Title: Non-invertible Global Symmetries in the Standard Model

Speakers: Shu-Heng Shao

Collection: Global Categorical Symmetries

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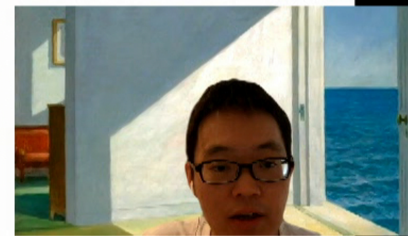
Abstract: We identify infinitely many non-invertible generalized global symmetries in QED and QCD for the real world in the massless limit. In QED, while there is no conserved Noether current for the axial symmetry because of the ABJ anomaly, for every rational angle, we construct a conserved and gauge-invariant topological symmetry operator. Intuitively, it is a composition of the axial rotation and a fractional quantum Hall state coupled to the electromagnetic U(1) gauge field. These conserved symmetry operators do not obey a group multiplication law, but a non-invertible fusion algebra over TQFT coefficients. These non-invertible symmetries lead to selection rules, which are consistent with the scattering amplitudes in QED. We further generalize our construction to QCD, and show that the neutral pion decay can be understood from a matching condition of the non-invertible global symmetry.

SCGCS



Non-invertible Global Symmetries in the Standard Model

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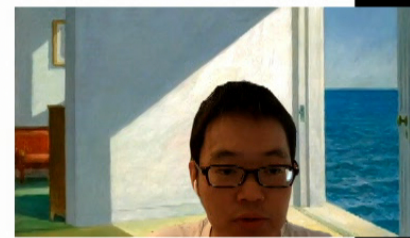
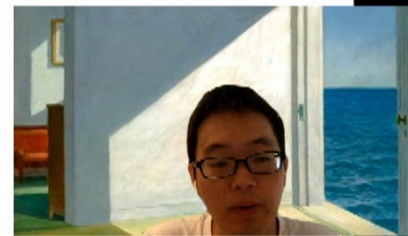
Mainly based on
[Choi-Lam-SHS 2205.05086]

See also
[Corodva-Ohmori 2205.06243]

And
[Roumpedakis-Seifnashri-SHS 2204.02407]
[Choi-Cordova-Hsin-Lam-SHS 2204.09025]



Axial symmetry in QED



- Consider QED with a massless, unit charge Dirac fermion.

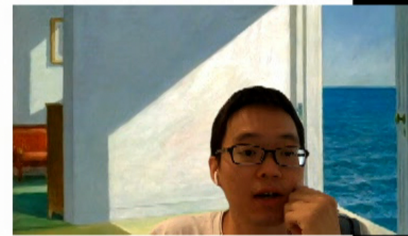
$$\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + i\bar{\Psi}\left(\partial_\mu - iA_\mu\right)\gamma^\mu \Psi$$

- The classical axial $U(1)_A$ symmetry acts as

$$\Psi \to \exp\left(\frac{i\alpha}{2}\gamma_5\right)\Psi \quad , \qquad \alpha \sim \alpha + 2\pi$$

- Note that $\alpha = 2\pi$ corresponds to the fermion parity, which is part of the gauge symmetry.
- The Adler-Bell-Jackiw anomaly implies that the classical $U(1)_A$ axial symmetry **fails** to be a global symmetry quantum mechanically.

ABJ anomaly



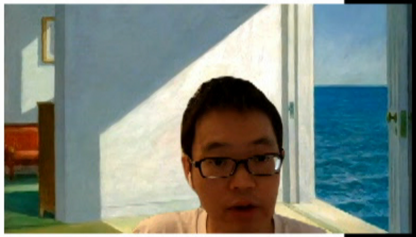
- The ABJ anomaly was discovered in the late 60s to explain the neutral pion decay, $\pi^0 \to \gamma\gamma$.

- It successfully determined the coupling

$$\frac{i}{8\pi^2 f_\pi}\pi^0 F \wedge F$$

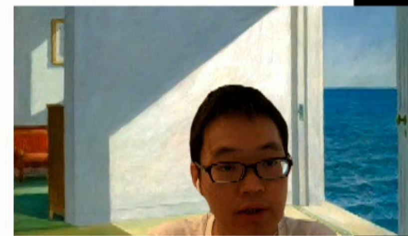
in the pion Lagrangian and predicted there are three colors of quarks.

ABJ anomaly?



- Conceptually, there is something *slightly* counterintuitive though.
- Usually we celebrate when we discover the **existence** of a global symmetry.
- ABJ anomaly states that there is **not** a global symmetry that one would have naively expected.
- So how come we can derive all these quantitative results from the **absence** of a global symmetry?
- Of course, since the fine structure constant is small, we can treat the electromagnetic gauge field as background. The $\pi^0 F \wedge F$ term follows from the Wess-Zumino term, which captures all the ‘t Hooft anomalies, in the chiral Lagrangian [Witten 1983].
- But wouldn’t it be nice if we can reinterpret these classic results from the existence of a generalized global symmetry (rather than the absence thereof)?

Non-invertible global symmetries



- We will show that the continuous, invertible $U(1)_A$ axial symmetry is broken by the ABJ anomaly to a discrete, non-invertible global symmetry labeled by the rational numbers.

- In the pion Lagrangian, the coupling $\pi^0 F \wedge F$ can be derived by matching the non-invertible global symmetry in the UV QCD.

- Therefore, the neutral pion decay $\pi^0 \to \gamma\gamma$ can be understood in terms of the non-invertible global symmetry.

QED

- In QED, the axial current is

$$j_\mu^A = \bar{\Psi}\gamma_5\gamma_\mu\Psi$$

- It obeys the anomalous conservation equation

$$d \star j^A = \frac{1}{4\pi^2} F \wedge F$$

- Naively, we can define the symmetry operator

$$U_\alpha(M) = \exp\left(\frac{i\alpha}{2}\oint_M \star j^A\right)$$

- However, it is **not** conserved (topological).

QED

- $U_\alpha(M) = \exp(\frac{i\alpha}{2}\oint_M \star j^A)$ is not conserved.
- Let us try a gauge non-invariant current [Adler 1969]:

$$\star \hat{j}^A \equiv \star j^A - \frac{1}{4\pi^2} AdA$$

- It is formally conserved, $d \star \hat{j}^A = 0$.
- But the symmetry operator is **not** gauge invariant on a general three-manifold M

$$\widehat{U}_\alpha(M) = \exp[\frac{i\alpha}{2}\oint_M (\star j^A - \frac{1}{4\pi^2} AdA)]$$

Dilemma

Operator	Gauge-invariant?	Conserved (topological)?
$U_\alpha(M) = \exp(\frac{i\alpha}{2}\oint_M \star j^A)$	✓	✗
$\hat{U}_\alpha(M) = \exp[\frac{i\alpha}{2}\oint_M (\star j^A - \frac{1}{4\pi^2} AdA)]$	✗	✓

Rational angles

- Let us be less ambitious, and assume the axial rotation angle is a fraction:

$$\alpha = \frac{2\pi}{N}$$

$$\hat{U}_{\frac{2\pi}{N}}(M) = \exp[\oint_M (\frac{2\pi i}{2N} \star j^A - \frac{i}{4\pi N} AdA)]$$

- The operator $\hat{U}_{\frac{2\pi}{N}}(M)$ is still not gauge invariant because of the fractional Chern-Simons term.

Fractional quantum Hall state

$$“-\frac{i}{4\pi N}\oint_M AdA”$$

- In condensed matter physics, this action is commonly used to describe the $\nu = 1/N$ fractional quantum Hall effect (FQHE) in 2+1d.
- It is however not gauge invariant. Fortunately, there is a well-known fix to this issue.
- The more precise, gauge invariant Lagrangian for the FQHE is

$$\oint_M \left(\frac{iN}{4\pi} ada + \frac{i}{2\pi} adA\right)$$

where a is a dynamical $U(1)$ gauge field living on the 2+1d manifold M.

Fractional quantum Hall state

$$\text{"}-\frac{i}{4\pi N}\oint_M AdA\text{"} \rightarrow \oint_M \left(\frac{iN}{4\pi} ada + \frac{i}{2\pi} adA\right)$$

- Naively, we integrate out a to obtain

$$\text{"}a = -\frac{A}{N}\text{"}$$

- Substituting this back to the Lagrangian returns the original fractional Chern-Simons term.
- This is illegal, however, since both a and A are properly quantized gauge fields obeying $\oint_\Sigma da \in 2\pi\mathbb{Z}$ and $\oint_\Sigma dA \in 2\pi\mathbb{Z}$. We can't divide them by N.
- The Lagrangian on the right is the precise, gauge invariant description of FQHE.

Back to QED

- Motivated by the discussion of FQHE in 2+1d, we define a new operator $\mathcal{D}_{1/N}(M)$ in 3+1d QED:

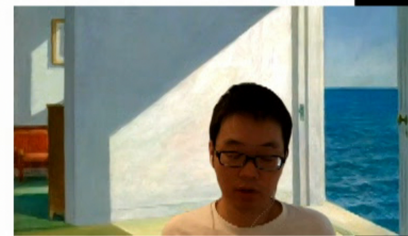
$$\widehat{U}_{\frac{2\pi}{N}}(M) = \exp[\oint_M (\frac{2\pi i}{2N} \star j^A - \frac{i}{4\pi N} AdA)]$$

$$\downarrow$$

$$\mathcal{D}_{1/N}(M) \equiv \exp[\oint_M (\frac{2\pi i}{2N} \star j^A + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA)]$$

- Here a only lives on the three-manifold M and A is the bulk electromagnetic gauge field. Here and throughout we omit the path integral over a in the definition of $\mathcal{D}_{1/N}$.
- Reminiscent of the η' sheet in [Komargodski 2018].

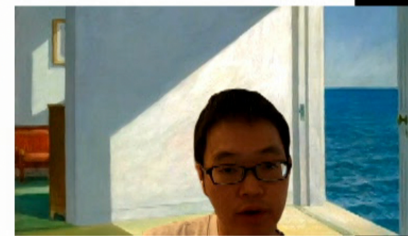
Non-invertible global symmetry



$$\mathcal{D}_{\frac{1}{N}}(M) \equiv \exp[\oint_M (\frac{2\pi i}{2N} \star j^A + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA)]$$

- This new operator is clearly gauge invariant because the Chern-Simons terms are properly quantized.
- It is conserved (topological). This is proved from gauging a discrete magnetic one-form symmetry. More on this later.
- The price we pay is that it is **not** an invertible symmetry.

Non-invertible fusion



$$\mathcal{D}_{1/N}(M) \equiv \exp[\oint_M (\frac{2\pi i}{2N} \star j^A + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA)]$$

$$\mathcal{D}_{1/N}^{\dagger}(M) \equiv \exp[\oint_M (-\frac{2\pi i}{2N} \star j^A - \frac{iN}{4\pi} \bar{a}d\bar{a} - \frac{i}{2\pi} \bar{a}dA)]$$

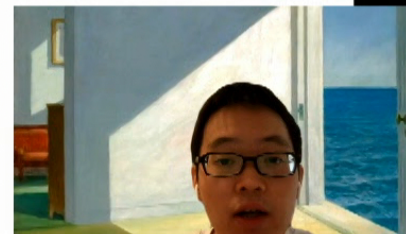
- The parallel fusion between $\mathcal{D}_{1/N}$ and $\mathcal{D}_{1/N}^{\dagger}$ is **not** the identity:

$$\mathcal{D}_{1/N} \times \mathcal{D}_{1/N}^{\dagger} = \exp[\oint_M (\frac{iN}{4\pi} ada - \frac{iN}{4\pi} \bar{a}d\bar{a} + \frac{i}{2\pi}(a - \bar{a})dA)] \equiv \mathcal{C} \neq 1$$

- We see that $\mathcal{D}_{1/N}$ is **not** a unitary operator.

Higher gauging

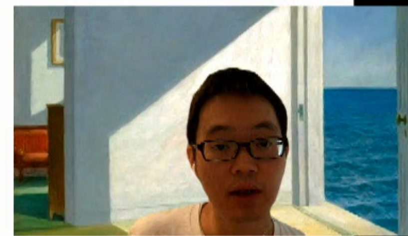
[Roumpedakis-Seifnashri-SHS 2022]



$$\mathcal{D}_{1/N}\times\mathcal{D}_{1/N}^{\dagger} = \exp[\oint_M (\frac{iN}{4\pi} ada - \frac{iN}{4\pi}\bar{a}d\bar{a} + \frac{i}{2\pi}(a-\bar{a})dA)] \equiv \mathcal{C}$$

- $\mathcal{C}$ is the condensation operator/defect from higher gauging of the $\mathbb{Z}_N$ subgroup of the $U(1)$ magnetic one-form symmetry.
- p-gauging [Roumpedakis-Seifnashri-SHS 2022]: gauge a q-form symmetry only along a codimension-p submanifold in spacetime. Higher gauging does not change the bulk QFT, but generates a codimension-p topological defect – the condensation defect [Kong-Wen 2014, Else-Nayak 2017, Gaiotto-JohnsonFreyd 2019,...].
- $\mathcal{C}(M)$ arises from one-gauging the $\mathbb{Z}_N$ magnetic one-form global symmetry along a codimension-one manifold M.

Non-invertible global symmetry

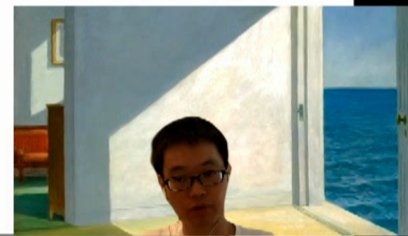


- It is easy to generalize this construction to an arbitrary rational axial rotation $\alpha = 2\pi p/N$ with $\gcd(p, N) = 1$.

$$\mathcal{D}_{\frac{p}{N}}(M) \equiv \exp[\oint_M (\frac{2\pi i p}{2N} \star j^A + \mathcal{A}^{N,p}[dA/N])]$$

where $\mathcal{A}^{N,p}$ is the 2+1d minimal $\mathbb{Z}_N$ TQFT [Hsin-Lam-Seiberg 2018].

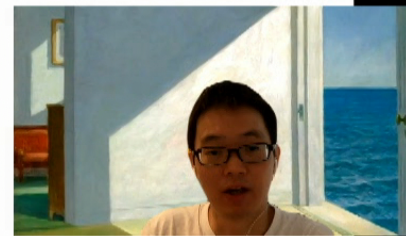
- Therefore, the continuous, invertible $U(1)_A$ axial symmetry is broken by the ABJ anomaly to a discrete, non-invertible global symmetry labeled by the rational numbers $\frac{p}{N} \in \mathbb{Q}/\mathbb{Z}$. It's a generalization of the construction in [Kaidi-Ohmori-Zheng 2021].



Non-invertible global symmetry

Operator	Gauge-invariant?	Conserved (topological)?	Invertible?
$U_\alpha(M) = \exp(\frac{i\alpha}{2}\oint_M \star j^A)$	✓	✗	N/A
$\hat{U}_\alpha(M) = \exp[\frac{i\alpha}{2}\oint_M (\star j^A - \frac{1}{4\pi^2} AdA)]$	✗	✓	✓
$\mathcal{D}_{\frac{1}{N}}(M) = \exp[\oint_M (\frac{2\pi i}{2N} \star j^A + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA)]$	✓	✓	✗

Non-invertible global symmetry



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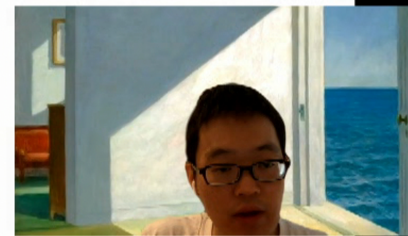
Non-invertible global symmetries



Why do these non-invertible topological operators qualify as generalized global symmetries?

See Kaidi's talk yesterday for a review!

Non-invertible operators as symmetri

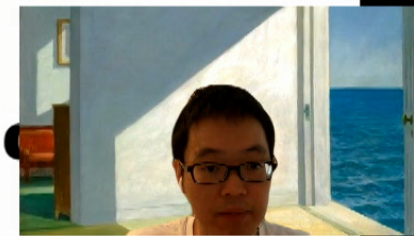
Why do these non-invertible topological operators qualify as generalized symmetries?

1. Some non-invertible symmetries can be gauged [Brunner-Carqueville-Plencner 2014].
2. They can have generalized anomalies, which lead to generalized 't Hooft anomaly matching conditions. They result in nontrivial constraints on the renormalization group flows [Chang-Lin-SHS-Wang-Yin 2018, Thorngren-Wang 2019+2021, Komargodski-Ohmori-Roumpedakis-Seifnashri 2020,...].
3. In quantum gravity, the no global symmetry conjecture is argued to be generalized to the absence of invertible *and* non-invertible global symmetries [Rudelius-SHS 2020, Heidenreich-McNamara-Montero-Reece-Rudelius-Valenzuela 2021, McNamara 2021].
4. Well, that's (basically) in the name of this Simons collaboration:

Global Categorical Symmetries

21

Non-invertible symmetries in 3+1

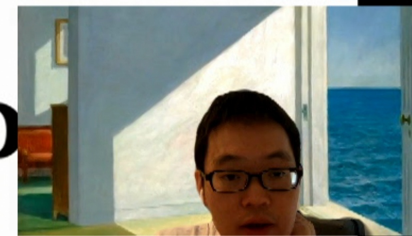


- In the past year, there has been a lot of developments on constructing non-invertible symmetries in familiar 3+1d lattice and continuum gauge theories [Koide-Nagoya-Yamaguchi 2021, Choi-Cordova-Hsin-Lam-SHS 2021+ 2022, Kaidi-Ohmori-Zheng 2021, Bhardwaj-Bottini-SchaferNameki-Tiwari 2022, Hayashi-Tanizaki 2022, Kaidi-Zafrir-Zheng 2022, Choi-Lam-SHS 2022, Cordova-Ohmori 2022].
- Some of these constructions apply to QFTs that are invariant under gauging a discrete one-form global symmetry $G^{(1)}$ (possibly with a discrete torsion phase).

QFT | $QFT/G^{(1)}$

Topological Dirichlet b.c.

Gauging in half spacetime in QED



- We will assume there is no monopole at the energy scale we are interested in.
- QED has a magnetic $U(1)^{(1)}$ one-form symmetry generated by a conserved two-form current [Gaiotto-Kapustin-Seiberg-Willett 2014]

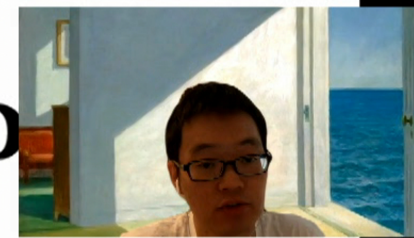
$$j^m = \frac{1}{2\pi} \star F \ , \qquad d \star j^m = 0$$

- When we gauge the $\mathbb{Z}_N^{(1)}$ subgroup with a specific choice of the discrete torsion phase that depends on p, the net effect is to shift the θ angle by

$$\theta \mapsto \theta - 2\pi p/N$$

- We can then undo this shift by performing an axial rotation on the fermions.
- The non-invertible symmetry $\mathcal{D}_{p/N}$ is realized by composing the gauging of $\mathbb{Z}_N^{(1)}$ and an axial rotation in half of spacetime, and impose the Dirichlet boundary condition.

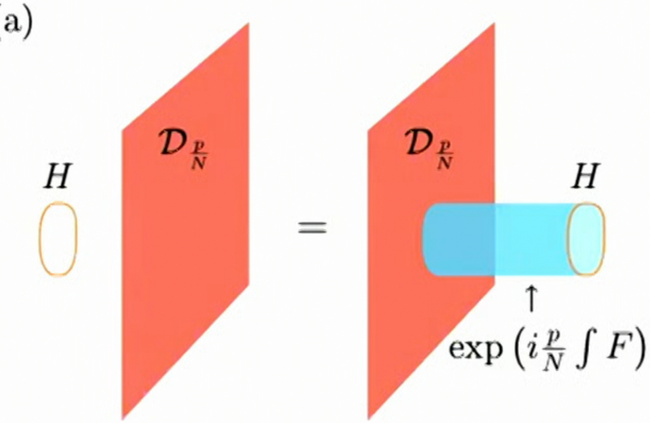
Gauging in half spacetime in QED



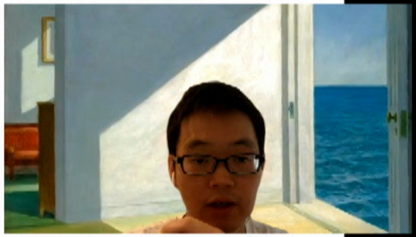
- Since the Dirichlet boundary condition for a discrete gauge theory is topological, this proves rigorously why $\mathcal{D}_{p/N}$ is conserved (topological).
- From the gauging construction, we see that $\mathcal{D}_{p/N}$ acts invertibly on the fermions as an axial rotation, but non-invertibly on the 't Hooft lines $H(\gamma)$ by the Witten effect:

$$H(\gamma) \mapsto H(\gamma)\exp\left(\frac{ip}{N}\int F\right)$$

(a)

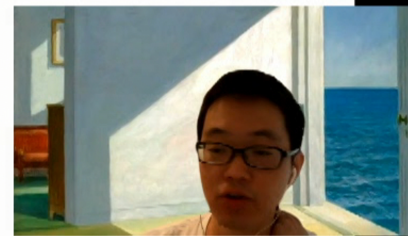


Electron mass



- Let us explore various consequences of the non-invertible symmetry in QED.
- Naturalness ['t Hooft 1980]: Impose a global symmetry group G. The Lagrangian should include all G-invariant terms with coefficients of order one with no fine-tuning.
- QED Lagrangian: $\mathcal{L} = \frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}(\partial_\mu - iA_\mu)\gamma^\mu\Psi$
- The electron mass term $m\bar{\Psi}\Psi$ violates the non-invertible global symmetry.
- Therefore, electron is naturally massless in QED because of the non-invertible global symmetry.
- See [Cordova-Ohmori 2022] for more discussions.

‘t Hooft Naturalness

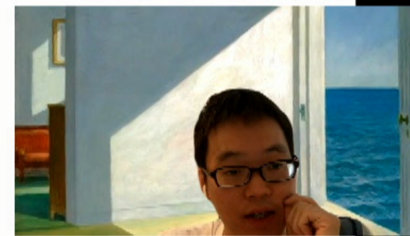


III2. NATURALNESS IN QUANTUM ELECTRODYNAMICS

Quantum Electrodynamics as a renormalizable model of electrons (and muons if desired) and photons is an example of a "natural" field theory. The parameters α, m_e (and m_μ) may be small independently. In particular m_e (and m_μ) are very small at large μ. The relevant symmetry here is chiral symmetry, for the electron and the muon separately. We need not be concerned about the Adler-Bell-Jackiw anomaly here because the photon field being Abelian cannot acquire non-trivial topological winding numbers[4].

‘t Hooft, *Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking* (1980)

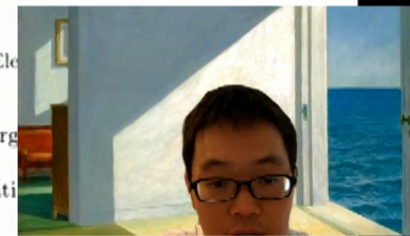
Selection rule in QED



- The operator $\mathcal{D}_{p/N}$ acts invertibly on the fermions as an axial rotation with $\alpha = 2\pi p/N$.
- The selection rule on the fermions on flat space amplitudes from $\mathcal{D}_{p/N}$ are the same as the naïve $U(1)_A$ symmetry.
- Note that there is no $U(1)$ instanton in flat space because $\pi_3\big(U(1)\big) = 0$.
- It implies that the total helicity of the electrons and positrons has to be conserved in massless QED.

Helicity conservation

- For example, in the electron-positron annihilation, the helicities of the electron and positron are opposite.
- The helicity conservation is usually explained using gamma matrices in QFT textbooks. It is satisfying to see that there is an underlying symmetry principle.



massless. (The calculation can be done for lower energ difficult and no more instructive.)†

Our starting point for both methods of calculati section is the amplitude

$$i\mathcal{M}\big(e^-(p)e^+(p') \to \mu^-(k)\mu^+(k')\big) = \frac{ie^2}{q^2}\Big(\bar{v}(p')\gamma^\mu u(p)\Big)\Big(\bar{u}(k)\gamma_\mu v(k')\Big). \quad (5.1)$$

We would like to use the spin sum identities to write the squared amplitude in terms of traces as before, even though we now want to consider only one set of polarizations at a time. To do this, we note that for massless fermions, the matrices

$$\frac{1+\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \frac{1-\gamma^5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad (5.17)$$

are *projection operators* onto right- and left-handed spinors, respectively. Thus if in (5.1) we make the replacement

$$\bar{v}(p')\gamma^\mu u(p) \longrightarrow \bar{v}(p')\gamma^\mu\Big(\frac{1+\gamma^5}{2}\Big)u(p),$$

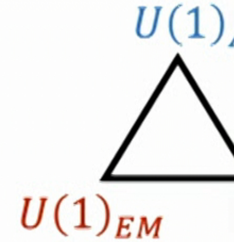
the amplitude for a right-handed electron is unchanged while that for a left-handed electron becomes zero. Note that since

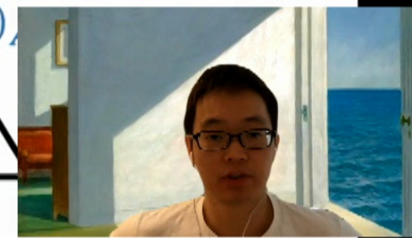
$$\bar{v}(p')\gamma^\mu\Big(\frac{1+\gamma^5}{2}\Big)u(p) = v^\dagger(p')\Big(\frac{1+\gamma^5}{2}\Big)\gamma^0\gamma^\mu u(p), \quad (5.18)$$

this same replacement imposes the requirement that $v(p')$ also be a right-handed spinor. Recall from Section 3.5, however, that the right-handed spinor $v(p')$ corresponds to a *left*-handed positron. Thus we see that the annihilation amplitude vanishes when both the electron and the positron are right-handed. In general, the amplitude vanishes (in the massless limit) unless the electron and positron have opposite helicity, or equivalently, unless their spinors have the same helicity.

Chapter 5 of Peskin&Schroeder

QCD



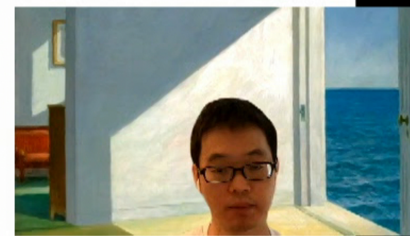


- Below the electroweak scale, the massless QCD Lagrangian for the up and down quarks has an axial global symmetry (corresponding to π^0)

$$U(1)_{A3}: \begin{pmatrix} u \\ d \end{pmatrix} \to \exp(i\alpha\gamma_5\sigma_3) \begin{pmatrix} u \\ d \end{pmatrix}$$

- It suffers from the ABJ anomaly with the electromagnetic $U(1)_{EM}$ gauge symmetry.
- By the exact same construction, we conclude that there is an infinite non-invertible global symmetry $\mathcal{D}_{p/N}$ in the UV QCD from $U(1)_{A3}$.
- How does the IR pion Lagrangian capture this non-invertible global symmetry?

Pion



- The pion Lagrangian

$$\mathcal{L}_{IR} = \frac{1}{2}\left(\partial_\mu \pi^0\right)^2 + ig\,\pi^0 F \wedge F + \cdots$$

- The pion field is compact, $\pi^0 \sim \pi^0 + 2\pi f_\pi$, where $f_\pi \sim 92.4 MeV$.
- The non-invertible global symmetry $\mathcal{D}_{p/N}$ shifts the pion field,

$$\pi^0 \to \pi^0 - 2\pi \frac{p}{N} f_\pi .$$

- The equations of motion in the presence of the non-invertible global symmetry $\mathcal{D}_{p/N}$ fix the coefficient g for $\pi^0 F \wedge F$, which gives the dominant contribution to the neutral pion decay $\pi^0 \to \gamma\gamma$.

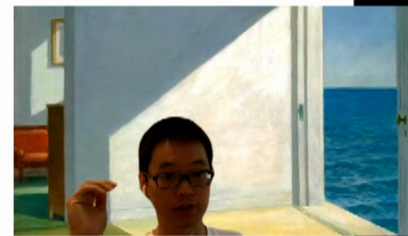
Pion

$$\mathcal{L}_{IR} = \frac{1}{2}(\partial_\mu \pi^0)^2 + ig\,\pi^0 F \wedge F$$

$$\mathcal{D}_{1/N}(M) = \exp[\oint_{x=0} (\frac{2\pi i}{N} \star j^{A3} + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA)]$$

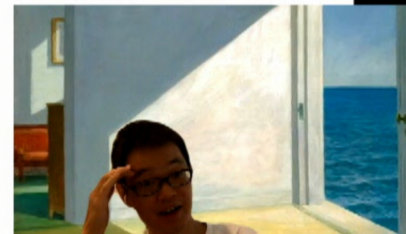
- Inserting $\mathcal{D}_{1/N}$ at $x = 0$ as a defect, the equations of motion are
  - π^0 EOM: $\pi^0|_{x=0^+} - \pi^0|_{x=0^-} = -\frac{2\pi}{N} f_\pi$
  - a EOM: $Nda + F = 0$
  - A EOM: $2ig(\pi^0|_{x=0^+} - \pi^0|_{x=0^-})F = \frac{i}{2\pi} da$
- Combining the above, it fixes $g = \frac{1}{8\pi^2 f_\pi}$.

Pion decay



- Conventionally, the pion decay $\pi^0 \to \gamma\gamma$ is explained by the ABJ anomaly. Of course, since the fine structure constant is small, we can treat the electromagnetic gauge field as background, then the $\pi^0 F \wedge F$ follows from the 't Hooft anomaly matching.
- We have provided an alternative explanation for the pion decay as a direct consequence from matching the non-invertible global symmetry in the UV QCD.
- The non-invertible global symmetry gives an invariant characterization of the ABJ anomaly in terms of the *existence* of a generalized global symmetry, rather than the *absence* thereof.

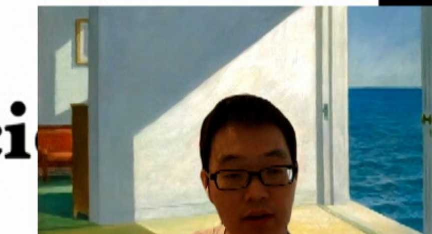
Goldstone boson?



- Even though the non-invertible symmetry is discrete, it is labeled by rational numbers, which are dense in $U(1)$. It's "almost" a continuous symmetry.
- Usually we think of π^0 as the Goldstone boson of the *anomalous* $U(1)_{A3}$ symmetry. It is so light but has a non-derivative coupling $\pi^0 F \wedge F$ at the same time.
- Perhaps π^0 can be viewed as a "Goldstone boson" for the non-invertible global symmetry $\mathcal{D}_{p/N}$.
- Indeed, the non-invertible symmetry shifts the pion field:

$$\mathcal{D}_{p/N}: \ \pi^0 \to \pi^0 - \frac{2\pi p}{N} f_\pi$$

Fusion algebra over TQFT coeffici



- For odd N, $\mathcal{D}_{1/N}(M)\times\mathcal{D}_{1/N}(M) = \mathcal{A}^{N,2}[M]\,\mathcal{D}_{2/N}(M)$
- The fusion "coefficient" is not a number, but a 2+1d TQFT.
- Generally, the fusion "coefficient" of d-dimensional topological defects is a d-dimensional TQFT [Roumpedakis-Seifnashri-SHS 2022, Choi-Cordova-Hsin-Lam-SHS 2022]:

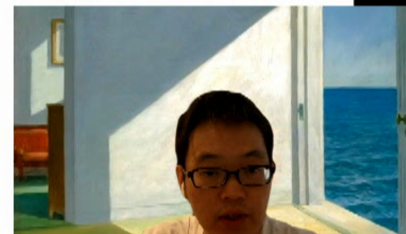
Fusion algebra over TQFT coefficients

$$\mathcal{D}(M)\times\mathcal{D}'(M) = \mathcal{T}(M)\,\mathcal{D}''(M)$$

Partition function of a TQFT $\mathcal{T}$ on M

- What does it mean mathematically??

Fusion of topological lines



- Example: When the topological defects are lines, the fusion coefficients N_{ab}^{c} are non-negative integers. The fusion algebra is a unital $\mathbb{Z}_+$-ring (plus some other conditions):

$$L_a \times L_b = \sum_c N_{ab}^{c} \; L_c \;, \qquad N_{ab}^{c} \in \mathbb{Z}_{\geq 0}$$

- The fusion coefficient $N_{ab}^{c} \in \mathbb{Z}_{\geq 0}$ should be viewed as a 0+1d topological quantum mechanics (i.e., a free qudit) with an N_{ab}^{c}-dim Hilbert space.
- For higher dim topological defects, the fusion coefficients are TQFTs, so

TQFTs are categorical generalizations of non-negative integers

3.1. Definition of a $\mathbb{Z}_+$-ring

Let $\mathbb{Z}_+$ denote the semi-ring of non-negative integers.

DEFINITION 3.1.1. Let A be a ring which is free as a $\mathbb{Z}$-module.

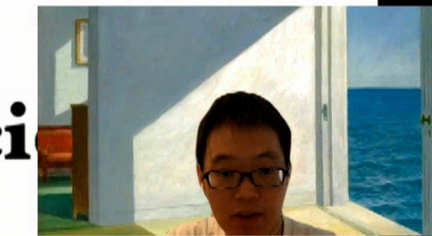
(i) A $\mathbb{Z}_+$-*basis* of A is a basis $B = \{b_i\}_{i \in I}$ such that $b_i b_j = \sum_{k \in I} c_{ij}^k b_k$, where $c_{ij}^k \in \mathbb{Z}_+$.

(ii) A $\mathbb{Z}_+$-*ring* is a ring with a fixed $\mathbb{Z}_+$-basis and with identity 1 which is a non-negative linear combination of the basis elements.

(iii) A *unital* $\mathbb{Z}_+$-*ring* is a $\mathbb{Z}_+$-ring such that 1 is a basis element.

Etingof-Gelaki-Nikshych-Ostrik
Tensor Categories

Fusion algebra over TQFT coeffici



- For odd N, $\mathcal{D}_{1/N}(M) \times \mathcal{D}_{1/N}(M) = \mathcal{A}^{N,2}[M]\, \mathcal{D}_{2/N}(M)$
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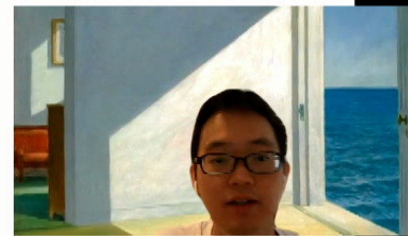
Conclusion

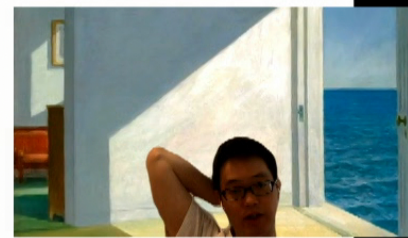


- In massless QED and QCD, the continuous, invertible $U(1)_A$ symmetry is broken by the ABJ anomaly into a discrete, non-invertible symmetry $\mathcal{D}_{p/N}$ labeled by rational numbers.

$$\mathcal{D}_{\frac{1}{N}}(M) \equiv \exp[\oint_M (\frac{2\pi i}{2N} \star j^A + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA)]$$

- The non-invertible symmetry is a composition of the naïve axial rotation with a fractional quantum Hall state.
- To put it in the maximally offensive way, the neutral pion decays $\pi^0 \to \gamma\gamma$ because of the non-invertible global symmetry.
- The axion-Maxwell theory has non-invertible global symmetries that shift the axion [Cordova-Ohmori 2022].

Conclusion



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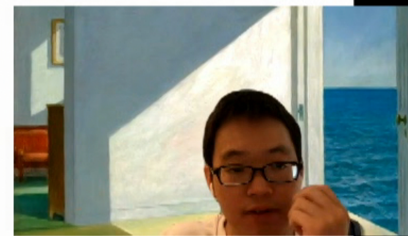
Thank you!



1960s	1988-	2010	2017-	2021	2022 Mar	2022 May	
Conserved charges in integrable systems	Non-inv lines in 1+1d RCFT 2+1d TQFT	Non-inv surfaces in 2+1d TQFT	Non-inv op as generalized sym Constraints on RG in 1+1d	3+1d gauge theories	Non-inv sym from higher-form sym	Non-inv sym in Nature	???

Above I mostly focus on codim-1 non-inv op.
Many many other developments not listed.