Title: Symmetries from string theory

Speakers: IÃ±aki GarcÃ-a-Etxebarria

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Abstract: "It is possible to construct interesting field theories by placing string theory on suitable singular geometries, and adding branes. In the fairly special cases where Lagrangians are known for the resulting theories, field theory arguments often show that these theories have generalised symmetry structures. In this talk I will review recent work developing a dictionary, valid even in the absence of a known Lagrangian description, between properties of the string theory geometry and generalised symmetries of the associated field theories."

Symmetries from string theory

Iñaki García Etxebarria

Based on

- 1908.08027 with B. Heidenreich and D. Regalado,
- 2112.02092 with F. Apruzzi, F. Bonetti, S. Hosseini and S. Schäfer-Nameki,
- 2203.10097 with M. Del Zotto and S. Schäfer-Nameki.

Durham University

Department of Mathematical Sciences

Simons Collaboration on Global Categorical Symmetries

QFTs from geometry

In this talk I will view string theory as a tool for associating Quantum Field Theories (QFTs) $\mathcal{T}[X]$ to (singular) manifolds X. This point of view goes under the name of "geometric engineering".

To any given theory $\mathcal{T}[X]$ we can associate a "symmetry TFT" $\mathrm{Symm}[\mathcal{T}[X]]$, a TFT in one dimension higher encoding symmetries and anomalies of the theory, and all its gaugings. (See Justin's talk on Monday.)

It turns out that $\mathrm{Symm}[\mathcal{T}[X]]$ is significantly easier to understand than $\mathcal{T}[X]$ itself, so our goal will be to construct $\mathrm{Symm}[X] := \mathrm{Symm}[\mathcal{T}[X]]$ directly from the geometry.

Why

Given a Lagrangian description of $\mathcal{T}[X]$ it is in principle possible (but subtle) to find its symmetries, and in this way reconstruct $\mathrm{Symm}[X]$.

Nevertheless, in the context of geometric engineering having a Lagrangian description of $\mathcal{T}[X]$ is more the exception than the rule: what we know is the topology (and sometimes metric) of X.

It is precisely in the cases where we don't know a Lagrangian that the information about symmetries and anomalies is most valuable.

Why

A more formal reason to care about this problem is that it hints towards a geometric version of the Landau paradigm: as we will see the map $X \to \mathrm{Symm}[X]$ is very sensitive to the details of X.

Geometric Landau question

Can we reconstruct X (modulo string dualities) given $\mathrm{Symm}[X]$?

An a priori weaker form is that a subset of the data associated to X, sufficient to reconstruct $\mathcal{T}[X]$, is determined by $\mathrm{Symm}[X]$. But the stronger form here seems plausible.

There is a categorical version of this question, where we ask about some category associated to X instead. For instance, in some cases we can associate a cluster category to X. The Grothendick group of this cluster category is easy to read from $\mathrm{Symm}[X]$. [Caorsi, Cecotti '17], [Del Zotto, IGE, Hosseini '20], [Del Zotto, IGE '22].

Geometric engineering

For reasons of analytic control we want to impose restrictions on the manifolds X that we consider. These are:

- X is non-compact, to decouple gravity. To make our life simpler I'll assume that X is a real cone over some base B.
- In order for the field theory to be supersymmetric, we assume that X has reduced holonomy (Calabi-Yau, for instance).

For instance, if X is a complex two-fold, we will assume that it is an ALE space of the form $\mathbb{C}^2/\Gamma_{\mathfrak{g}}$, with $\Gamma_{\mathfrak{g}} \subset SU(2)$. This is a cone over $S^3/\Gamma_{\mathfrak{g}}$, with $\Gamma_{\mathfrak{g}}$ acting freely on S^3. On $\mathbb{C}^2$ the origin is fixed by all elements of $\Gamma_{\mathfrak{g}}$, so we have an orbifold singularity there.

If we place IIB string theory (10d) on this geometry we obtain a $(2,0)$ SCFT $\mathfrak{g}_{(2,0)}$ in six dimensions, arising from modes at the singularity. These theories are believed to be indexed by $\Gamma_{\mathfrak{g}}$, or equivalently by an algebra $\mathfrak{g}$ of type $\mathfrak{a}_n$, $\mathfrak{d}_n$, $\mathfrak{e}_6$, $\mathfrak{e}_7$ or $\mathfrak{e}_8$.

The $(2,0)$ theory in 6d

This is a very important theory of a very strange kind: it is an interacting conformal theory in six dimensions.

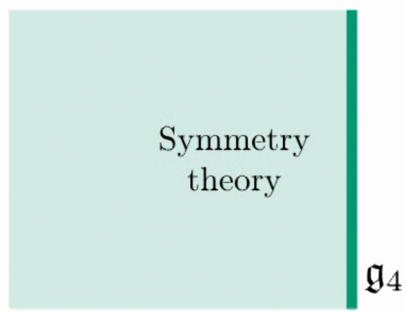
The existence of such theories is fairly surprising from a Lagrangian point of view: by dimensional reasons any d-dimensional gauge theory becomes free as we go to large distances. The $(2,0)$ SCFTs, on the other hand, remain interacting at all scales.

One important property of the $(2,0)$ theory with algebra $\mathfrak{g}$ is that upon reduction on T^2 with complex structure τ it gives rise to 4d $\mathcal{N}=4$ SYM with algebra $\mathfrak{g}$ and complexified gauge coupling τ. Let me call this object $\mathfrak{g}_4$.

$\mathfrak{g}_4$ as a relative theory

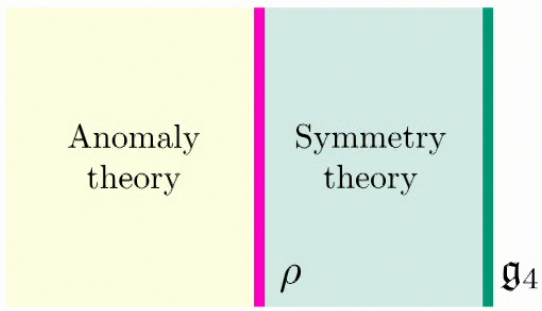
What I have just described fully specifies the behaviour of local operators, but it does not fully fix the theory. For example it does not fully fix the partition function on K3.

The right way of thinking about $\mathfrak{g}_4$ is as a "relative theory" [Freed, Teleman '12]: in physical terms it is a set of boundary gapless modes for a TFT in one dimension higher ($4 + 1 = 5$ here). This TFT includes information about the potential symmetries, anomalies and gaugings of all theories with local dynamics given by $\mathfrak{g}_4$. We refer to this TFT as $\mathrm{Symm}[\mathfrak{g}_4]$.



The absolute $\mathcal{N} = 4$ theories

We can make obtain more familiar objects by introducing a second gapped interface ρ between $\mathrm{Symm}[\mathfrak{g}_4]$ and an invertible TFT, the anomaly theory.



Colliding ρ and $\mathfrak{g}_4$ we obtain what we usually think of as $\mathcal{N} = 4$ SYM *theories* in $d = 4$. The possible choices of ρ were classified by [Aharony, Seiberg, Tachikawa '13] (from a different viewpoint). The connection with the picture above was essentially done (for $SU(N)$) in [Witten '98], and extended to the $\mathfrak{d}_i$, $\mathfrak{e}_i$ cases in [IGE, Heidenreich, Regalado '19].

Back to 10d

Our starting point was not directly the 4d theory $\mathfrak{g}_4$ on $\mathcal{M}_4$, but rather 10d string theory on $\mathcal{M}_4 \times T^2 \times \mathbb{C}^2/\Gamma_{\mathfrak{g}}$. How do we reproduce the previous discussion from the string theory perspective? Where is $\mathrm{Symm}[\mathfrak{g}_4]$?

My goal will be to derive $\mathrm{Symm}[\mathfrak{g}_4]$ (*) without using any knowledge about the Lagrangian of $\mathcal{N} = 4$.

(*) In this talk I will work out some aspects of the symmetry theories. Eventually we would like to have a complete description as a fully extended TFT instead.

The geometric engineering perspective

There are multiple formulations of string theory, connected by duality. This means that they are all secretly the same theory, and we have rules for mapping from one formulation to another.

In this talk I will work in the M-theory ($D = 11$) and IIB ($D = 10$) duality frames. In either frame, the basic picture is the same. We place the string theory on $X^{2n} \times \mathcal{M}^{D-2n}$, where X^{2n} is a Calabi-Yau manifold of complex dimension n, which is also a real cone with base B^{2n-1}. There is a singularity at the base of the cone, where we have a field theory $\mathcal{T}[X^{2n}]$ on $\mathcal{M}^{D-2n}$.

Branes

In the context of geometric engineering we treat the background geometry $X^{2n} \times \mathcal{M}^{D-2n}$ as given, and ignore its dynamics. All the dynamical field theory behaviour comes from "(p-)branes" and their associated fluxes moving in the given geometry.

"p-branes" are dynamical supersymmetric solitons, localised on $(p+1)$-dimensional manifolds S^{p+1}. The spectrum of branes depends on the theory:

- In M-theory we have M2 and M5 branes. ($p = 2$ and $p = 5$.)
- In IIB we have Dp branes with $p \in \{-1, 1, 3, 5, 7, 9\}$ and F1, NS5 branes. In this talk I will focus on D3 branes, with $p = 3$.

Fluxes

Each p-brane coupled electrically to some background $(p+1)$-form gauge field C_{p+1}

$$S = \ldots + \int_{S^{p+1}} C_{p+1} \,.$$

We can measure the charge by integrating the field strength $F_{p+2} = dC_{p+1}$ on a manifold $R^{D-(p+2)}$ linking S^{p+1} once in the D-dimensional total space:

$$\int_{R^{D-(p+2)}} \star F_{p+2} = 1 \,.$$

There is a duality relation on fluxes $F_{p+2} = \star F_{D-(p+2)}$, so we can view this last equation as saying that the p-brane is charged magnetically with respect to the dual gauge field $C_{D-(p+3)}$:

$$\int_{R^{D-(p+2)}} F_{D-(p+2)} = 1 \,.$$

Blowing up the singularity

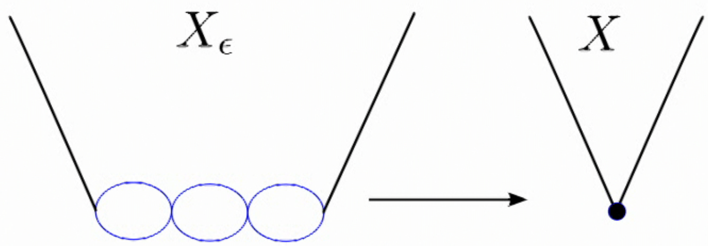
To see how the QFT arises, we can consider a family of smooth Calabi-Yau spaces $\tilde{X}_\epsilon$, such that $\lim_{\epsilon\to 0}\tilde{X}_\epsilon = X$. The singularity of X arises from cycles in $\tilde{X}_\epsilon$ going to zero size.

As an example, consider the $\mathbb{C}^2/\mathbb{Z}_n$ singularity (for me $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$), with $p \in \mathbb{Z}_n$ acting as

$$p\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \omega^p & 0 \\ 0 & \omega^{-p} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and $\omega = \exp(2\pi i/n)$.

A family of spaces $\tilde{X}_\epsilon$ can be constructed by "blowing-up" the singularity: the singular point gets replaced by $n-1$ copies of S^2 (of size ϵ) intersecting according to the Dynkin diagram of A_{n-1}:



Light branes

Wrapping a p-brane on a small q-cycle of size ϵ gives a propagating object in the field theory of dimension $p+1-q$ and mass ϵ.

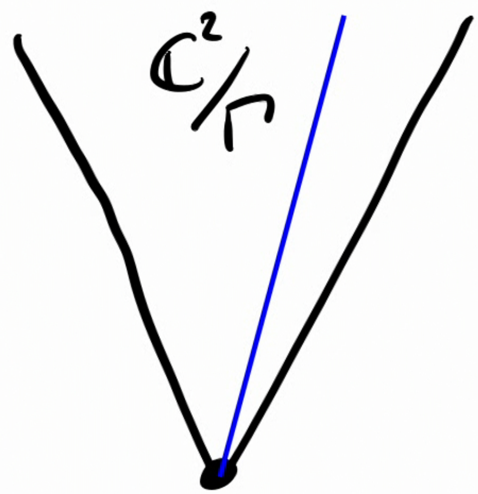
In the singular limit these objects becomes massless. Two examples:

- M-theory on $\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{R}^7$: M2 branes wrapping the S^2 cycles appear as light particle states propagating on $\mathbb{R}^7$. In the massless limit these give rise to Yang-Mills theory in 7d, with the branes giving the non-abelian structure.
- IIB on $\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{R}^6$. In this case, if we wrap D3 branes, we get light strings propagating in 6d. In the massless limit we have the "non-abelian" $(2,0)$ theory of type $\mathfrak{a}_{n-1}$.

Heavy branes

So far I have described the dynamical states, but for understanding the symmetry of the theory we are more interested in the behaviour of extended defect operators.

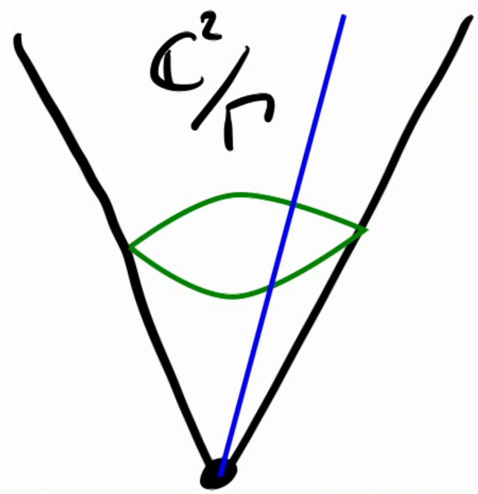
We can view these as infinitely heavy objects inserted into our configuration. The mass of the object, for the wrapped brane, is proportional to the volume wrapped in X. So defects will arise from branes wrapping non-compact cycles ending on the singular point.



Charge operators

Now we have a geometric characterisation of the defect operators (generalised Wilson/'t Hooft lines) in the field theory as branes wrapping non-compact cycles. These are in general *not* topological.

The symmetry operators are rather the flux operators measuring which non-compact lines we have in our configuration:



Boundary conditions and flux non-commutativity

In order to define the string theory fully on the non-compact space $X^{2n} \times \mathcal{M}^{D-2n}$ we need to specify the boundary conditions at infinity, which is (assuming $\mathcal{M}^{D-2n}$ compact) of the form $B^{2n-1} \times \mathcal{M}^{D-2n}$.

In the field theory these boundary fluxes appear as background fluxes for higher form symmetries.

We do this by giving a state in the Hilbert space associated to $B^{2n-1} \times \mathcal{M}^{D-2n}$. There are many subtleties in making this statement precise, but the crucial one for us is due to flux non-commutativity [Moore '04], [Freed, Moore, Segal '06], due to the fact that the non-compact brane wraps a torsional cycle of B^{2n-1} (the base of X^{2n}), and therefore the flux sourced by it should be measured on a torsional cycle.

Charge operators

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The geometric engineering perspective

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Non-commutativity of fluxes in M-theory

Let us put M-theory on $\mathcal{M}_{11} = \mathcal{N}_{10} \times \mathbb{R}$. We will try to understand the Hilbert space $\mathcal{H}(\mathcal{N}_{10})$, or more precisely its grading by flux. This was done in [Freed, Moore, Segal '06].

M-theory contains 3-form gauge fields C_3. The magnetic charge is measured by the topological class of C_3. To measure the electric charge, recall that in the Hamiltonian formulation of the theory the canonical momentum Π_{C_3} conjugate to C_3 is $\star G_4$. This is what we integrate to measure the electric charge. If we express states in $\mathcal{H}(\mathcal{N}_{10})$ in terms of their wavefunctions $\psi(C_3)$, then a state of definite electric charge is an eigenstate of momentum:

$$\psi(C_3 + \lambda) = e^{2\pi i \int_{\mathcal{N}_{10}} Q_e \lambda} \psi(C_3)$$

for all flat λ. Here $Q_e \in H^7(\mathcal{N}_{10})$ is the electric charge.

$$\psi(x) = e^{ipx}$$

$$\psi(x+\alpha) = e^{ip\alpha}\,\psi(x)$$

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So we cannot simultaneously measure electric and magnetic charges, if there are flat topologically non-trivial λ. This is the case iff $\mathrm{Tor}\, H^4(\mathcal{N}_{10}) \neq 0$.

Non-commutativity of fluxes in M-theory

This can be restated in terms of the flux operators, as follows: for every $\sigma \in \mathrm{Tor}\, H_6(\mathcal{N}_{10}; \mathbb{Z}) = \mathrm{Tor}\, H^4(\mathcal{N}_{10}; \mathbb{Z})$ there is a unitary flux operator Φ_σ. Similarly for any $\sigma' \in \mathrm{Tor}(H_3(\mathcal{N}_{10}; \mathbb{Z})) = \mathrm{Tor}\, H^7(\mathcal{N}_{10}; \mathbb{Z})$.

These operators in general do not commute:

$$\Phi_\sigma \Phi_{\sigma'} = e^{2\pi i\, \mathsf{L}(\sigma,\sigma')} \Phi_{\sigma'} \Phi_\sigma$$

where $\mathsf{L}(\sigma, \sigma')$ is the linking pairing on $\mathcal{N}_{10}$: choose $n \in \mathbb{Z}$ such that $n\sigma = \partial D$. Then

$$\mathsf{L}(\sigma, \sigma') = \frac{1}{n} D \cdot \sigma' \mod 1\,.$$

Non-commutativity of fluxes in M-theory

The pairing $\mathsf{L}(\cdot,\cdot)$ is *perfect*, which implies that if $\mathrm{Tor}(H_3(\mathcal{N}_{10};\mathbb{Z})) = \mathrm{Tor}(H_6(\mathcal{N}_{10};\mathbb{Z})) \neq 0$, then for each $\sigma \neq 0$ there is some σ' such that $\mathsf{L}(\sigma,\sigma') \neq 0$, and thus

$$\Phi_\sigma \Phi_{\sigma'} = e^{2\pi i\,\mathsf{L}(\sigma,\sigma')} \Phi_{\sigma'} \Phi_\sigma \neq \Phi_{\sigma'} \Phi_\sigma \,.$$

What this all implies, it that whenever $\mathrm{Tor}(H_3(\mathcal{N}_{10};\mathbb{Z})) \neq 0$ it is not possible to simultaneously diagonalize all Φ_σ. In particular, it is not consistent to take the simple "fluxless" choice $\Phi_\sigma = 1$ for all σ. We need to turn on *some* flux at infinity!

Maximal isotropic subspaces

The final algebraic structure is fairly simple: we have a Hilbert space, and a set of non-commuting operators acting on it.

We can specify a state in the Hilbert space as usual: by choosing a maximal subspace $\mathcal{I} \subset \mathrm{Tor}(H_3(\mathcal{N}_{10});\mathbb{Z}) \times \mathrm{Tor}(H_6(\mathcal{N}_{10});\mathbb{Z})$ such that the corresponding group of operators $\{\Phi_x\}$ for $x \in \mathcal{I}$ is abelian, and imposing that

$$\Phi_x \left|0;L\right\rangle = \left|0;L\right\rangle \qquad \forall x \in \mathcal{I}$$

In our M-theory setting, this corresponds to setting to zero on the boundary as many fluxes as possible. (This is the sector with vanishing Φ_x flux, for non-zero background flux for the higher form symmetries choose non-zero eigenvalues.)

Back to M-theory on $\mathbb{C}^2/\Gamma_{\mathfrak{g}}$

We want to consider M-theory on a space $\mathcal{M}_{11} = \mathbb{C}^2/\Gamma_{\mathfrak{g}} \times \mathcal{M}_7$ with $\Gamma_{\mathfrak{g}}$ a discrete subgroup of $SU(2)$. Let us apply our methods to classify the space of possible theories for a fixed $\mathfrak{g}$.

We have that $\mathbb{C}^2/\Gamma_{\mathfrak{g}}$ is a cone over $S^3/\Gamma_{\mathfrak{g}}$, so in order to understand the boundary conditions at infinity we want to quantize the flux sector of M-theory on $\mathbb{R} \times S^3/\Gamma_{\mathfrak{g}} \times \mathcal{M}_7$.

Back to M-theory on $\mathbb{C}^2/\Gamma_{\mathfrak{g}}$

$\Gamma_{\mathfrak{g}}$ acts freely on S^3, so $\pi_1(S^3/\Gamma_{\mathfrak{g}}) = \Gamma_{\mathfrak{g}}$. By Hurewicz's theorem

$$H_1(S^3/\Gamma_{\mathfrak{g}}) = \frac{\pi_1(S^3/\Gamma_{\mathfrak{g}})}{[\pi_1(S^3/\Gamma_{\mathfrak{g}}), \pi_1(S^3/\Gamma_{\mathfrak{g}})]} = \Gamma_{\mathfrak{g}}^{\text{ab}}\,.$$

The group $\Gamma_{\mathfrak{g}}^{\text{ab}}$ is easy to determine:

$\Gamma_{\mathfrak{g}} \subset SU(2)$	$\mathfrak{g}$	$\Gamma_{\mathfrak{g}}^{\text{ab}}$
$\mathbb{Z}_N$	A_{N-1}	$\mathbb{Z}_N$
Binary dihedral $\text{Dic}_{(2k-2)}$	D_{2k}	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$
Binary dihedral $\text{Dic}_{(2k-1)}$	D_{2k+1}	$\mathbb{Z}_4$
Binary tetrahedral $2T$	E_6	$\mathbb{Z}_3$
Binary octahedral $2O$	E_7	$\mathbb{Z}_2$
Binary icosahedral $2I$	E_8	1

(Notice that $\Gamma_{\mathfrak{g}}^{\text{ab}} = Z(G_{\mathfrak{g}})$, with $G_{\mathfrak{g}}$ the simply connected Lie group with algebra $\mathfrak{g}$.)

Back to M-theory on $\mathbb{C}^2/\Gamma_{\mathfrak{g}}$

From here

$$H_*(S^3/\Gamma_{\mathfrak{g}}) = \{\mathbb{Z}, \Gamma_{\mathfrak{g}}^{\mathsf{ab}}, 0, \mathbb{Z}\}\,.$$

To make my life easier I will assume that $\mathcal{M}_7$ is closed and has no torsion in homology. Then Künneth's formula implies

$$\begin{aligned}\mathrm{Tor}(H_3(\mathcal{M}_7 \times S^3/\Gamma_{\mathfrak{g}})) = H_2(\mathcal{M}_7) \otimes H_1(S^3/\Gamma_{\mathfrak{g}}) = H_2(\mathcal{M}_7) \otimes \Gamma_{\mathfrak{g}}^{\mathsf{ab}} \\ = H_2(\mathcal{M}_7; \Gamma_{\mathfrak{g}}^{\mathsf{ab}})\,.\end{aligned}$$

and similarly

$$\mathrm{Tor}(H_6(\mathcal{M}_7 \times S^3/\Gamma_{\mathfrak{g}})) = H_5(\mathcal{M}_7; \Gamma_{\mathfrak{g}}^{\mathsf{ab}})\,.$$

Given elements $\sigma_a = a \otimes \ell_a$, $\sigma_b = b \otimes \ell_b$, we have the linking form

$$\mathsf{L}(\sigma_a, \sigma_b) = (a \cdot b)\, \mathsf{L}_{S^3/\Gamma_{\mathfrak{g}}}(\ell_a, \ell_b)\,.$$

Back to M-theory on $\mathbb{C}^2/\Gamma_{\mathfrak{g}}$

It is not difficult to compute the linking form on $S^3/\Gamma_{\mathfrak{g}}$, we find:

$\Gamma_{\mathfrak{g}}$	$G_{\mathfrak{g}}$	$\Gamma_{\mathfrak{g}}^{\mathsf{ab}}$	$\mathsf{L}_{\mathfrak{g}}$
$\mathbb{Z}_N$	$SU(N)$	$\mathbb{Z}_N$	$\frac{1}{N}$
$\mathrm{Dic}_{(4N-2)}$	$\mathrm{Spin}(8N)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathrm{Dic}_{(4N-1)}$	$\mathrm{Spin}(8N+2)$	$\mathbb{Z}_4$	$\frac{3}{4}$
$\mathrm{Dic}_{(4N)}$	$\mathrm{Spin}(8N+4)$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\mathrm{Dic}_{(4N+1)}$	$\mathrm{Spin}(8N+6)$	$\mathbb{Z}_4$	$\frac{1}{4}$
$2T$	E_6	$\mathbb{Z}_3$	$\frac{2}{3}$
$2O$	E_7	$\mathbb{Z}_2$	$\frac{1}{2}$
$2I$	E_8	0	0

Back to M-theory on $\mathbb{C}^2/\Gamma_{\mathfrak{g}}$

Classification

The possible global forms of the $d = 7$ theories on $\mathcal{M}_7$ are given by maximal commuting subspaces of $H_2(\mathcal{M}_7; \Gamma_{\mathfrak{g}}^{\mathsf{ab}}) \times H_5(\mathcal{M}_7; \Gamma_{\mathfrak{g}}^{\mathsf{ab}})$, with commutators as above.

This result agrees with what one obtains from applying the ideas in [Gaiotto, Moore, Neitzke '10], [Aharony, Seiberg, Tachikawa '13].

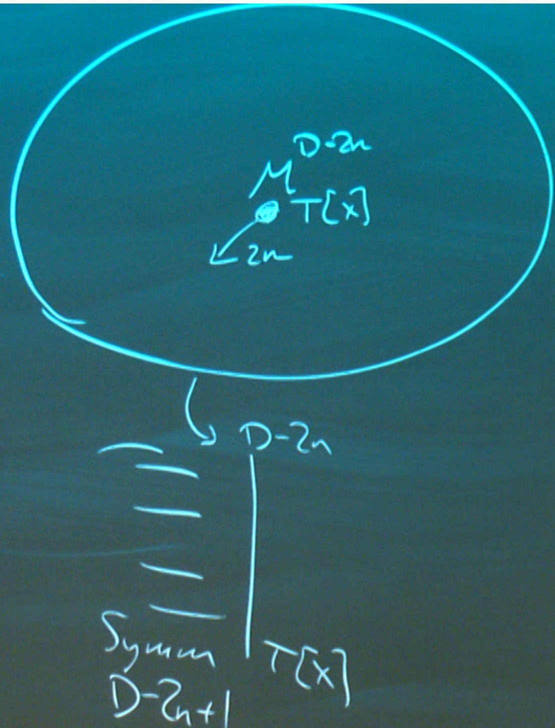
Screened charges and non-locality of defects

An alternative derivation of this result can be obtained by thinking about screening of line operators, closely following [Aharony, Seiberg, Tachikawa '13]. This was done in geometric language in [Del Zotto, Heckman, Park, Rudelius '15], where they introduce the *defect group*, which in this case is

$$\mathbb{D} = \frac{H_2(\mathbb{C}^2/\Gamma_{\mathfrak{g}}, S^3/\Gamma_{\mathfrak{g}})}{H_2(\mathbb{C}^2/\Gamma_{\mathfrak{g}})} \times \frac{H_2(\mathbb{C}^2/\Gamma_{\mathfrak{g}}, S^3/\Gamma_{\mathfrak{g}})}{H_2(\mathbb{C}^2/\Gamma_{\mathfrak{g}})}$$

It is easy to show that

$$\frac{H_2(\mathbb{C}^2/\Gamma_{\mathfrak{g}}, S^3/\Gamma_{\mathfrak{g}})}{H_2(\mathbb{C}^2/\Gamma_{\mathfrak{g}})} = H_1(S^3/\Gamma_{\mathfrak{g}}) = \Gamma_{\mathfrak{g}}^{\mathsf{ab}} \,.$$



$$\psi(x) = e^{ipx}$$

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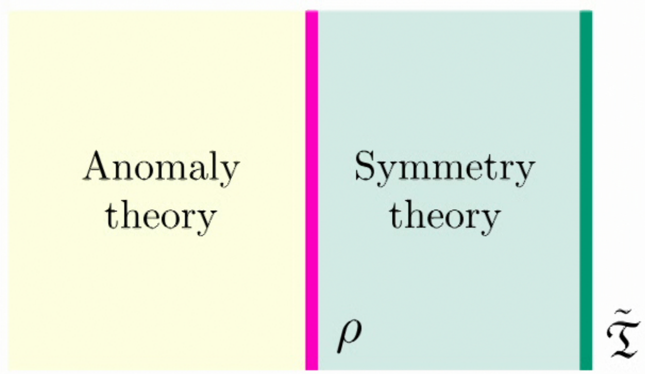
Relative theories

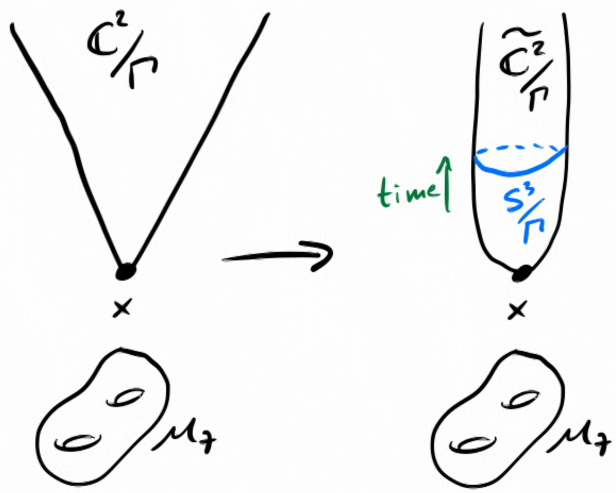
A high level summary of the previous discussion is that in geometric engineering we have something like a "QFT on a singularity relative to the string theory bulk": the full theory is only defined only after specifying boundary values for the supergravity fields, even in the deep IR limit where dynamical excitations for the bulk decouple.

In general this relates a $D-2n$-dimensional field theory to a D-dimensional supergravity bulk, with $n > 1$. I would now like to relate this picture to the better understood notion of relative QFTs by Freed and Teleman:



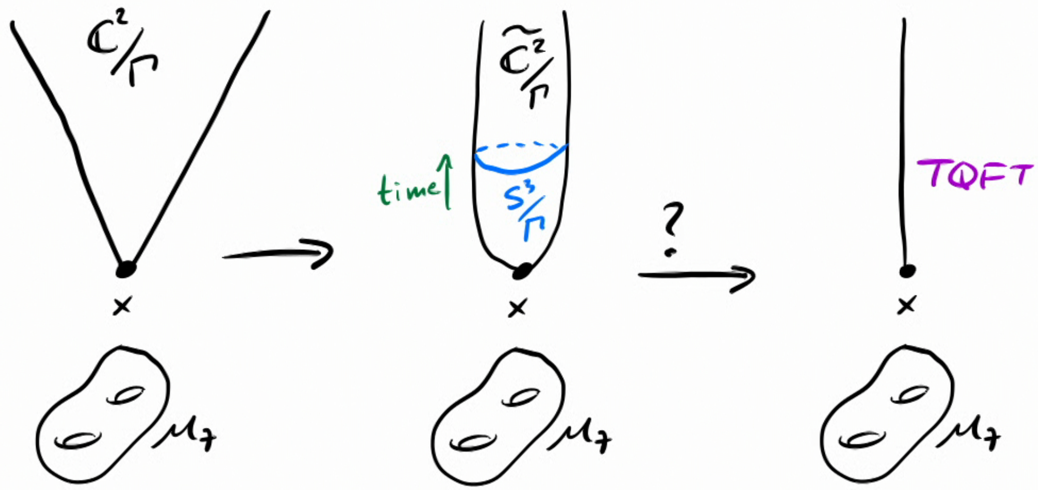
How symmetry theories appear in string theory

The derivation in [IGE, Heidenreich, Regalado '19] uses a modified asymptotic structure.



How symmetry theories appear in string theory

The derivation in [IGE, Heidenreich, Regalado '19] uses a modified asymptotic structure. This suggests a strategy for deriving the symmetry theory associated to the field theory: dimensional reduction on the link of the singularity: [Apruzzi, Bonetti, IGE, Hosseini, S. Schäfer-Nameki '21]



The anomaly theory

As an example, for 5d SCFTs the resulting symmetry theory is:

$$S_{\mathsf{Sym}} = \int_{\mathcal{W}_6} \Big(K_{ij} B_2^{(i)} \cup \delta C_3^{(j)} + \Omega_{ijk} B_2^{(i)} \cup B_2^{(j)} \cup B_2^{(k)} + \Upsilon_{ij\alpha} B_2^{(i)} \cup B_2^{(j)} \cup F_2^{(\alpha)} \Big)$$

where the K, Ω, Υ coefficients are classical spin-Chern-Simons invariants on the (5d) link.

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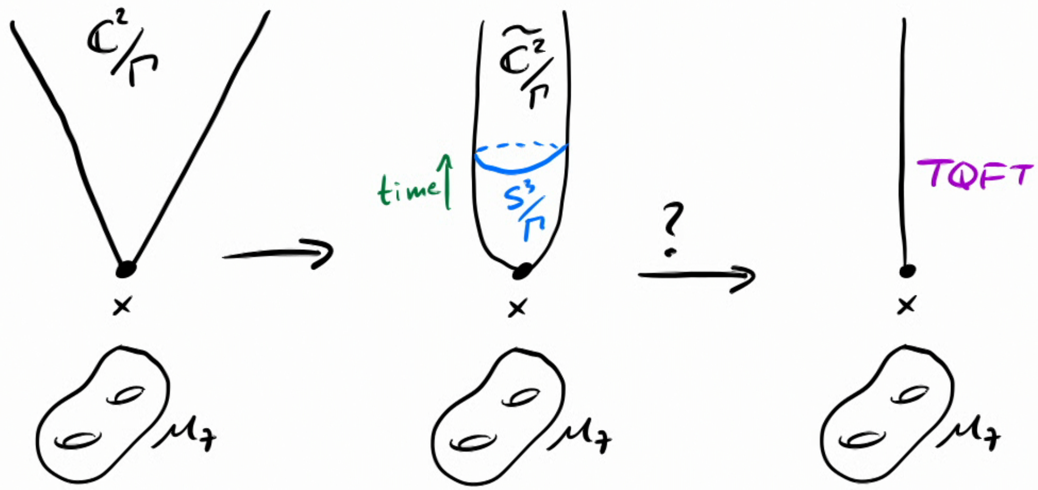
The derivation in [IGE, Heidenreich, Regalado '19] uses a modified asymptotic structure. This suggests a strategy for deriving the symmetry theory associated to the field theory: dimensional reduction on the link of the singularity: [Apruzzi, Bonetti, IGE, Hosseini, S. Schäfer-Nameki '21]



In this picture the boundary conditions at infinity that we need to specify in string theory correspond to ρ, so the object that arises naturally is the symmetry theory. ("Symmetry inflow" instead of "anomaly inflow".)

The anomaly theory

As an example, for 5d SCFTs the resulting symmetry theory is:

$$S_{\mathsf{Sym}} = \int_{\mathcal{W}_6} \Big(K_{ij} B_2^{(i)} \cup \delta C_3^{(j)} + \Omega_{ijk} B_2^{(i)} \cup B_2^{(j)} \cup B_2^{(k)} + \Upsilon_{ij\alpha} B_2^{(i)} \cup B_2^{(j)} \cup F_2^{(\alpha)} \Big)$$

where the K, Ω, Υ coefficients are classical spin-Chern-Simons invariants on the (5d) link. We can compute these geometrically using differential cohomology (see also [Cvetič, Dierigl, Lin, Zhang '21]), and in cases where there is a geometric interpretation we can compare against field theory predictions. For instance, for $SU(p)_q$ we get

$$K_{11} = \gcd(p,q) \; ; \; \Omega_{111} = \frac{q\,p\,(p-1)\,(p-2)}{6\gcd(p,q)^3} \; ; \; \Upsilon_{111} = \frac{p\,(p-1)}{2\gcd(p,q)^2}$$

in agreement with [Gukov, Pei, Hsin '20].

The BF theory

See Federico's talk tomorrow and Saghar's poster/gong-show talk for the details of how to derive the anomaly part of the symmetry theory.

The BF theory

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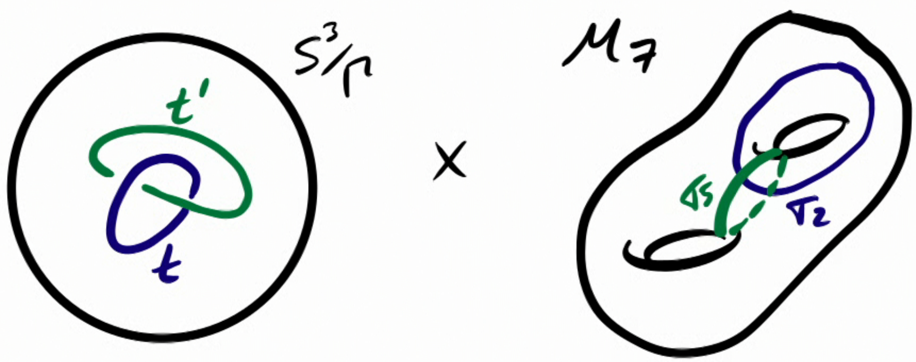
In the full theory on $S^3/\Gamma \times X^8$ there are non-commuting flux operators [Freed, Moore, Segal '06] wrapping $t \times \sigma_2$ and $t' \times \sigma_5$, with $t, t' \in H_1(S^3/\Gamma) = \Gamma^{\text{ab}}$ and $\sigma_i \in H_i(X^8)$. Their commutation relations (on a spatial slice $\mathcal{M}_7$ of X^8) are

$$\Phi(t \times \sigma_2)\Phi(t' \times \sigma_5) = e^{2\pi i L(t,t')\sigma_2 \cdot \sigma_5}\Phi(t' \times \sigma_5)\Phi(t \times \sigma_2)\,.$$



2-groups

So far I have discussed higher form symmetries only, but the same picture is expected to hold for the full symmetry structure of the geometrically engineered theories.

2-groups

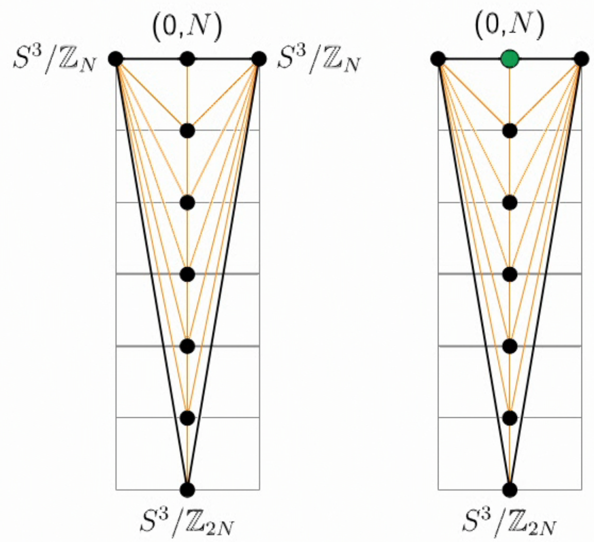
Consider a theory with discrete 1-form symmetry $\Gamma^{(1)}$ and a connected continuous global 0-form symmetry $\mathscr{F}^{(0)} = F/C$, where F is the simply connected form of $\mathscr{F}^{(0)}$. We define $\mathscr{F}^{(0)}$ to be the symmetry acting faithfully on local operators, or equivalently the most general structure group that we can take for 0-form backgrounds.


Symm
D-2n+1

Computation of 2-groups

The problem of computing 2-groups therefore reduces to the computation of $H_1(L^d)$ and $H_1(L^d - S)$, for L^d the base of the toric Calabi-Yau cone, and S a neighbourhood of the singular locus in L^d.

Thanks to toric technology developed in [IGE, Heidenreich '16] this can be done easily by counting triangles. For instance, for $SU(N)_N$



Conclusions

For geometrically engineered theories there is a close connection between the symmetries of a theory and the geometry. But crucially, the symmetries are often much easier to understand than the field theory itself.

I have focused on the developments I understand best. There is a lot of recent literature developing complementary approaches, and older related literature on anomaly inflow. (Works by Andrea, Dewi, Federico, Ibou, Lakshya, Mathew, Marieke, Michele, Saghar, Sakura, and many others)

We don't quite have a full systematic dictionary yet, but the general picture is gradually becoming clear.