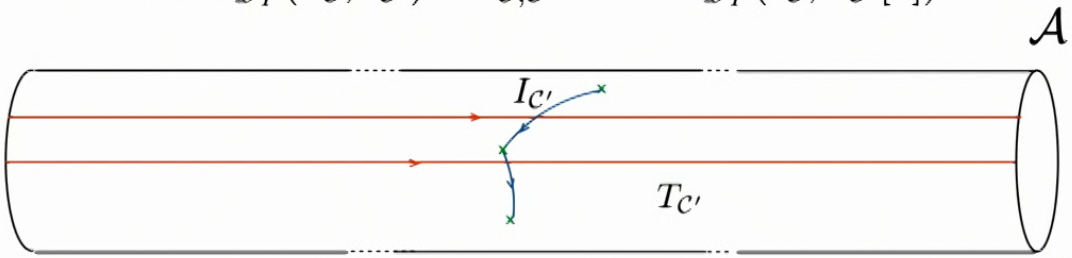
$$I_{\mathcal{C}'}$$

and the real line Lagrangians

$$T_{\mathcal{C}}$$

form a Koszul dual set

$$Hom_{\mathscr{D}_Y}(T_{\mathcal{C}}, I_{\mathcal{C}'}) = \delta_{\mathcal{C},\mathcal{C}'} = H\grave{o}m_{\mathscr{D}_Y}(I_{\mathcal{C}}, T_{\mathcal{C}'}[d])$$



$$^L\mathfrak{g} = \mathfrak{su}_2$$



The vanishing cycle branes

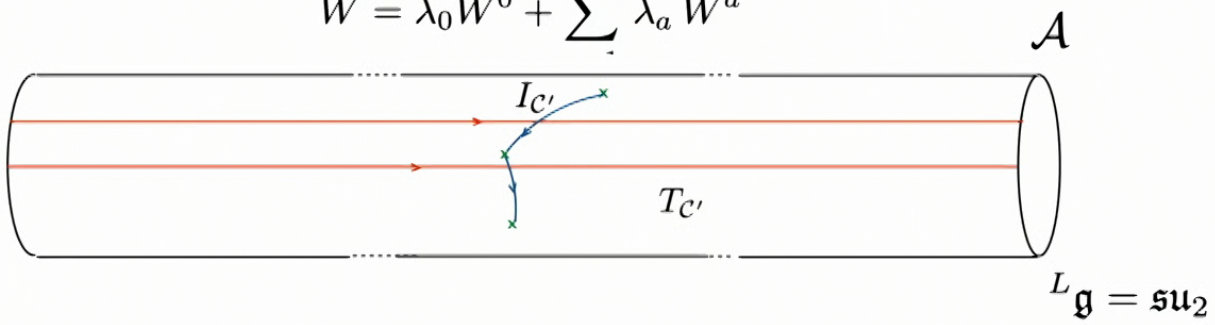
$I_{\mathcal{C}'}$

and the real line Lagrangians

$T_{\mathcal{C}}$

are the right and left thimbles of the potential,

$$W = \lambda_0 W^0 + \sum^{\mathrm{rk}} \lambda_a W^a$$



and correspond to simple and projective modules of the algebra A .

From the complex describing the brane

$$\mathscr{B}E_{\mathcal{U}} \cong \ldots \xrightarrow{e_1} \mathscr{B}E_1(T) \xrightarrow{e_0} \mathscr{B}E_0(T)$$

we get for free a complex of vector spaces

$$0 \to hom_A(\mathscr{B}E_0, I_{\mathcal{U}}\{J\}) \xrightarrow{e_0} hom_A(\mathscr{B}E_1, I_{\mathcal{U}}\{J\}) \xrightarrow{e_1} \ldots.$$

with the action of the differential

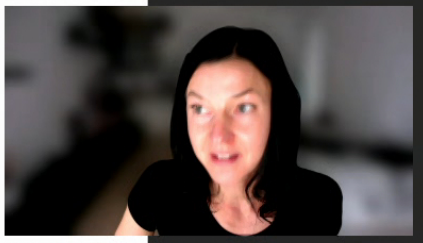
$$Q = \sum_k e_k$$

that squares to zero.

The link homology,

$$Hom^*_{\mathscr{D}_Y}(E_{\mathcal{U}}, \mathscr{B}I_{\mathcal{U}}\{\vec{J}\}) = H^*\big(hom_A(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}}\{J\})\big)$$

is the cohomology of this complex.



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For any simple Lie algebra ${}^L\mathfrak{g}$,

$$Hom^{*,*}_{\mathscr{D}_Y}(\mathscr{B}E_{\mathcal{U}}, I_{\mathcal{U}})$$

the link homology is a small, known "piece" of the complex

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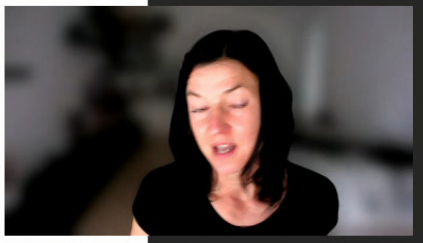

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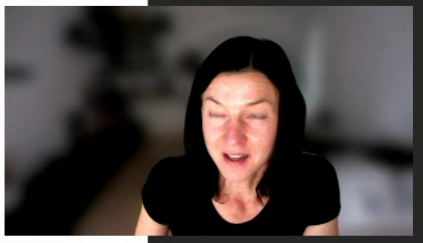
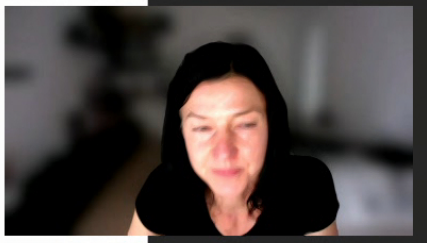


We need one more step to make the theory computable.



In general, finding the projective resolution of a brane

$$\mathscr{B}E_{\mathcal{U}} \cong \ldots \xrightarrow{e_1} \mathscr{B}E_1(T) \xrightarrow{e_0} \mathscr{B}E_0(T)$$

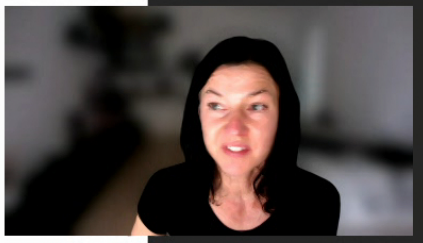
is difficult, even though solvable in principle.

The first, one has to compute which module of the algebra

the brane maps to by the Yoneda functor,

$$\mathscr{B}E \in \mathscr{D}_Y \to Hom^{*,*}_{\mathscr{D}_Y}(T, \mathscr{B}E) \in \mathscr{D}_A$$

and then find the (infinite) "bar resolution" of this module.



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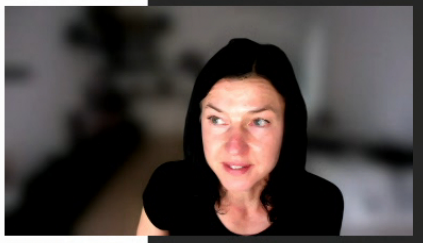


In our case, both of these problems are solvable

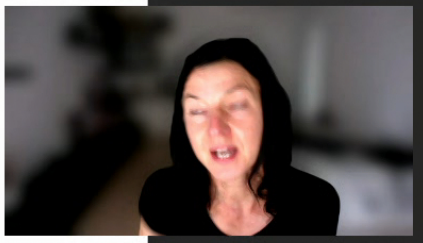
simultaneously, thanks to the fact

$$Y$$

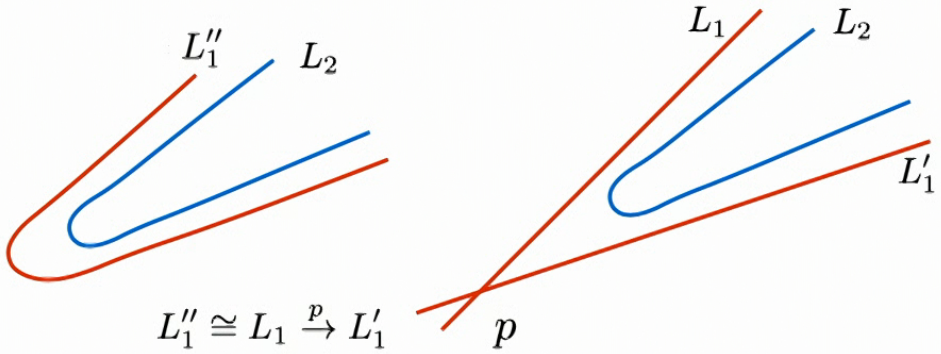
is a cousin of configuration space of colored points



on the Riemann surface.



In one dimension, all cone maps are geometric



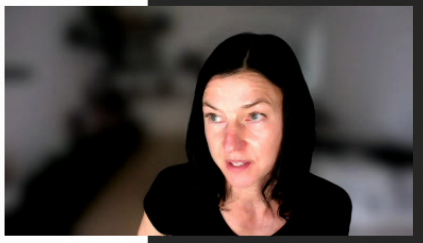
and all our Lagrangians are products of one dimensional ones.

It follows that almost all maps in the complex are geometric,

The few remaining ones are found by asking that

$$Q = \sum_k e_k$$

squares to zero.



In general, finding the projective resolution of a brane

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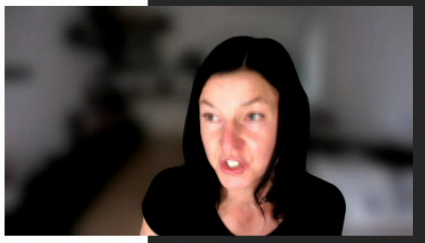
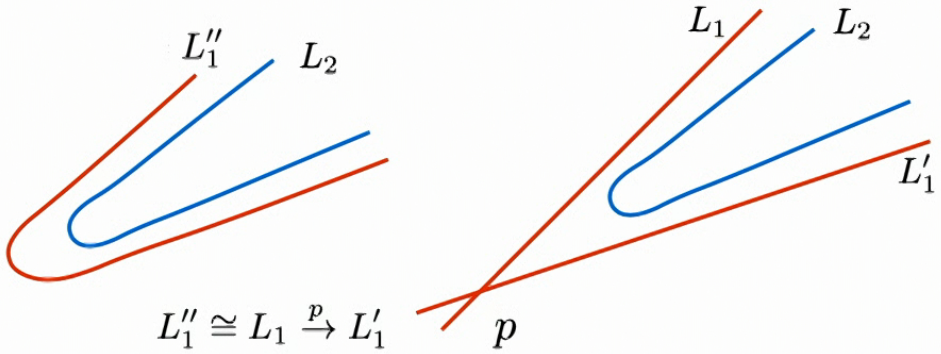


In fact, the downstairs theory knows everything,



so it determines the resolutions not just in the downstairs algebra

$$A$$

but also upstairs in ,

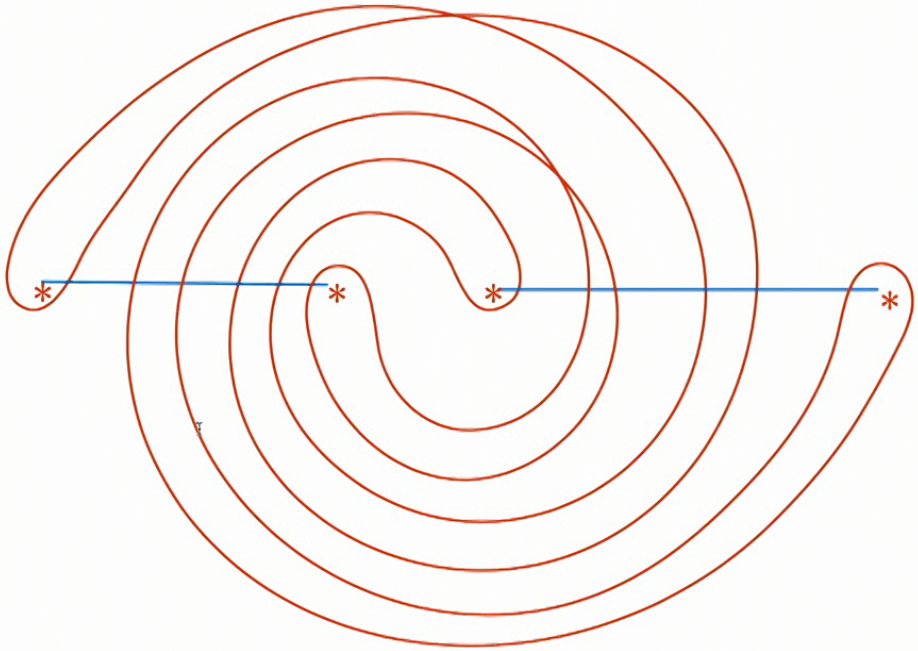
$$\mathscr{A}$$

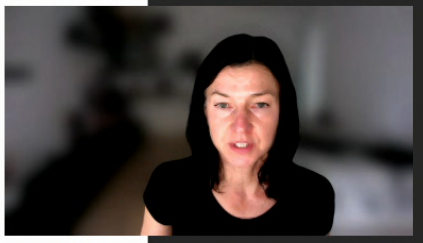
again asking that $Q = \sum_k e_k$ squares to zero.

As an example, for the trefoil,

$$^{L}\mathfrak{g} = \mathfrak{su}_2$$

we would get the following brane configuration:

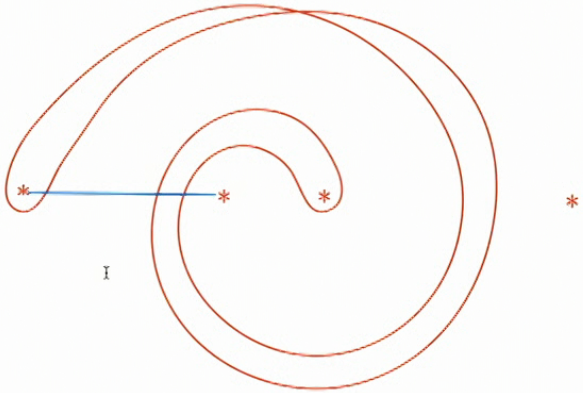




For simplicity, we can ask about the reduced Khovanov homology

where the unknot homology is set to be trivial.

This lets us erase a pair of Lagrangians and simplify the problem to $d = 1$



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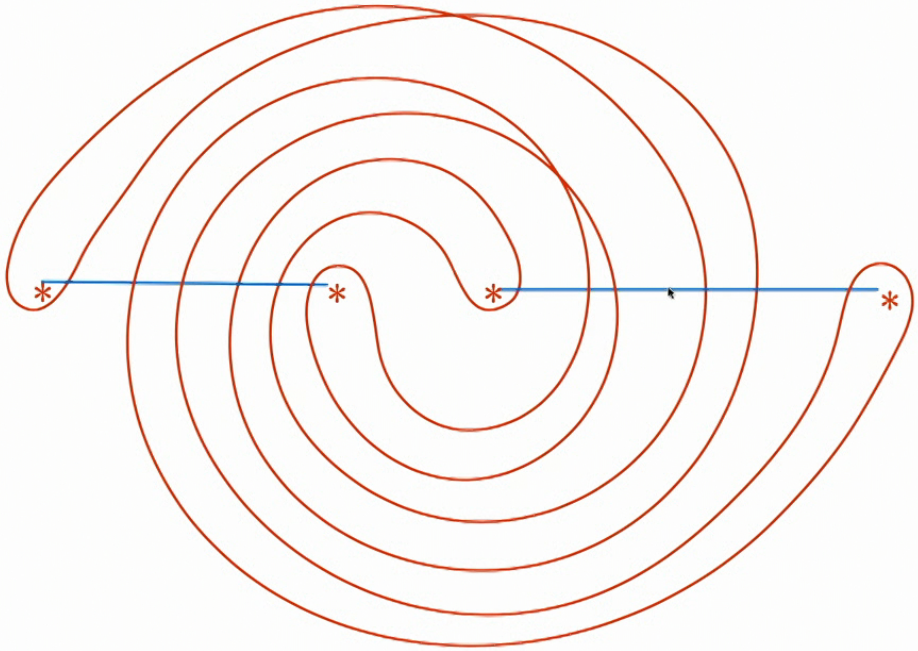
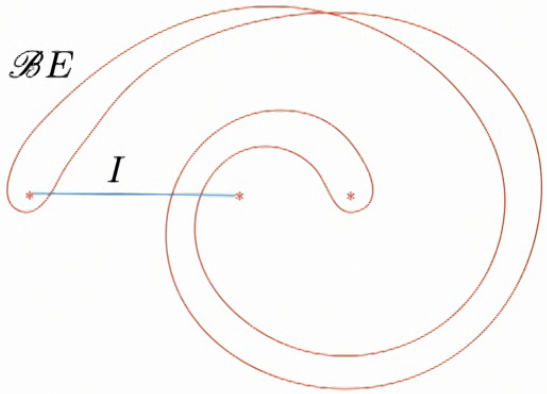
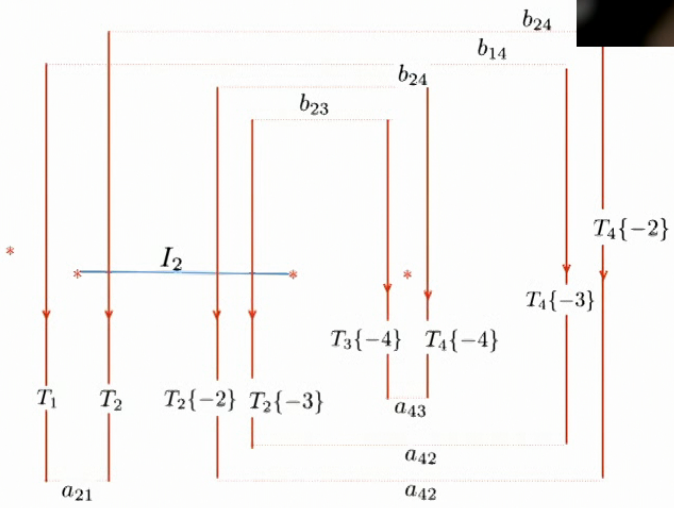


The $\mathscr{B}E$ brane has the following resolution





$$\mathscr{B}E \cong \; T_3\{-4\} \xrightarrow{\begin{pmatrix} a_{43} \\ b_{23} \end{pmatrix}} \begin{matrix} T_4\{-4\} \\ \oplus \\ T_2\{-3\} \end{matrix} \xrightarrow{\begin{pmatrix} b_{24} & 0 \\ 0 & a_{42} \end{pmatrix}} \begin{matrix} T_2\{-2\} \\ \oplus \\ T_4\{-3\} \end{matrix} \xrightarrow{\begin{pmatrix} a_{42} & 0 \\ 0 & b_{14} \end{pmatrix}} \begin{matrix} T_4\{-2\} \\ \oplus \\ T_1 \end{matrix} \xrightarrow{\begin{pmatrix} b_{24} \\ a_{21} \end{pmatrix}} T_2$$

$$I \cong I_2$$

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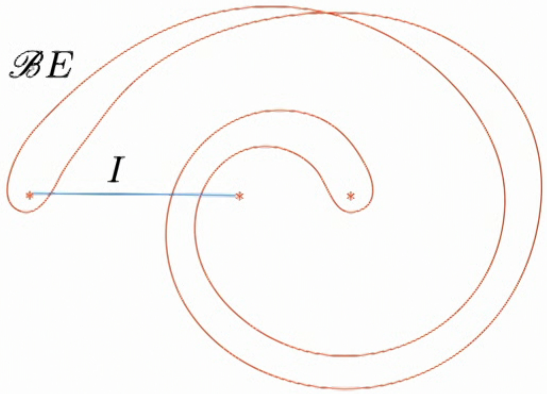

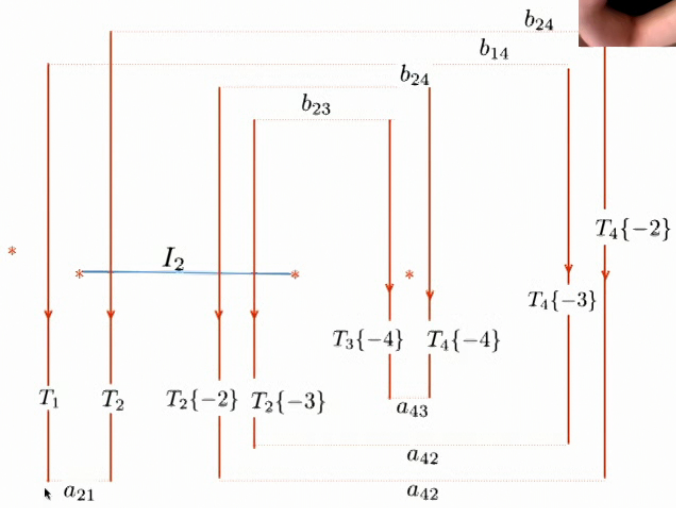

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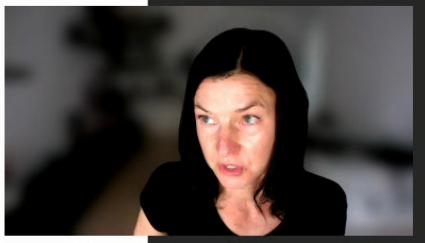

Out of the complex describing the brane

a small part contributes to the homology

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