

Title: A (kind of) monoidal localization theorem for the small quantum group

Speakers: Cris Negron

Collection: Global Categorical Symmetries

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Abstract: " I will talk about a monoidal localization theorem for the small quantum group $u_q(G)$, where G is a reductive algebraic group and q is a root of unity. In joint work with Julia Pevtsova, we show that the category of representations for $u_q(G)$ admits a fully faithful tensor embedding into the category of coherent sheaves over a "quantum" flag variety. This quantum flag variety is, essentially, some finitely fibered space over the classical flag variety G/B . I will explain how this embedding theorem codifies certain relationships between the small quantum group and its quantum Borels.
"

A kind of monoidal localization theorem for the
small quantum group.

C. Negrón (W.J. Tenebrera) 22.03.10764

Δ (kind of monoidal localization) then for the
small quantum group.

C. Negrón (w/ J. Reutsov) 2203.10764

- Reminder on $G_q / u_1(G)$
- Promise of the Pavele
- Constructing the Pavele
- The total space + localization

Δ (kind of monoidal localization) then for the
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- Reminder on $G_q / u_1(\mathbb{Q})$
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char 0

$$K = \bar{K} \quad (k = \mathbb{C})$$

$$g \in \mathbb{C}^{\times} \text{ of } (f_n)$$

order of

Δ (kind of monoidal localization) then after the Reminder on G_q
small quantum group.

C. Negrón (w/ J. Reutsov) 22.03.10764

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$g \in \mathbb{P}^X$ of (f_n)

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Reminder on G_q

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Take $G = \text{almost-simple}/\mathbb{C}$
(Just think $G = SL(n, \mathbb{C})$)

$G = \prod_{\lambda} \bar{H}_{\lambda}$ form of G

char 0

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order of

ration

han. Turn off the

Reminder on G_q

2203.10764

char 0

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$(\cdot)$

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φ

Take $G = \text{almost-simple}/\mathbb{Q}$
(Just think $G = \text{SL}(n, \mathbb{Q})$)

$G = \prod_{\lambda} G_{\lambda}$ form of G . We can

think of G_q in two ways:

Version I) G_q is a char 0 lift of
 $G/\mathbb{F}_q$. More precisely, rep G_q are

in order the

Reminder on G_q

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($k = \mathbb{C}$)
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Take G = almost-simple / $\mathbb{C}$
(Just think $G = SL(n, \mathbb{C})$)

$G = \overline{\mathbb{F}_q}$ form of G . We can

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Version I) G_q is a char 0 lift of
 $G/\overline{\mathbb{F}_q}$. More precisely, $\text{rep } G_q$ are
char 0 lift of $\text{rep } G$.

Version II) $\text{rep } G_q$ is a
braided defo. of the symm mon. cat

$\text{rep } G / \mathbb{C}$



in for the

Reminder on G_q

10764

($k = \mathbb{C}$)
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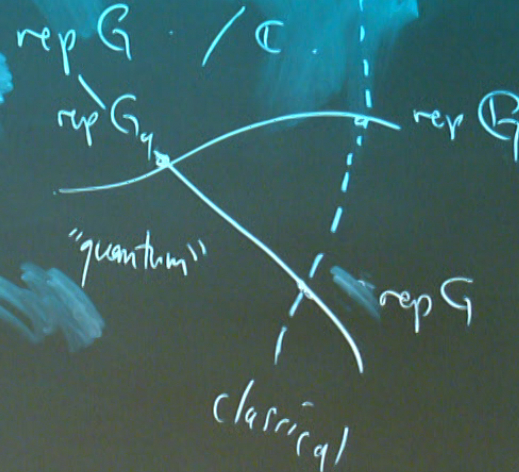
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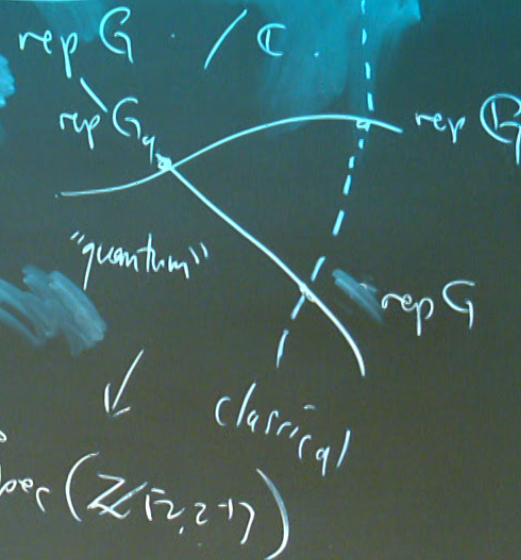
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Have at $\text{ord}(q^2) = l$, a big 'classical' part

$$\text{rep } G \xrightarrow{q\text{-Fr}} \text{rep } G_q \xrightarrow{\quad} \text{rep}_{\mathbb{F}_q}(G)$$

this map has a kernel called the small
quantum group

all 'quantum' info
for $\text{rep } G_q$

Reminder on G_q

Take $G = \text{almost-}$
(Just think $G = \text{SL}(n, \mathbb{C})$)
 $G = \overline{\text{IF}}_q$ form of G .
Think of G_q in two w
Version I) G_q is a char of
 $G/\overline{\text{IF}}_q$. More precisely, rep
char 0 lift of $\text{rep } G$.

Have at $\text{ord}(q^2) = l$, a big 'classical' part

$$\text{rep } G \xrightarrow{q-\text{Fr}} \text{rep } G_q \rightarrow \text{rep } u_q(G)$$

This map has a kernel called the small quantum group

$$u_q(SL_2) = \{ \text{all 'quantum' info for rep } G_q \}$$

$$\text{Ex: } \mathbb{C}\langle E, F, K \rangle$$

$$(E^l, F^l, K^l - 1, K E K^{-1} = q^2 E$$

$$R = \left(\sum_{i=0}^{l-1} \text{coeff}(q, i) E^i \otimes F^{l-i} \right) q^{\text{cilling}}$$

Reminder on G_q

Take G = almost -
(Just think $G = SL_n$)
 $G = \overline{H}_1$ form of G .
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Version I) G_q is a char of
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Reminder on G_q

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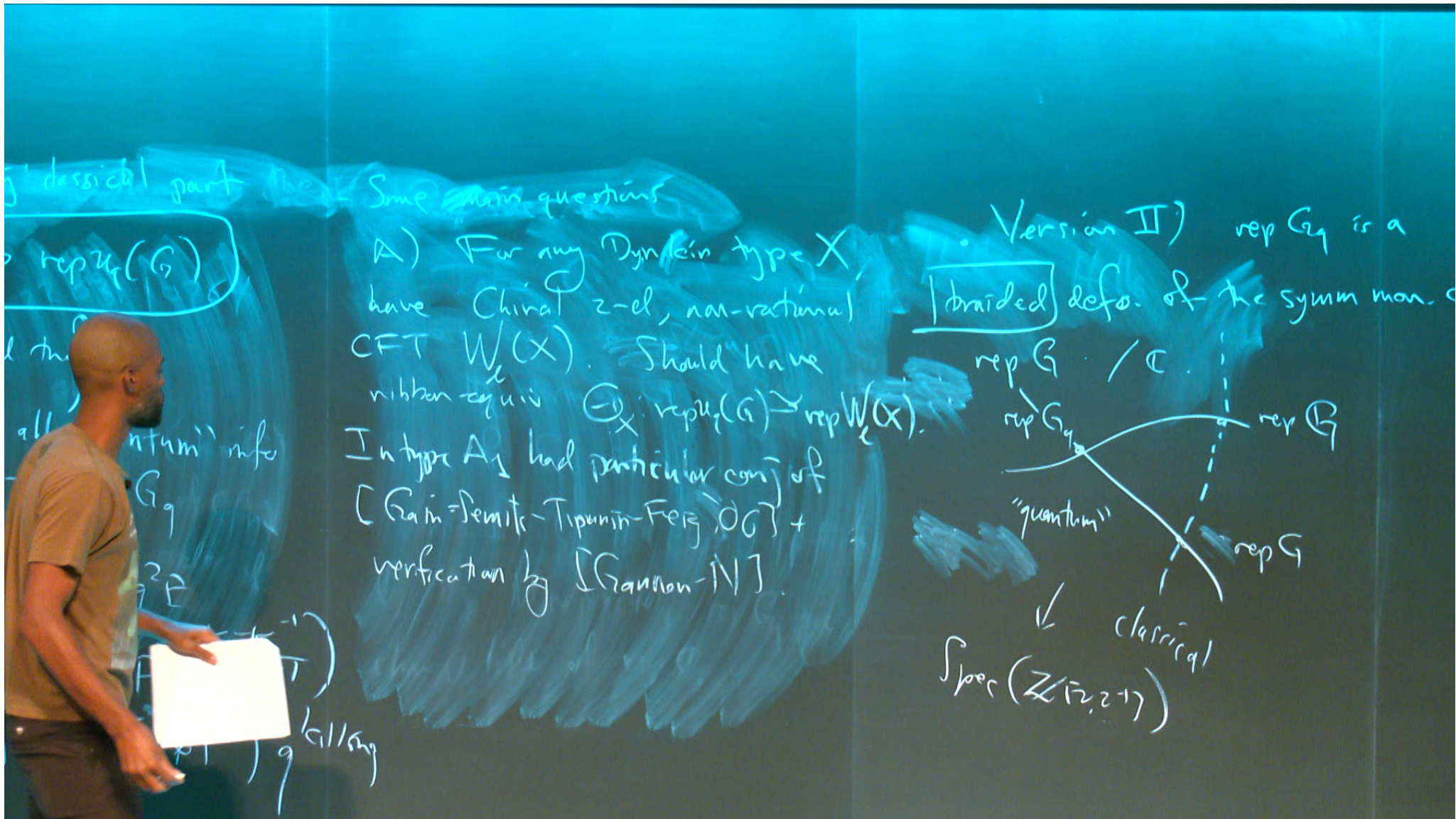
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classical part then

$\text{rep}_q(G)$

all 'quantum' info

G_q

$q^2 E$

(T^{-1})

g killing

Some main questions

A) For any Dynkin type X , have Chiral 2-d, non-rational CFT $W(X)$. Should have ribbon equiv $\oplus_X \text{rep}_q(G) \rightarrow \text{rep} W(X)$.

In type A_1 had particular conj of $[G_{\text{ain}} = \text{Fermi-Timurin-Ferj } OG] + \text{verification by [Gannon-N]}$.

Version II) $\text{rep } G_q$ is a braided defo. of the symm mon. $\text{rep } G$.

$\text{rep } G \quad / \quad \mathbb{C}$

$\text{rep } G_q$

"quantum"

classical

$\text{Spec}(\mathbb{Z}[2, 2+1])$

main questions

For any Dynkin type X ,

Chiral 2-d, non-rational

$V(X)$. Should have

equiv $\bigoplus_X \text{rep}_q(G) \rightarrow \text{rep} W_q(X)$.

A_1 had particular conj of

equiv-Terwilliger-Frobenius $\{OG\} +$

an by [Gannon-N].

B) Should be a "non-compact"
3-d TQFT w/ circle value

$Z(S^1) = D^b(\mathcal{U}_q(G))$. This
would be the "bulk theory" to (A).
Need to construct ...

Spec

main questions

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had particular conj of

-Turaev-Frenkel, DG +

[Gannon-N]

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$Z(S^1) = D^b(\mathcal{U}_q(G))$. This

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Need to construct ...

- Today won't deal dir. w/ these

but speak on some (current?) infrastructure

for $\mathcal{U}_q(G)$.

The promise of the Borels

For G have Borel(s) $B \subseteq G$.

E.g. $SL(n, \mathbb{C})$ has upper Δ in mat

Some main questions

A) For any Dynkin type X ,
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$$SL(n) \begin{bmatrix} & & \\ & B & \\ & & \end{bmatrix}$$

Jordan normal form says all elem in G lie in some Borel.

(Σ)

Some main questions

A) For any Dynkin type X , have Chiral 2-d, non-rational CFT $W(X)$. Should have ribbon equiv $\oplus_X \text{rep}_\ell(G) \rightarrow \text{rep} W_\ell(X)$.

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$$SL(n) \begin{bmatrix} & & \\ & B & \\ & & \end{bmatrix}$$

Jordan normal form says all elem in G lie in some Borel.

Practical $\Rightarrow$ "version" of reducing to the Borel

($\leq$)

Some main questions

A) For any Dynkin type X , have Chiral 2-d, non-rational CFT $W(X)$. Should have ribbon equiv $\oplus_X \text{rep}_\ell(G) \rightarrow \text{rep } W_\ell(X)$. In type A_1 had particular conj of $[G_{\text{irr}} = \text{Semis} - \text{Tipunin-Fierz}, OG] +$ verification by [Gannon-N].

Borel's

Borel(s) $B \subseteq G$.

has upper Δ in mat

Jordan normal

Arn says all

in G lie in
ant Borel.

Far more intricate ex,
(Arb-Roz-Ginz p3) rays
have a Δ equiv

$$D^b(G_q)_{\text{principal}} \xrightarrow{\text{block}} D^b(N)_{\text{coh}}^G$$

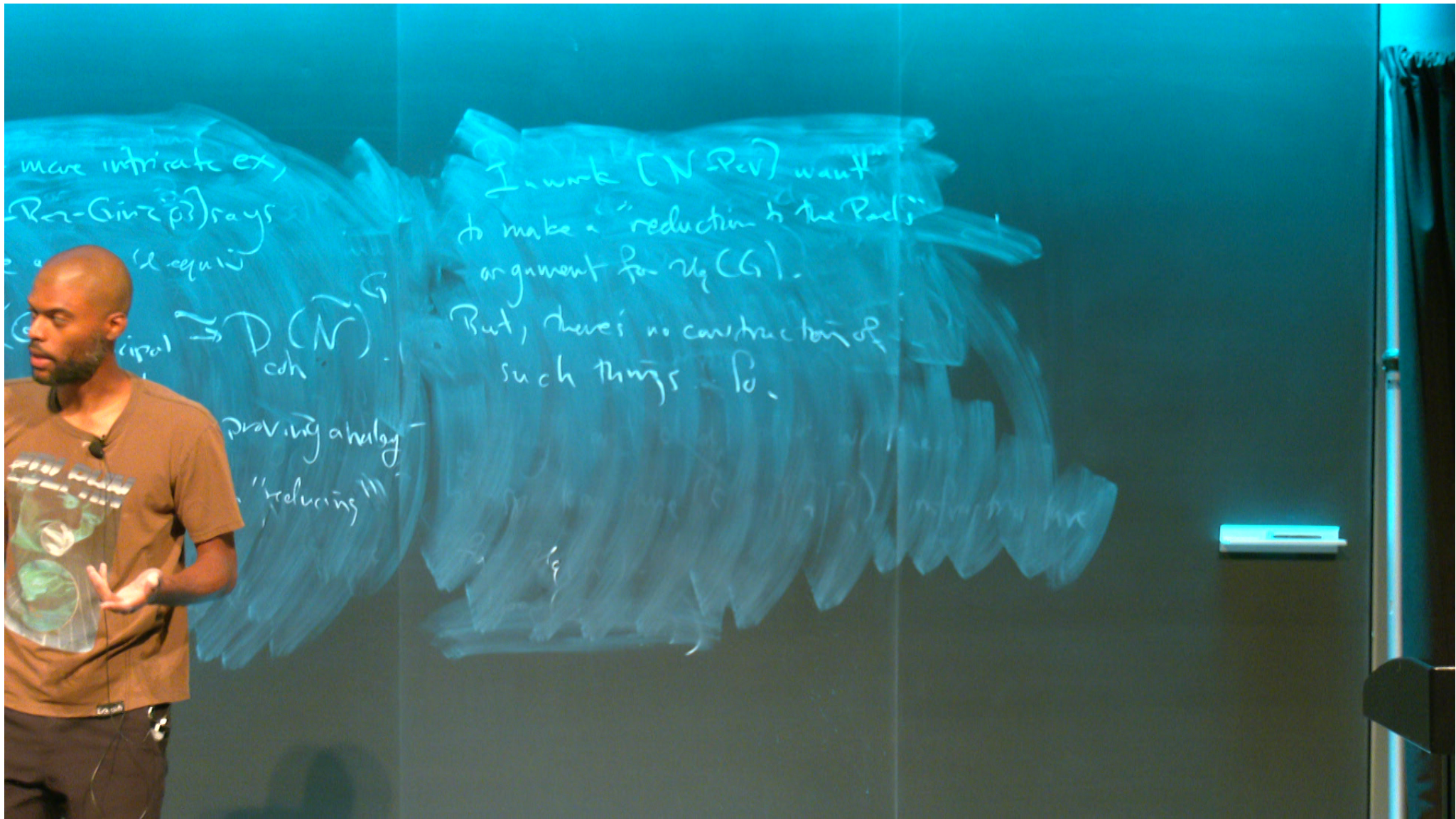
The proof follows by proving a key
result for P_q , then "reducing"
to Borel.

B) Should be a "non-compact"
3-d TQFT w/ circle value

$Z(S^1) = D^b(\mathcal{U}_q(G))$. This
would be the "bulk theory" to (A).

(X). Need to construct...

Today want deal dir. w/ these
but speak on same (correcting/?) infra
for $\mathcal{U}_q(G)$.



more intricate ex,
 $\text{Per-Ginz}(\mathfrak{p})$ rays
 Δ (equi)

(G_q) principal $\rightarrow D(N)$
 block coh

proof follows by proving analogy
 H for $\mathfrak{P}_q$, then "reducing"
 Barel.

I think [N-Per] want
 to make a "reduction to the Pael's"
 argument for $\mathcal{U}_q(G)$.

But, there is no construction of
 such things. $\mathfrak{P}_0$.

$$\mathcal{U}_q(q) = (P \in E_\alpha, F_\alpha, K_\alpha^{-1}; \alpha \text{ simple})$$

the Borels

are Borel(s) $B \subseteq G$.

n, C has upper Δ in mat

B Jordan normal

form says all
elem in G lie in
some B orel.

time of reducing
to the Borel

$$\begin{array}{c} B_{\text{ad}} \quad T_{\text{ad}} \\ \downarrow \\ G(\mathbb{C}) \hookrightarrow G \rightarrow G^{(1)} \end{array}$$

Far more intricate ex,
(Art-Ree-Ginzburg) says
have a Δ equiv

$$D^b(G_q)_{\text{principal block}} \xrightarrow{\sim} D^b(N)_{\text{coh}}^G$$

The proof follows by proving analog
result for P_q , then "reducing"
to Borel.

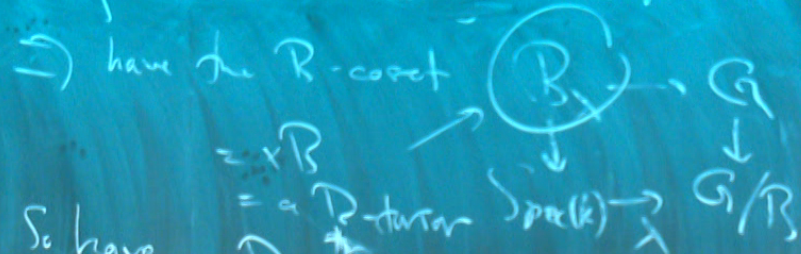
I think [N-Rev] want
to make a "reduction to the Borel"
argument for $\mathcal{U}_q(G)$.

But, there is no construction of
such things. So.

$$\mathcal{U}_q(\mathfrak{g}) = (P(E_q, F_q, K_q^{-1}; \alpha, \beta))$$

Δ (Small) quantum Borel, according (N-Red)

Def¹: Consider a pt $x \in \text{Spec}(k) \rightarrow G/B$



So have

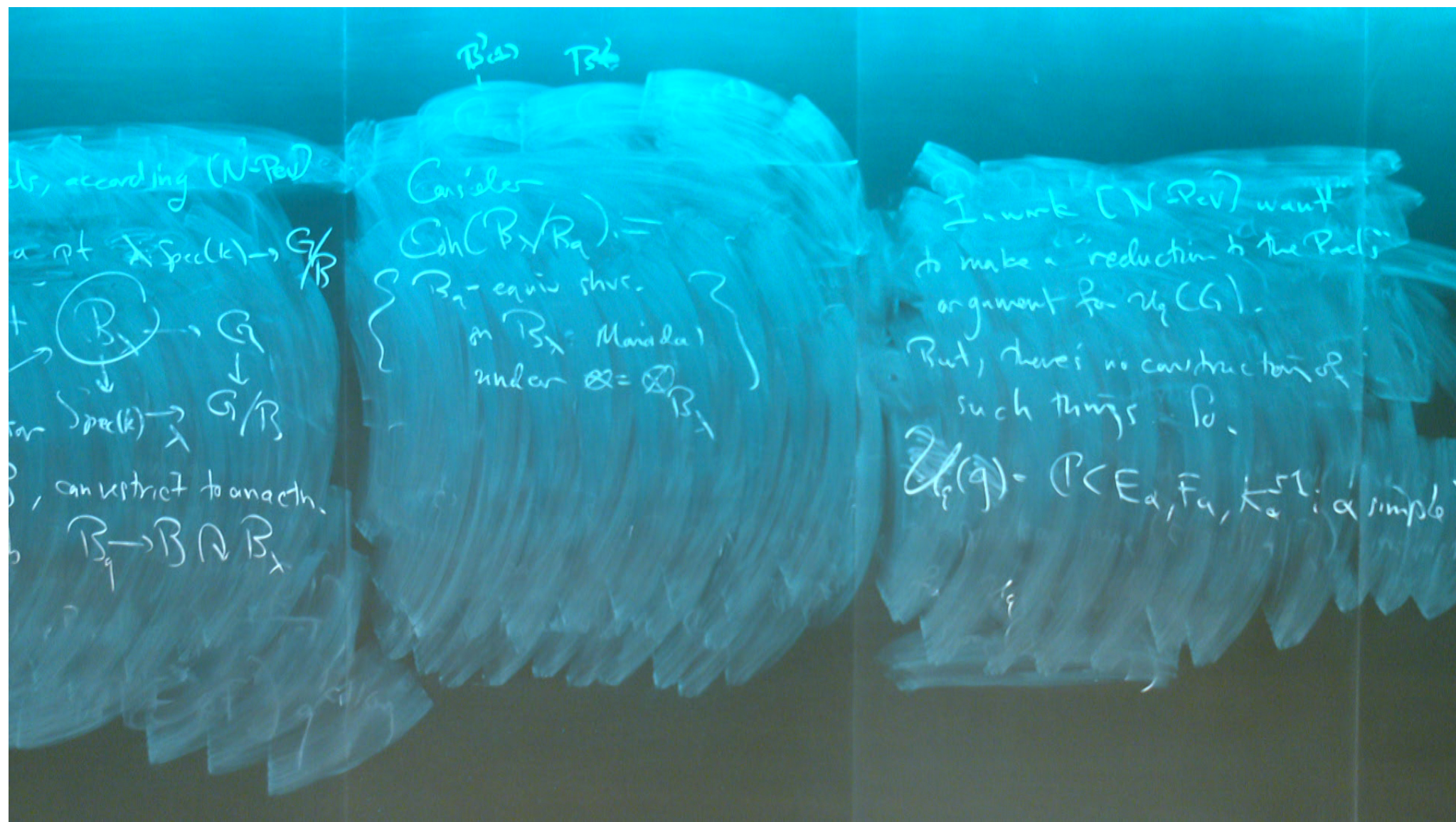
$B_x \cap B$, can restrict to an action of B_q via q -Frob $B_q \rightarrow B \supset B_x$

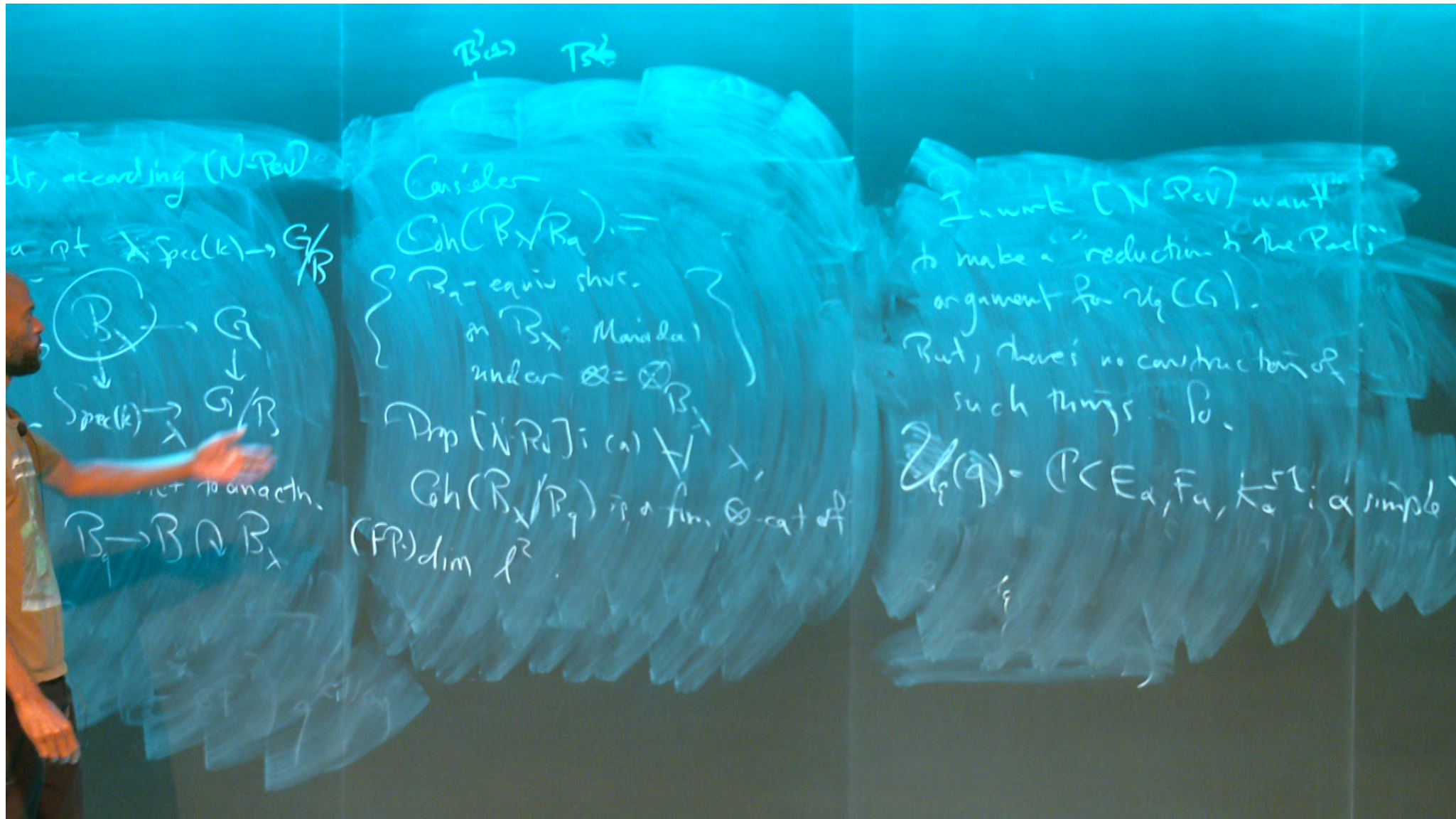
$$\begin{array}{ccc} B_x & & T_x \\ \downarrow & & \downarrow \\ G/B & \hookrightarrow & G \rightarrow G^{(1)} \end{array}$$

Far more intricate ex, (Art-Re-Ginz'03) rays have a Δ (2 equiv)

$$D^b(G_q)_{\text{principal}} \xrightarrow{\sim} D^b(A)_{\text{coh}}$$

The proof follows by proving result for B_q , then "reduce to Borel".





dr, according (N-Rev)

$$\alpha: \text{pt } \lambda: \text{Spec}(k) \rightarrow G/B$$

$$\begin{array}{ccc} \text{pt } \lambda & \xrightarrow{\alpha} & G/B \\ \downarrow & & \downarrow \\ \text{Spec}(k) & \xrightarrow{\lambda} & G/B \end{array}$$

, can restrict to an action.

$$B_q \rightarrow B \cap B_\lambda$$

B_λ T_λ

Consider

$$\text{Coh}(B_\lambda/B_q) :=$$

$$\left\{ \begin{array}{l} B_q \text{ equiv shvs.} \\ \text{on } B_\lambda: \text{Maurice} \\ \text{under } \mathcal{O} = \mathcal{O}_{B_q} \end{array} \right\}$$

$$\text{Prop } [N\text{-Rev}] \text{ (a) } \forall \lambda,$$

$$\text{Coh}(B_\lambda/B_q) \text{ is a fin. } \mathcal{O}\text{-cat of (FP) dim } l^2$$

$$(b) \forall \lambda \text{ have canonical, central, inv. J.}$$

$$\mathcal{O}\text{-fun } \mathcal{O}_{B_\lambda}: \text{rep } \mathcal{U}_\lambda(G) \rightarrow \text{Coh}(B_\lambda/B_q)$$

I think [N-Rev] want to make a "reduction to the Paoletti" argument for $\mathcal{U}_\lambda(G)$.

But, there is no construction of such things. Po.

$$\mathcal{U}_\lambda(G) = (P \in E_\alpha, F_\alpha, K_\alpha^{-1}; \alpha \text{ simple})$$

B_n T36

consider
 $\text{Cat}(B_n/R_n) :=$

B_n -equiv shv.
 on B_n : Manifold
 under $\otimes = \otimes_{B_n}$

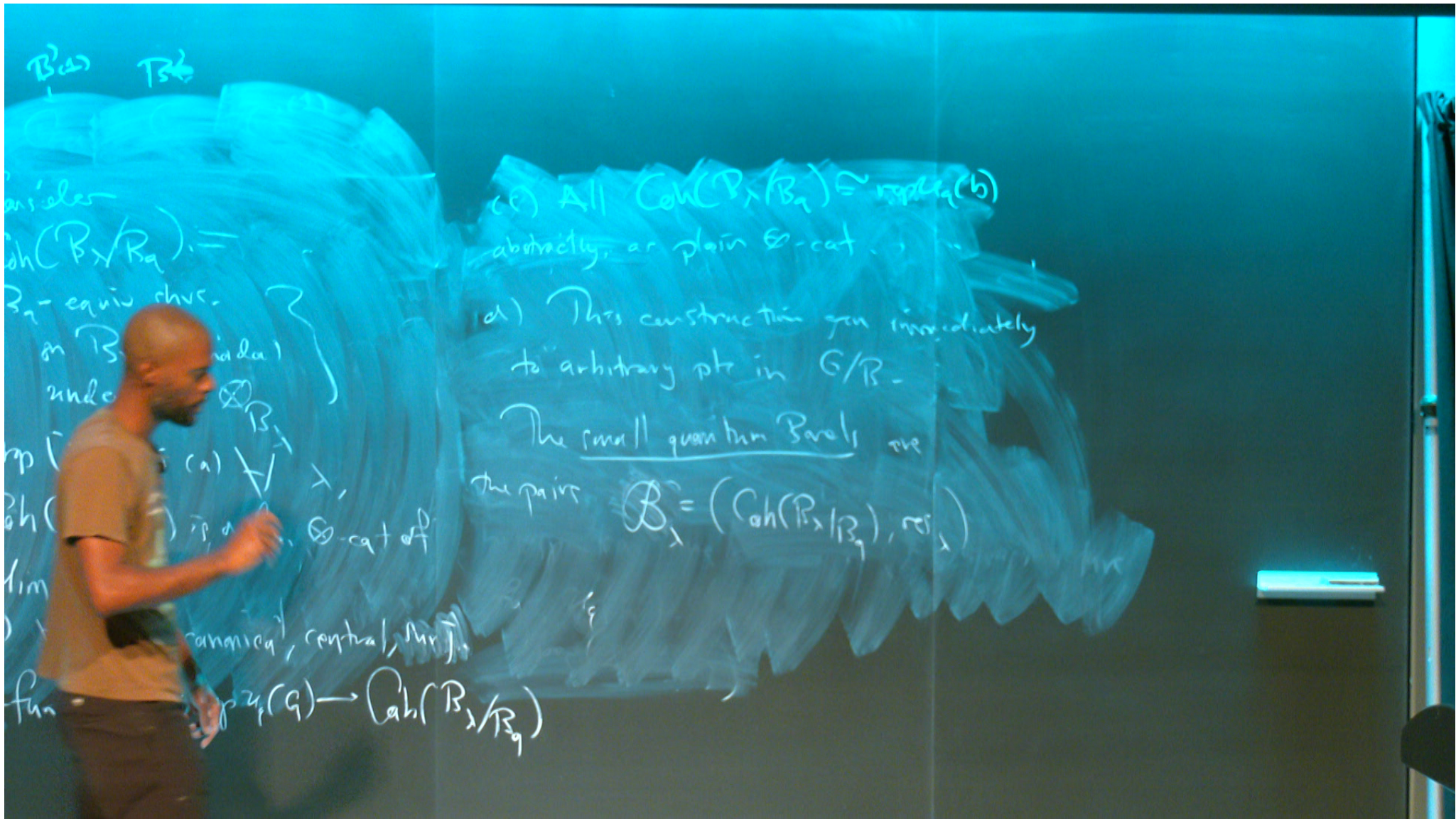
$\text{rep}(N(R_n))$: (a) $\forall \lambda$,

$\text{Cat}(B_n/R_n)$ is a fin. $\otimes$ -cat of
 dim ℓ^2

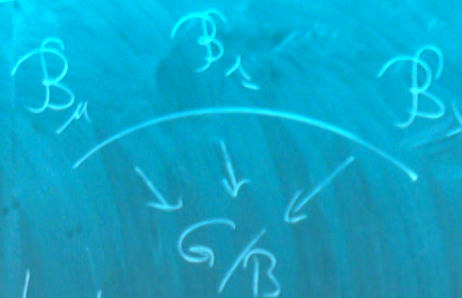
(b) $\forall \lambda$ have canonical, central, surj.

fun $\otimes_{B_n} : \text{rep } \mathcal{U}_\ell(\mathfrak{g}) \rightarrow \text{Cat}(B_n/R_n)$

(c) All $\text{Cat}(B_n/R_n) \cong \text{rep } \mathcal{U}_\ell(\mathfrak{g})$
 abstractly, as plain $\otimes$ -cat.



What I have then is a flat family



which has a total space

$$\mathcal{B}_{\text{tot}} = \text{Coh}(G/B)$$

(5)

$\mathcal{B}_{\text{tot}}$ $\mathcal{T}_{\mathcal{B}}$

Consider

$$\text{Coh}(\mathcal{B}_\lambda / \mathcal{R}_\lambda) :=$$

$\left\{ \begin{array}{l} \mathcal{R}_\lambda \text{ - equiv shv.} \\ \text{on } \mathcal{B}_\lambda \text{ - Manifolds} \\ \text{under } \otimes = \otimes_{\mathcal{R}_\lambda} \end{array} \right\}$

$\text{Prop } [N, \mathcal{R}_\lambda] : (a) \forall \lambda >$

$\text{Coh}(\mathcal{B}_\lambda / \mathcal{R}_\lambda)$ is a fin. $\otimes$ -
(FP) dim ℓ^2 .

(b) $\forall \lambda$ have canonical, canon.

$\otimes$ -fun $\otimes_{\mathcal{R}_\lambda} : \text{rep } \mathcal{Z}_\ell(G) \rightarrow$

I have then a flat family



hard total space

$$\mathcal{B}_{\text{tot}} = \text{Coh}(G/B)$$

$\mathcal{B}_{\text{tot}}$ TSC

Then (VPR-3):

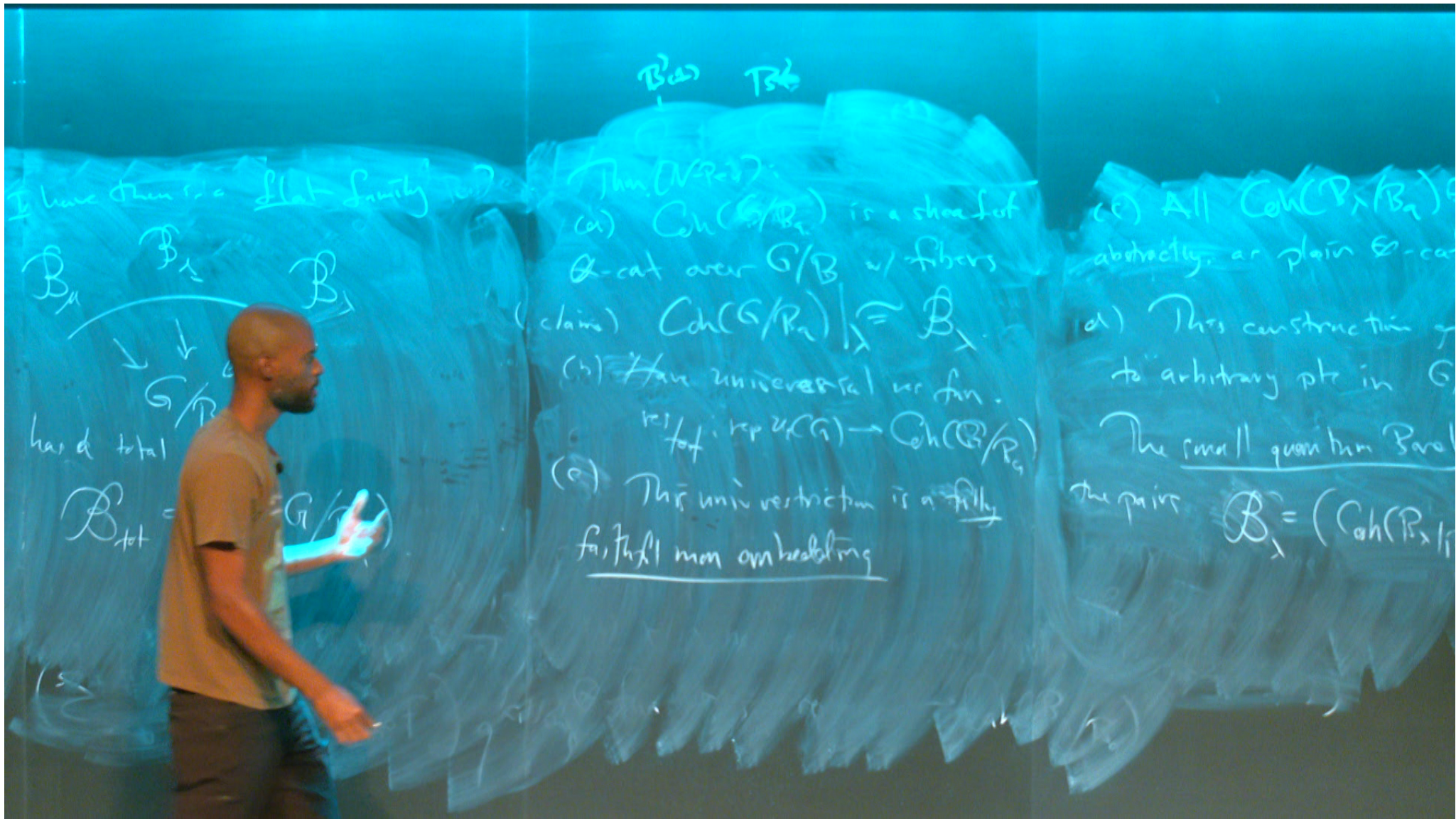
(a) $\text{Coh}(G/B)$ is a sheaf of $\mathcal{O}$ -cat over G/B w/ fibers

$$\text{Coh}(G/B)|_X \cong \mathcal{B}_X$$

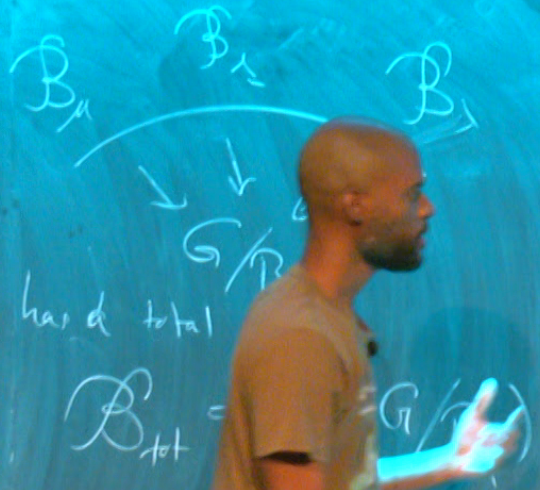
(c) All $\text{Coh}(P_x/B_x)$ abstractly, as plain $\mathcal{O}$ -cat

(d) This construction goes to arbitrary pts in G

The small quantum Borel
the pair $\mathcal{B}_X = (\text{Coh}(P_x/B_x), \dots)$



I have then a flat family



$\mathcal{B}_{\text{tot}}$ T36

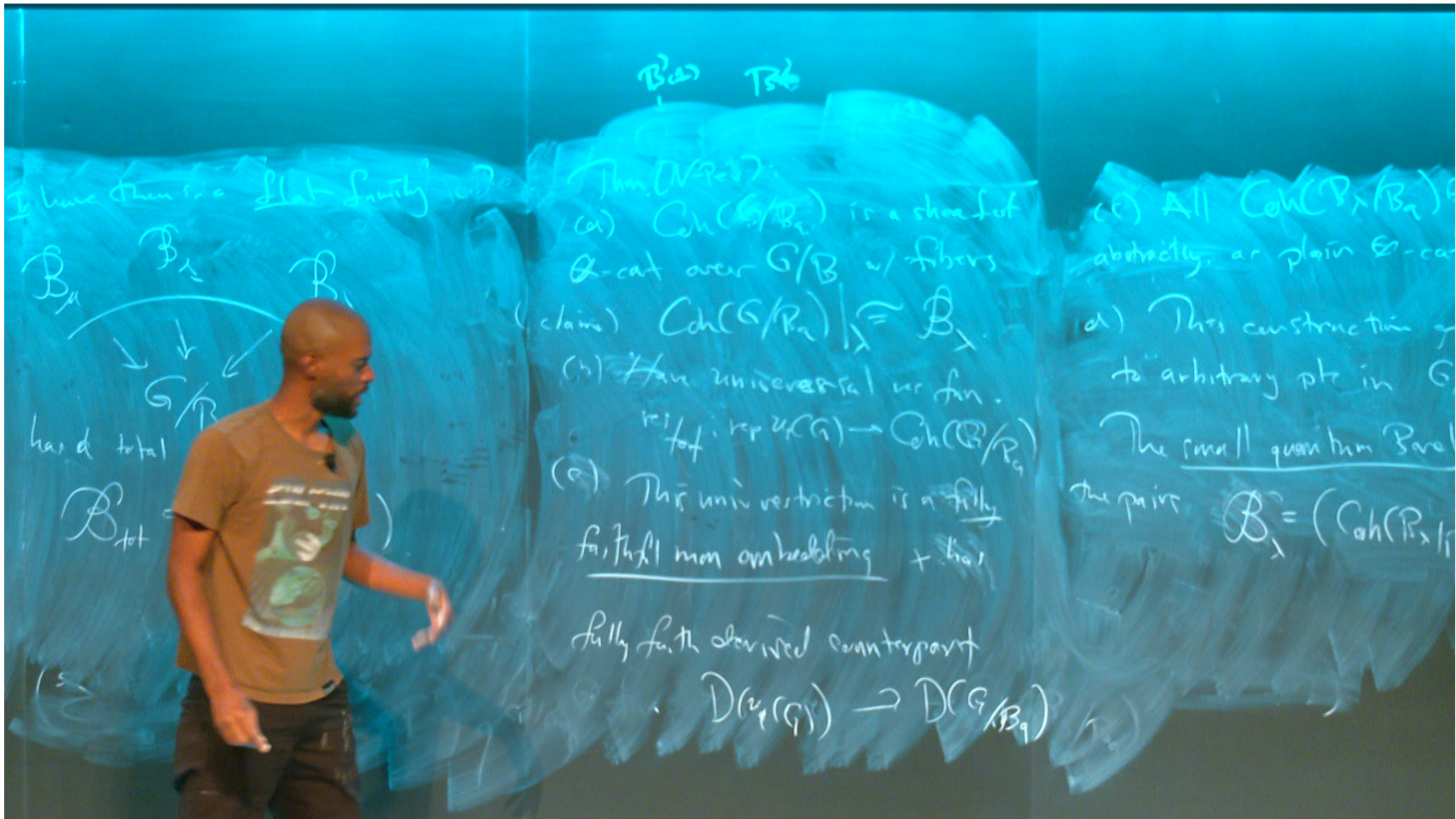
Then (V9-3):

- (a) $\text{Coh}(G/B)$ is a sheaf of $\mathcal{O}$ -cat over G/B w/ fibers
- (claim) $\text{Coh}(G/B)|_{\lambda} \cong \mathcal{B}_{\lambda}$
- (b) ~~Have~~ universal rep for $\text{rep } U(G) \rightarrow \text{Coh}(G/B)$
- (c) This universal restriction is a fully faithful monoidal embedding

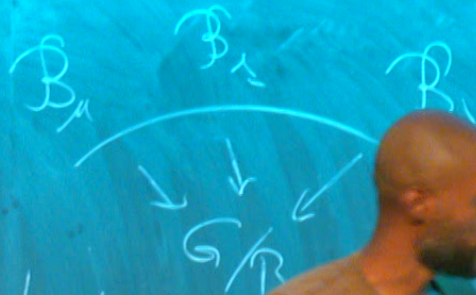
(c) All $\text{Coh}(P_{\lambda}/B_{\lambda})$ abstractly, as plain $\mathcal{O}$ -cat

(d) This construction goes to arbitrary pts in G

The small quantum Borel the pair $\mathcal{B}_{\lambda} = (\text{Coh}(P_{\lambda}/B_{\lambda})$



I have then a flat family



hard total

B_{tot}

B_{tot} $T_{B_{tot}}$

Then (V.9.2):

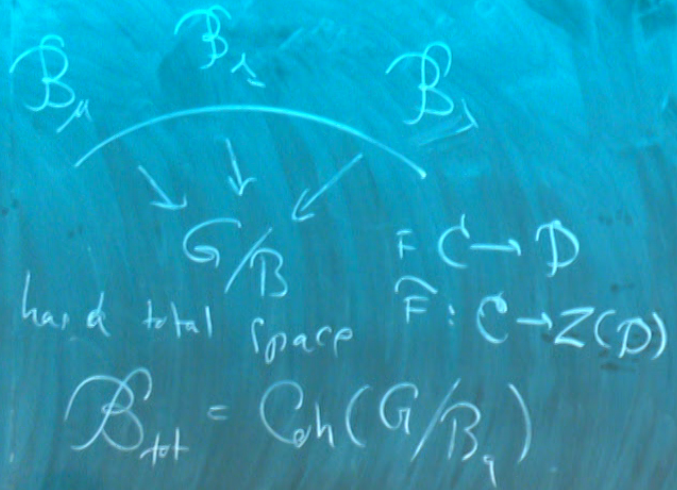
- (a) $Coh(G/B_q)$ is a sheaf of $\mathcal{O}$ -cat over G/B w/ fibers
- (claim) $Coh(G/B_q)|_{\lambda} \cong B_{\lambda}$
- (b) ~~Have~~ universal wr fun. $rep \mathcal{U}(G) \rightarrow Coh(B_q/B_q)$
- (c) This restriction is a fully faithful mon embedding + has fully faith derived counterpart $D(\mathcal{U}(G)) \rightarrow D(G/B_q)$

(c) All $Coh(B_{\lambda}/B_q)$ abstractly, as plain $\mathcal{O}$ -cat

(d) This construction goes to arbitrary pts in G

The small quantum Borel the pair $B_{\lambda} = (Coh(B_{\lambda}/B_q)$

I have then a flat family



B_{tot} T36

Then (VPR-3):

- (a) $\text{Coh}(G/B_q)$ is a sheaf of $\mathcal{O}$ -cat over G/B w/ fibers
- (claim) $\text{Coh}(G/B_q)|_{\lambda} \cong B_{\lambda}$
- (b) ~~Have~~ universal wr fun. $\text{rep } \mathcal{U}(G) \rightarrow \text{Coh}(G/B_q)$
- (c) This restriction is a fully faithful mon embedding + has

$\text{rep } \mathcal{U}(G) \rightarrow Z_{\text{Coh}(G/B)}^{(G/B_q)}$ fully faith derived counterpart

$$D(\mathcal{U}(G)) \rightarrow D(G/B_q)$$

(c) All $\text{Coh}(P_{\lambda}/B_q)$ abstractly, as plain $\mathcal{O}$ -cat

(d) This construction goes to arbitrary pts in G

The small quantum Borel the pair $B_{\lambda} = (\text{Coh}(P_{\lambda}/B_q)$