Title: All unitary 2D QFTs share the same state space

Speakers: Andre Henriques

Collection: Global Categorical Symmetries

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Abstract: "A unitary 1d QFT consists of a Hilbert space and a Hamiltonian. A group acting on a 1d QFT is a group acting on the Hilbert space, commuting with the Hamiltonian. Note that the *data* of an action only involves the Hilbert space. The Hamiltonian is only there to provide a constraint. Moreover, all 1d QFT have isomorphic Hilbert spaces (except in special cases, e.g. in the case of a 1d TQFT, when the Hilbert space is finite dimensional).

A unitary 2d QFT consists of the 0-dimensional and 1-dimensional part of the QFT, along with the data of the Stress-energy tensor. An action of a fusion category on a 2d QFT is again something where the *data* only involves the 0-dimensional and 1-dimensional part of the QFT, while the Stress-energy tensor is only there to provide a constraint. The upshot is that it makes sense to act on the 0-dimensional and 1-dimensional part of the QFT. Moreover, I conjecture that all 2d QFTs have isomorphic 0-dimensional + 1-dimensional parts (except in special cases, e.g. in the case of a chiral CFT)."

All unitary 2D QFTs share the

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† : All QFTs in this talk will be assumed unitary, non-anomalous, UV complete, and with ∞-dimensional state spaces. All 2D QFTs are furthermore assumed to satisfy cluster decomposition (to avoid direct sums of QFTs).

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Functorial QFT

1D QFT (Quantum Mecanics):

A quantum mechanical system is fully characterised by a Hilbert space $\mathscr{H}$ and a Hamiltonian H (a self-adjoint operator on $\mathscr{H}$, with bounded below spectrum).

In functorial QFT, instead of $\mathscr{H}$ and H, one declares:

$\mathscr{H}$ (the state space)

e^{-tH} (the Wick-rotated time evolution)

to be most fundamental.

One says that $\mathscr{H}$ is associated to a point, and e^{-tH} is associated to an interval of length t.

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[The partition function $Z(t) = tr(e^{-tH})$ is what the theory associates to a circle of length t>0. Note that this might fail to converge. When it does converge, one sais that the theory is compact – I will not insist that my QFTs be compact.]

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The observables (a.k.a. perators) of the theory are "things one can put at points".
Every bounded linear map $A : \mathscr{H} \to \mathscr{H}$ gives an example of an observable, where

corresponds to the linear map $e^{-t_2 H} A\, e^{-t_1 H} : \mathscr{H} \to \mathscr{H}$.

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But not every observable is of that form.
We call observables of the above form *bounded observables*.

[General observables are continuous linear maps $\mathscr{H}^\vee \to \mathscr{H}^\wedge$, where
$\mathscr{H}^\vee := \mathrm{colim}(\mathscr{H} \xrightarrow{e^{-\frac{1}{2}H}} \mathscr{H} \xrightarrow{e^{-\frac{1}{4}H}} \mathscr{H} \xrightarrow{e^{-\frac{1}{8}H}} \mathscr{H} \ldots)$ and
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Equivalently, $\mathscr{H}^\vee$ is the union of the images of e^{-tH} for $t>0$, and
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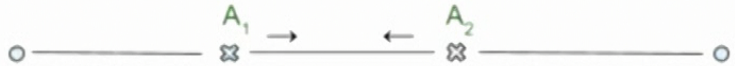
Unlike general observables, bounded observables *do not develop singularities* when the points collide:



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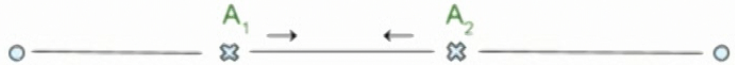

<u>Observation:</u>
Any two quantum mechanical systems have isomorphic algebras of bounded observables. (Because all separable ∞-dimensional Hilbert spaces are isomorphic.)

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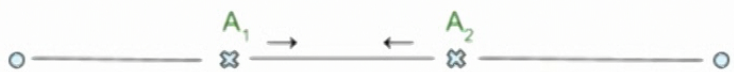

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2D QFT:

In 2D QFT, the state space depends on a choice of circle manifold, i.e., a choice of length. In other words, there's not a single Hilbert space $\mathscr{H}$, but rather a collection of Hilbert spaces $\mathscr{H}(t)$, one for every real number $t>0$.

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Factorization tells us that whenever we have:

the correlator $\langle O_- O_+ \rangle_M$ can be written as

$$\langle O_- O_+ \rangle_M = \sum \langle O_- | e_i \rangle_{M_-} \langle e_i | O_+ \rangle_{M_+}$$

where the sum is over a basis $\{ e_i \}$ of the Hilbert space associated to S.

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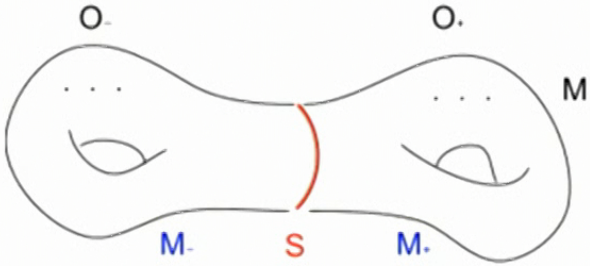

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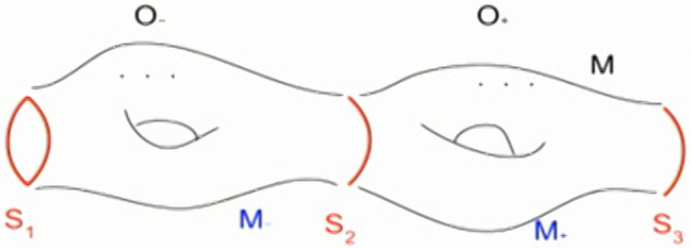
More generally, given composable cobordisms



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$$\langle \xi | O_- O_+ | \eta \rangle_M = \sum \langle \xi | O_- | e_i \rangle_{M_-} \langle e_i | O_+ | \eta \rangle_{M_+} \qquad (1)$$

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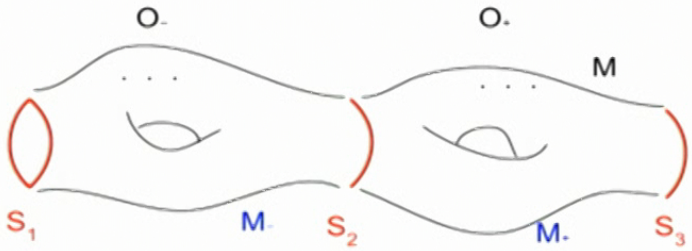

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And equation (1) is the statement that $Z(M; O_- O_+) = Z(M_+;O_+) \circ Z(M_-;O_-)$.

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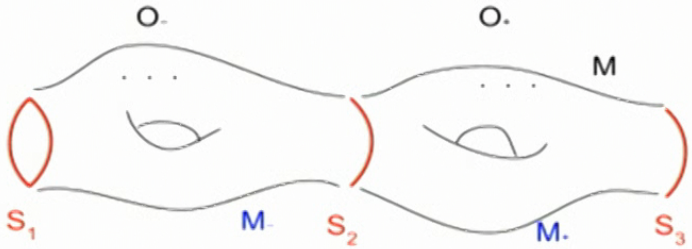

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All in all, we have:

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$Z_M : \mathscr{H}_{in} \to \mathscr{H}_{out}$ a Wick-rotated time evolution for ~~every~~ 2-dimensional cobordisms M.

[every 2-dimensional cobordisms ⇔ the theory is compact]

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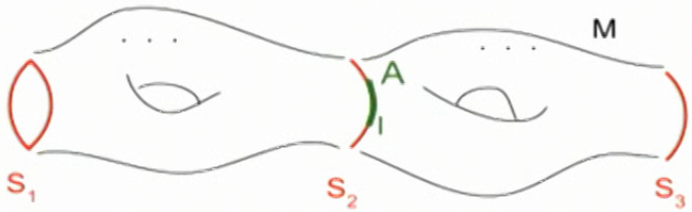
$$\mathscr{H}_1 \xrightarrow{Z(M_-)} \mathscr{H}_2 \xrightarrow{Z(M_+)} \mathscr{H}_3$$

a bounded linear map $A : \mathscr{H}_2 \to \mathscr{H}_2$ is typically supported on the whole curve S_2. But it coud be that the action of A is localised on some interval $I \subset S_2$:



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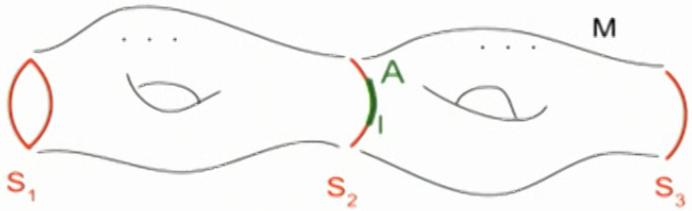

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The setup which allows to express this most comfortably is that of ***extended functorial QFT***. In tha tsetup, we're allowed to *cut 2-manifold along curves*, but also *1-manifolds along points*.

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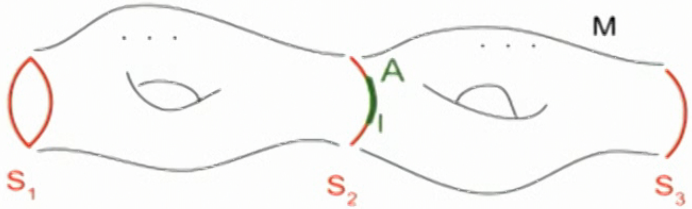

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Extended 2D QFT

Points:

In addition to state spaces associated to 1-manifolds and time evolution operators associated to cobordisms, we now also have a linear category associated to a point:

$\mathscr{B}$: the category of boundary conditions.

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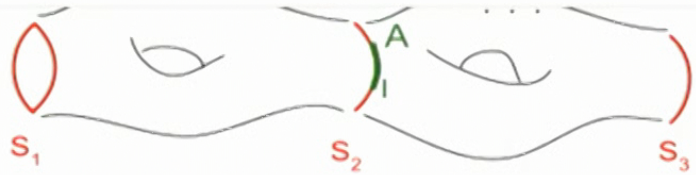

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Let us describe the category $\mathscr{B}$:
Its objects are the boundary conditions of our 2D QFT.
Given a $\in \mathscr{B}$, **$\mathrm{End}_{\mathscr{B}}(a)$:= the algebra of bounded operators on $\mathscr{H}_a^{\,a}$ localised in [-1,0].**

Trick: In a category with direct sums, if you know all endomorphisms of objects, then you also know all homs between objects:
Given a,b $\in \mathscr{B}$, one recovers $\mathrm{Hom}_{\mathscr{B}}(a,b)$ as the lower left corner in

$$\mathrm{End}_{\mathscr{B}}(a\oplus b) = \begin{pmatrix} \mathrm{Hom}_{\mathscr{B}}(a,a) & \mathrm{Hom}_{\mathscr{B}}(b,a) \\ \mathrm{Hom}_{\mathscr{B}}(a,b) & \mathrm{Hom}_{\mathscr{B}}(b,b) \end{pmatrix}$$

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Conjecture: For all 2D QFTs and all non-zero boundary condition a $\in \mathscr{B}$, $\mathrm{End}_{\mathscr{B}}(a)$ is a hyperfinite III_1 factor *[this is known in 2D CFT – follows from the Bisognano-Wichmann theorem]*. Moreover, for any a $\in \mathscr{B}$, we have $\mathscr{B} \simeq \mathrm{End}_{\mathscr{B}}(a)$-Mod.

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$$\mathrm{End}_{\mathscr{B}}(a\oplus b) = \begin{pmatrix} \mathrm{Hom}_{\mathscr{B}}(a,a) & \mathrm{Hom}_{\mathscr{B}}(b,a) \\ \mathrm{Hom}_{\mathscr{B}}(a,b) & \mathrm{Hom}_{\mathscr{B}}(b,b) \end{pmatrix}$$

where $\mathrm{End}_{\mathscr{B}}(a\oplus b)$ is assumed to be known.

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Here, the "***hyperfinite III_1 factor***" is a certain von Neumann algebra which is unique up to isomorphism.
We write R for the unique hyperfinite III_1 factor.

⇒ For any 2D QFT its category of boundary conditions is equivalent to the category R-Mod of modules over the hyperfinite III_1 factor. (✣)

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What the 2D QFT assigns to points doesn't remember anything about the QFT.

(*Note:* This is in sharp contrast with the case of extended TQFTs.)

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But in order to identify $\mathscr{B}$ with R-Mod, we had to chose a boundary condition $a \in \mathscr{B}$, and an isomorphism $\mathrm{End}_{\mathscr{B}}(a)$-Mod = R.

Useful (non-trivial) fact:
Linear functors R-Mod → R-Mod correspond bijectively to R-R-bimodules.

I need to describe an R-R-bimodule $\mathscr{H}(t)$ for every interval of length $t \geq 0$.
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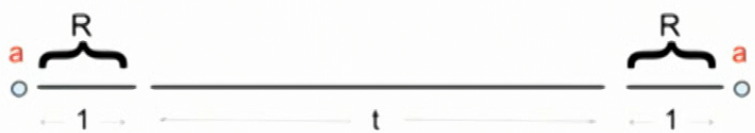

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Cultural remark:
In the theory of subfactors, there's lot of uniqueness results about actions of various stuff on the category of modules over the hyperfinite III_1 factor. For example:

Theorem: (Popa)
Given a unitary fusion category C, there exists a unique faithful representation of C on R-Mod, up to isomorphism.
Equivalently, any two faithful tensor functors C → Bim(R) are conjugate.

We cannot apply the above theorem to $\mathbb{R}_+$ because it's not a fusion category.
But can take it as inspiration and conjecture that actions $\mathbb{R}_+ \to \mathrm{Bim}(R)$ are similarly unique [assuming a technical condition called "split property"]:

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Consequences of the conjecture

Non-smooth space-times:

Usually, people working in functorial QFT insist that their (D-1)-dimensional Cauchy surfaces be equipped with a germ of D-manifold.
Otherwise, the composition of two cobordisms might fail to be smooth.

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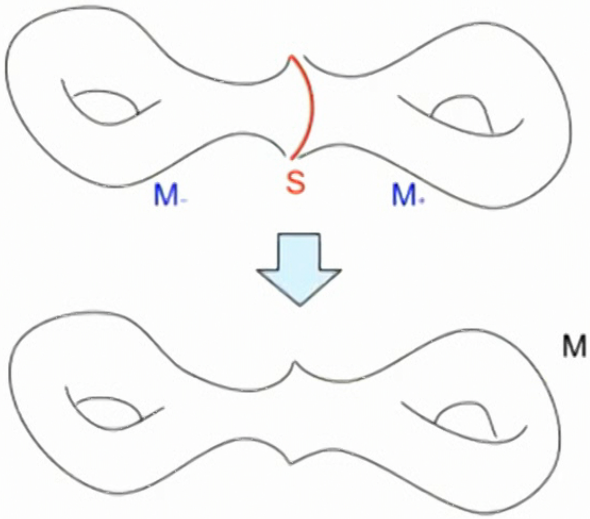
Part of the conjecture says that the Hilbert space of a 2D QFT only depends on the length of the circle (or length of the interval in the case of QFT with boundary conditions).

Turing this around, if you believe the conjecture, this means that: every unitary 2D QFT can be evaluated on the kind of non-smooth space-times that one obtains by gluing cobordisms along an isometry of their boundary:



[For non-compact 2D QFT, one should also insist that $\partial_{in}M$ and $\partial_{out}M$ be non-empty]

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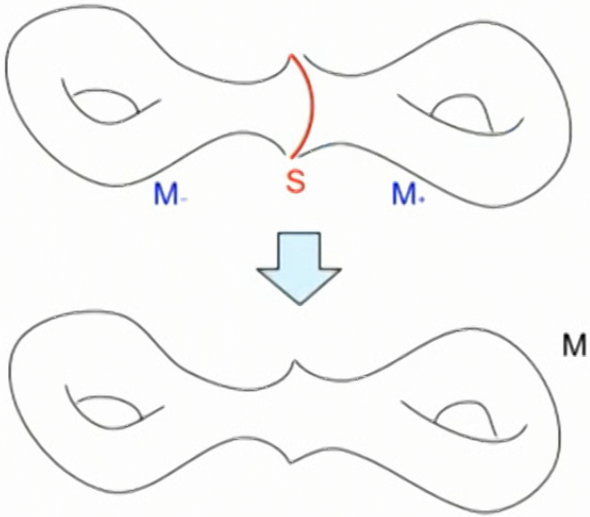

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Sanity check:

At least in the case of a free massive boson, this seems to be ok.

The partition function of the free boson theory on a Riemannian surface M is the ζ-regularised determinant:

$$Z(M) = \mathrm{Det}_\zeta\,(\Delta+m^2)$$

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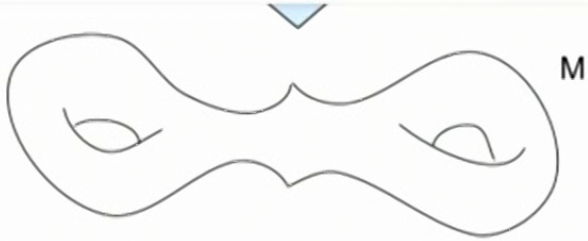

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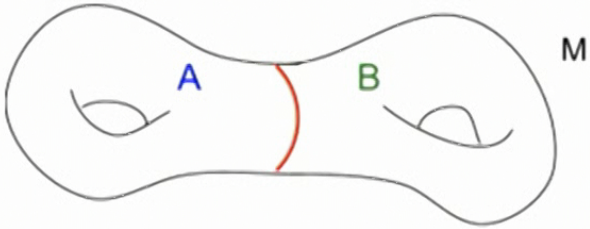
A plethora of domain walls:

If QFT A and QFT B assign the same stuff to points and to 1-manifolds, then any isomorphism φ between the corresponding stuff produces an invertible interface between A and B.

⚠ Not a topological interface! ⚠



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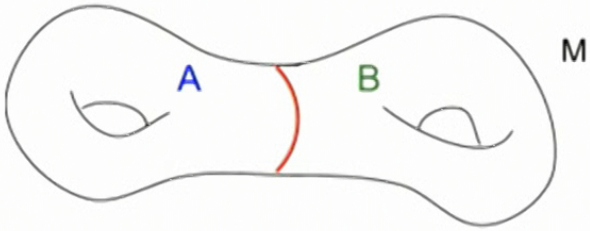

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If M is split in two regions M_- and M_+ labelled by A and B, we first get vectors $\langle Z^A(M_-) | \in \mathscr{H}^A(\partial M_-)$ and $\langle Z^B(M_+) | \in \mathscr{H}^B(\partial M_+)$. We then use the isomorphism $\varphi : \mathscr{H}^A \simeq \mathscr{H}^B$ (and the isometry between ∂M_- and ∂M_+) to identify

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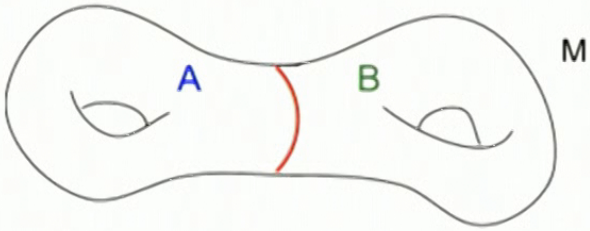

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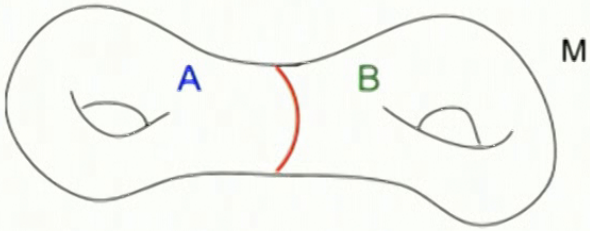

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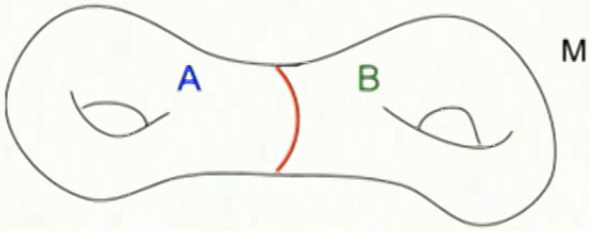

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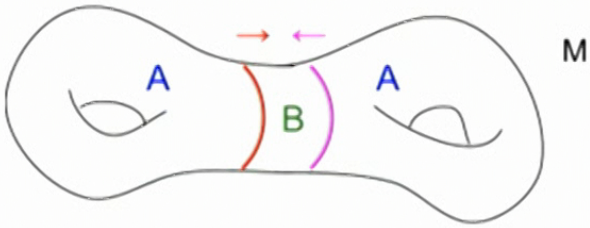
(1) the system doesn't develop singularities

(2) it converges to the identity defect (the transparent defect)



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(Here, "symmetry" means a *non-topological* defect)

Fix a 2D QFT. Let $\mathscr{B}$ = R-Mod be its category of boundary conditions, and ${}_R\mathscr{H}(t)_R$ the bimodule associated to an interval of length $t \geq 0$.

In the formalism of extended 2D QFT, an *action of a fusion category C* by <u>topological</u> defects consists of:

- For every $c \in C$, an R-R-bimodule ${}_RX(c)_R$, along with isomorphisms ${}_RX(c \otimes d)_R \simeq {}_RX(c) \boxtimes_R X(d)_R$ subject to various coherences...
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- For $f_{\mu\nu} \in C^\infty[0,t]$ a smooth function with support away from the boundary, the smeared stress energy tensor $T^{\mu\nu}[f_{\mu\nu}] := \int_{[0,t]} T^{\mu\nu}(x) f_{\mu\nu}(x)\, dx$ [which is an unbounded operator on $\mathscr{H}(t)$] should commute with $u(t)$:

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A plethora of (non-invertible) symmetries:

(Here, "symmetry" means a *non-topological* defect)
Fix a 2D QFT. Let $\mathscr{B}$ = R-Mod be its category of boundary conditions, and ${}_R\mathscr{H}(t)_R$ the bimodule associated to an interval of length $t \geq 0$.

In the formalism of extended 2D QFT, an *action of a fusion category C* by <u>topological</u> defects consists of:

- For every $c \in C$, an R-R-bimodule ${}_RX(c)_R$, along with isomorphisms ${}_RX(c \otimes d)_R \cong {}_RX(c) \boxtimes_R X(d)_R$ subject to various coherences...
- For every $c \in C$, a unitary bimodule isomorphism:

$$u(t) : X(c) \boxtimes_R \mathscr{H}(t) \cong \mathscr{H}(t) \boxtimes_R X(c)$$

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If we remove the last condition, we get the notion of *an action of the fusion category C by* ***not necessarily topological*** *defects*.

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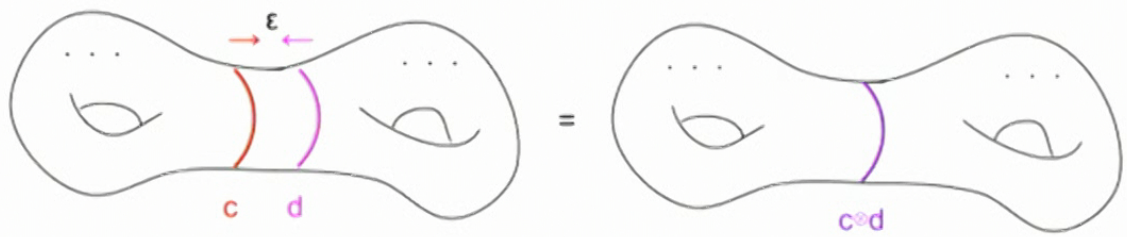
The fusion of defects is given, as usual, by:

$\lim_{\varepsilon \to 0}$



(for topological defects, the equality would hold true even before taking the limit).

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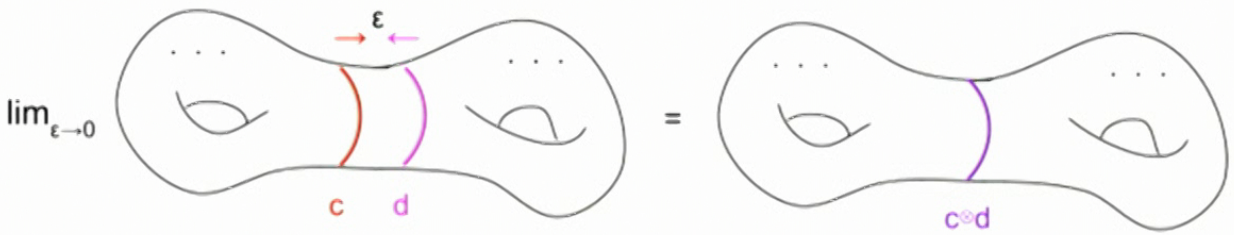

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<u>*Observe:*</u>

The formulation of an acton of C by not necessarily topological defects *only refers to the stuff that the QFT assigns to points and to 1-manifolds*. If that' stuff is independent of the 2D QFT, we get:

Any fusion category which acts on *some* 2D QFT acts on *every* 2D QFT (by not necessarily topological defects)

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Fix a represenation $\mathbb{R}_+ \to \mathrm{Bim}(R)$, which we think of as the stuff that a 2D QFT assigns to points and to 1-manifolds. We write $\mathscr{H}(t)$ for the bimodule associated to $t \in \mathbb{R}_+$.

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Definition:
A Wightman field φ (a.k.a. operator-valued distribution) is a gadget which associates to every test function $f \in C^\infty[0,t]$ with support away from the boundary, an operator $\varphi[f]$ on $\mathscr{H}(t)$ (typically unbounded) that commutes with the two actions of R on $\mathscr{H}(t)$.

The assignment $f \mapsto \varphi[f]$ should satisfy the *locality axiom*:

$$\varphi[f \cup g] = \varphi[f] \boxtimes_R \varphi[g]$$

where $f \cup g \in C^\infty[0,t_1+t_2]$ denotes the concatenation of $f \in C^\infty[0,t_1]$ and $g \in C^\infty[0,t_2]$.

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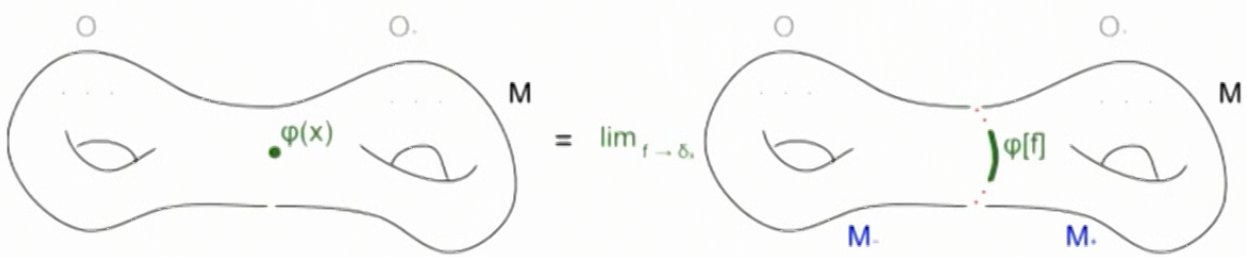

But the limit might fail to exist.

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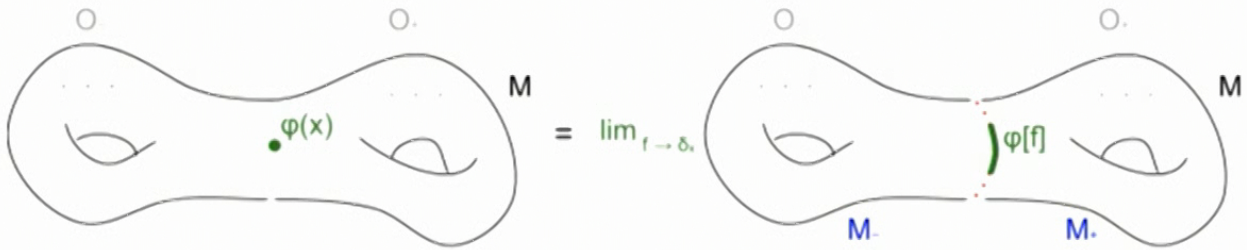

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In the free Fermion CFT, we have a Wightman field ψ satisfying

$$\psi[f]\psi[g] + \psi[g]\psi[f] = \int f(x)g(x)dx. \tag{2}$$

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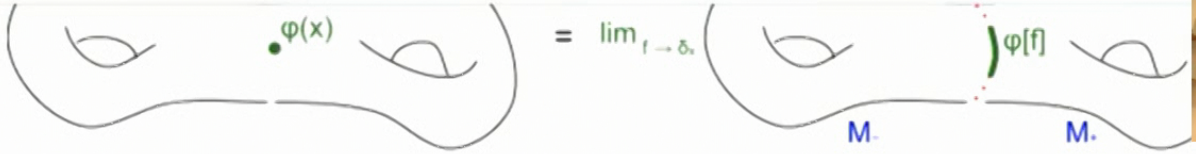

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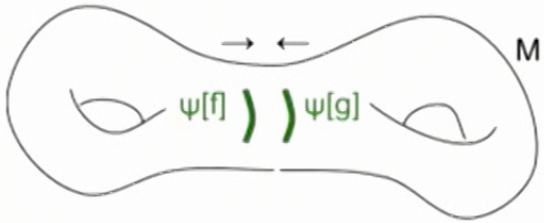

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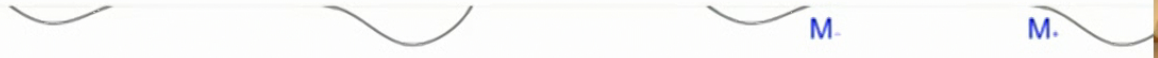

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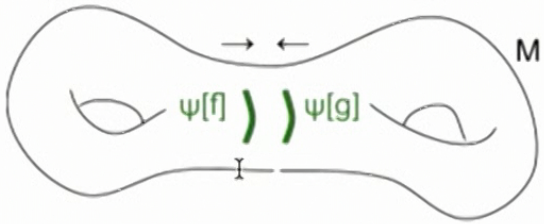

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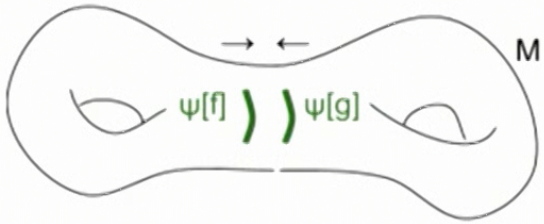

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Thus, a consequence of the conjecture ist:

***Any* 2D QFT admits observables ψ[f] that satisfy the same commutation relations (2) as smeared free fermions†**

†: assuming the QFT is not purely bosonic.

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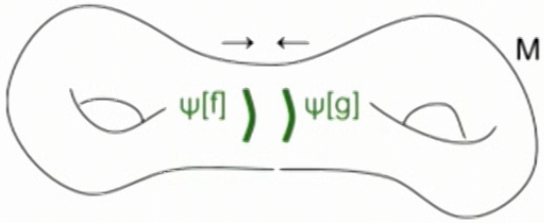

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