Title: Lessons from SU(N) Seiberg-Witten Geometry

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Abstract: "Motivated by applications to soft supersymmetry breaking, we revisit the Seiberg-Witten solution for N=2 super Yang-Mills theory in four dimensions with gauge group SU(N). We present a simple exact Taylor series expansion for the periods obtained at the origin of moduli space, thereby generalizing earlier results for SU(2) and SU(3). With the help of these analytic results and others, we analyze the global structure of the Kahler potential, presenting evidence for a conjecture that the unique global minimum is the curve at the origin of moduli space.

Two applications of these results are considered. Firstly, we analyze candidate walls of marginal stability of BPS states on special slices for which the expansions of the periods simplify. Secondly, we consider soft supersymmetry breaking of the N=2 theory to non-supersymmetric four-dimensional SU(N) gauge theory with two massless adjoint Weyl fermions (""adjoint QCD""). The Seiberg-Witten Kahler potential and strong coupling spectrum play a crucial role in this analysis, which ultimately leads to an exploration of the adjoint QCD phase diagram."

Lessons from $SU(N)$ Seiberg-Witten geometry

Emily Nardoni

based on [2206.xxxxx]
with Eric D'Hoker and Thomas Dumitrescu

see also [2012.11843], [2207.xxxxx]
with the above and Efrat Gerchkovitz

KAVLI IPMU

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Solving for exact low energy dynamics with supersymmetry

- Seiberg and Witten famously solved for the low energy effective QFT on the Coulomb branch of pure *SU*(2) $\mathcal{N} = 2$ gauge theory in four dimensions.

- Today we revisit the pure *SU*(*N*) solution.

The effective action on the moduli space

- The Coulomb branch $\mathcal{M}$ is $N-1$ complex-dimensional, described by gauge-invariant coordinates u_n.

- At a generic point, $SU(N) \to U(1)^{N-1}$ and the low energy dynamics are described by **abelian vector multiplets** $A_{I=1,\ldots,N-1}$ with leading interactions are determined by the holomorphic $\mathcal{F}(A)$.

- The **Kähler potential** is given in terms of $\mathcal{F}(A)$ by,

$$K(A,\bar{A}) = \frac{1}{2\pi}\mathrm{Im}\left(\bar{A}_I \frac{\partial \mathcal{F}(A)}{\partial A_I}\right)$$

  K determines the metric on $\mathcal{M}$,

$$g_{I\bar{J}} = \frac{\partial^2 K}{\partial a^I \partial \bar{a}^{\bar{J}}} = \frac{1}{2\pi}\mathrm{Im}(\tau_{IJ}) \qquad (a_I = \langle A_I \rangle)$$

The Seiberg-Witten solution

[Seiberg, Witten '94][Argyres, Faraggi '94]
[Klemm, Lerche, Theisen, Yankielowicz '94]

- The solution is given in terms of a set of 2(*N*-1) holomorphic special coordinates $a_{DI}(u), a_I(u)$, where $a_{DI} = \partial\mathcal{F}(a)/\partial a_I$.

- These are the **periods** of a meromorphic one-form $\lambda_{\rm SW}$ on a canonical basis of homology cycles (A_I, B_I) of a family of curves $\mathcal{C}(u)$,

$$a_I = \frac{1}{2\pi i}\oint_{A_I} \lambda_{\rm SW} \ , \quad a_{DI} = \frac{1}{2\pi i}\oint_{B_I} \lambda_{\rm SW}$$

- For given moduli, $\mathcal{C}(u)$ is given by:

$$y^2 = A(x)^2 - \Lambda^{2N} \ , \quad A(x) = x^N - \sum_{n=0}^{N-2} u_n x^n$$

with $\lambda_{\rm SW} = \dfrac{xA'(x)dx}{y}$

Computing the periods is a challenge

- In *SU*(2), (a_D, a) are given by hypergeometric functions ${}_2F_1$.
  [Seiberg, Witten '94]
- In higher *N* they can be formulated as solutions to Picard-Fuchs differential equations. [Klemm, Lerche, Theisen '95][Ito, Yang '95]…
- In *SU*(3), in certain regions (a_{DI}, a_I) can be expressed as Appell F_4 functions, defined

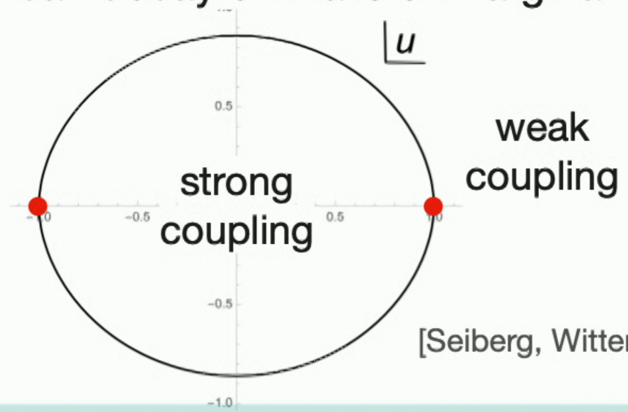
$$F_4(a,b,c_1,c_2;\alpha,\beta) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c_2)}{\Gamma(a)\Gamma(b)\Gamma(c_2+n)n!}\, \beta^n\, {}_2F_1(a+n, b+n; c_1; \alpha)$$

- One of our key results is a simple, exact Taylor series expansion for the periods in the region near the origin of $\mathcal{M}$, generalizing the earlier results for *N*=2 and *N*=3. [D'Hoker, Dumitrescu, **EN**-to appear]

What do we hope to learn?

- The region near the origin is at strong coupling.
- The masses of BPS states in this chamber are determined by the (a_{DI}, a_I) periods, which we've computed.
- These BPS states can decay on walls of marginal stability.

e.g. in *SU*(2):



[Seiberg, Witten '94][Bilal, Ferrari '96]

Motivating question: can we obtain more insight into the border of the strong coupling region for higher *N*?

Probing the IR phase of non-supersymmetric QFTs

N=2: [Cordova, Dumitrescu '18] *N*>2: [D'Hoker, Dumitrescu, Gerchkovitz, **EN**-to appear]

$\mathcal{N}=2$ super Yang Mills with $G=SU(N)$

UV $V_{\not{\text{SUSY}}} \sim M^2 \mathrm{Tr}\bar{\phi}\phi$

(ϕ the vector multiplet scalar)

IR

$SU(N)$ gauge theory with $N_f = 2$ adjoint Weyl fermions $\lambda_\alpha^{i=1,2}$

- Standard lore suggests a phase in which $\langle \mathrm{tr}\lambda^i\lambda^j\rangle \neq 0$, leading to chiral symmetry breaking and confinement.

for *N*=2 see also: [Unsal '07][Bi, Senthil '18][Anber, Poppitz '18][Cordova, Dumitrescu '18]

Motivating question: Can we use the embedding in SUSY to constrain the IR phase of this non-supersymmetric QFT?

The role of the Kähler potential

- The SUSY-breaking deformation can be **tracked** on the Coulomb branch, where it is identified with K.

  [Luty, Rattazzi '99][Abel, Buican, Komargodski '11][Cordova, Dumitrescu '18]

- Then, the global structure of the Kähler potential can inform us on the possible vacuum structure of adjoint QCD.

Outline for the rest of the talk

I) A Taylor expansion of the $(a_{DI}(u), a_I(u))$ periods near the origin of the Coulomb branch.

II) What we learn about the Kähler potential for N=2, N=3, N>3.

III) An application to analyzing walls of marginal stability near the strong coupling chamber in $SU(3)$.

IV) An application to soft supersymmetry-breaking.

I) A Taylor expansion of the periods around the origin of moduli space

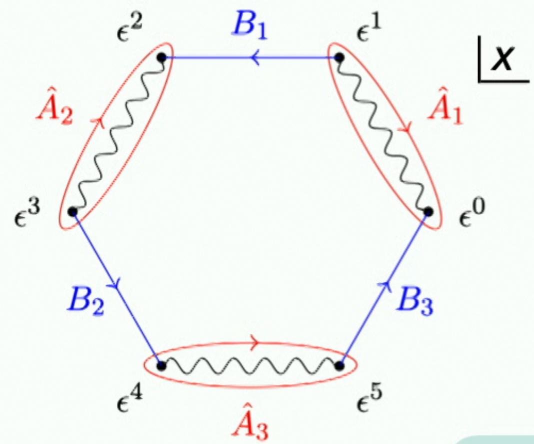
The curve at the origin

- The $\mathbb{Z}_{2N}$-symmetric curve is $y^2 = x^{2N} - 1$, with branch points at *2N*-th roots of unity $\epsilon = e^{\frac{2\pi i}{2N}}$.

Conventions (*N*=3):

$$A_I = \sum_{J=1}^{N} \hat{A}_J$$

$$\langle A_I, B_J \rangle = \delta_{IJ}$$



- The periods are then determined in terms of $Q(\xi) = \frac{1}{2\pi i} \int_0^{\xi} \lambda_{\mathrm{SW}}$,

$$a_I = 2 \sum_{J=1}^{I} \left\{ Q(\epsilon^{2J-1}) - Q(\epsilon^{2J-2}) \right\} \ , \quad a_{DI} = 2 \left\{ Q(\epsilon^{2I}) - Q(\epsilon^{2I-1}) \right\}$$

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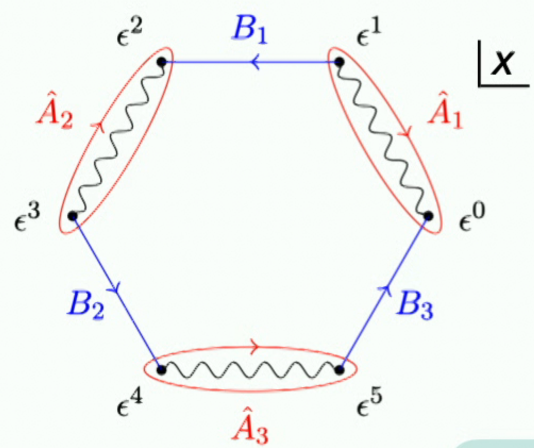

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A Taylor expansion of the periods

[D'Hoker, Dumitrescu, **EN**-to appear]

- We find a Taylor series expansion of $Q(\xi)$ in powers of u_n,

$$Q(\xi) = \sum_{\substack{\ell_n=0 \\ n=0,\cdots,N-2}}^{\infty} V_{L,M}(\xi) \, \frac{u_0^{\ell_0} \cdots u_{N-2}^{\ell_{N-2}}}{\ell_0! \cdots \ell_{N-2}!}$$

$$V_{L,M}(\xi) = \xi^{NM+L+N+1} \left[\frac{2^{M-(L+1)/N}}{4\pi^2 N} \Gamma\left(\tfrac{L+1}{N}\right) \Gamma\left(\tfrac{NM-L-1}{2N}\right)^2 \sin^2\left(\pi \tfrac{NM-L-1}{2N}\right) \right]$$

$$L = \sum_{j=0}^{N-2} j\ell_j \, , \qquad M = \sum_{j=0}^{N-2} \ell_j$$

- This form of the expansion is optimal, as the coefficient of each monomial in u_n consists of a **single factorized** expression.

- These functions are a nontrivial generalization of Appell functions.

Simplification on a special slice

- The sums reduce to hypergeometric functions on the slice parameterized by only u_0,

$$Q(\xi) = \xi\,\hat{Q}_1 + \xi^{N+1}\,\hat{Q}_{N+1} \quad \text{for} \quad u_{n\neq 0} = 0$$

$$\hat{Q}_1 = \frac{\Gamma(1+\frac{1}{2N})}{4\,\Gamma(\frac{3}{2})\Gamma(\frac{1}{2}+\frac{1}{2N})}\, u_0\, {}_2F_1\left(\tfrac{N-1}{2N}, \tfrac{N-1}{2N}; \tfrac{3}{2}; u_0^2\right)$$

$$\hat{Q}_{N+1} = \frac{\Gamma(\frac{1}{2}+\frac{1}{2N})}{4\,\Gamma(\frac{1}{2})\Gamma(1+\frac{1}{2N})}\, {}_2F_1\left(-\tfrac{1}{2N}, -\tfrac{1}{2N}; \tfrac{1}{2}; u_0^2\right)$$

- For *N*=2, these reproduce the well-known expressions.

Reproducing Appell functions for *N*=3

- For instance, in *SU*(3) there are two moduli $u_0 \equiv v,\ u_1 \equiv u$, with

$$y^2 = \left(x^3 - ux - v\right)^2 - 1$$

- Decomposing the series into irreducible representations of $\mathbb{Z}_6$, we recover the Appell functions,

$$Q(\xi) = \sum_{n=0}^{5} \xi^n \hat{Q}_n$$

$$\hat{Q}_1 \sim v\, F_4\left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{3}{2}, \tfrac{4u^3}{27}, v^2\right)$$
$$\hat{Q}_2 \sim u\, F_4\left(\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{4}{3}, \tfrac{1}{2}, \tfrac{4u^3}{27}, v^2\right)$$
$$\hat{Q}_3 = 0$$
$$\hat{Q}_4 \sim F_4\left(-\tfrac{1}{6}, -\tfrac{1}{6}, \tfrac{2}{3}, \tfrac{1}{2}, \tfrac{4u^3}{27}, v^2\right)$$
$$\hat{Q}_5 \sim uv\, F_4\left(\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{4}{3}, \tfrac{3}{2}, \tfrac{4u^3}{27}, v^2\right)$$

Convergence of the period expansion for *N*=3

- The region of absolute convergence of the F_4 is

$$\sqrt{\left|\frac{4u^3}{27}\right|} + \sqrt{|v^2|} < 1$$

which extends to the multi-monopole points $(v = 0,\ \frac{4u^3}{27} = 1)$, and to the Argyres-Douglas points $(v^2 = 1,\ u = 0)$.

[Douglas, Shenker '95][Argyres, Douglas '95]

- These can also be analytically continued to regions of large *u* and small *v*, and vice versa.

- On the special slices when either *u* or *v* is zero, the F_4's **reduce** to ${}_2F_1$'s, and we have full control of the analytic continuation.

II) A study of the Kähler potential

Computing the Kähler potential [D'Hoker, Dumitrescu, **EN**-to appear]

- The Kähler potential is determined in terms of the periods as

$$K = \frac{1}{2\pi} \sum_{I=1}^{N-1} \mathrm{Im}\left(\bar{a}_I a_{DI}\right)$$

- In terms of the decomposition of Q onto $\mathbb{Z}_{2N}$ representations,

$$Q(\xi) = \sum_{j=0}^{2N-1} \xi^j \hat{Q}_j$$

K takes a diagonal form

$$K = \frac{2N}{\pi} \sum_{\substack{j=1 \\ j\neq N}}^{2N-1} |\hat{Q}_j|^2 \tan\left(\frac{\pi j}{2N}\right)$$

Computing the Kahler potential

- Another useful quantity is $\partial K/\partial u_n$. This can be computed using the identification of the derivatives of λ_{SW} with the holomorphic abelian differentials ω_n,

$$\frac{\partial \lambda}{\partial u_n} = \omega_n + d(..) \quad \Rightarrow \quad \frac{\partial a_I}{\partial u_n} = \frac{1}{2\pi i} \oint_{A_I} \omega_n$$

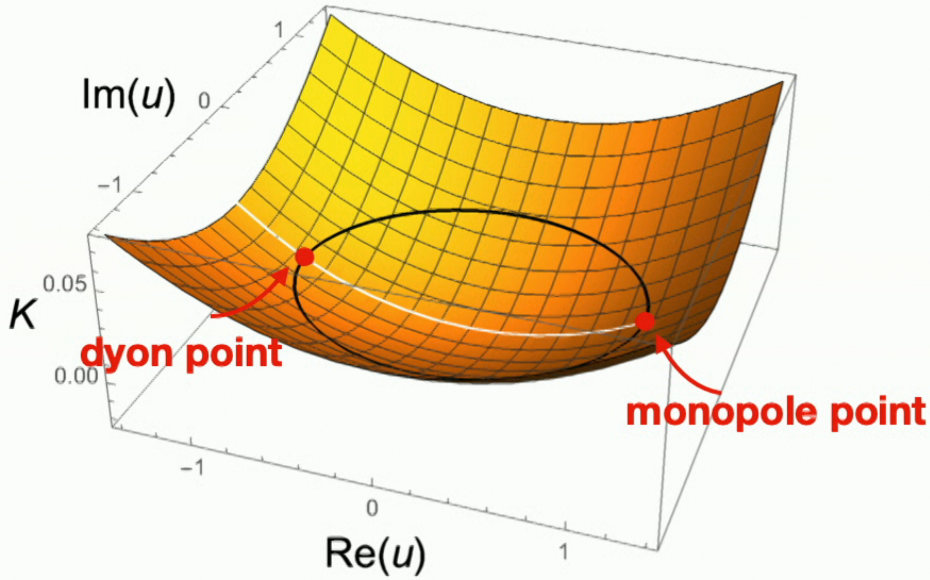
- Exploiting this, the derivatives are recast as surface integrals,

$$\frac{\partial K}{\partial \bar{u}_n} = \frac{i}{16\pi^3} \int_{\Sigma} \lambda \wedge \bar{\omega}_n$$

- In addition to analytic applications, this form is useful for numerical evaluations of *K*.

The Kahler potential for *SU*(2)

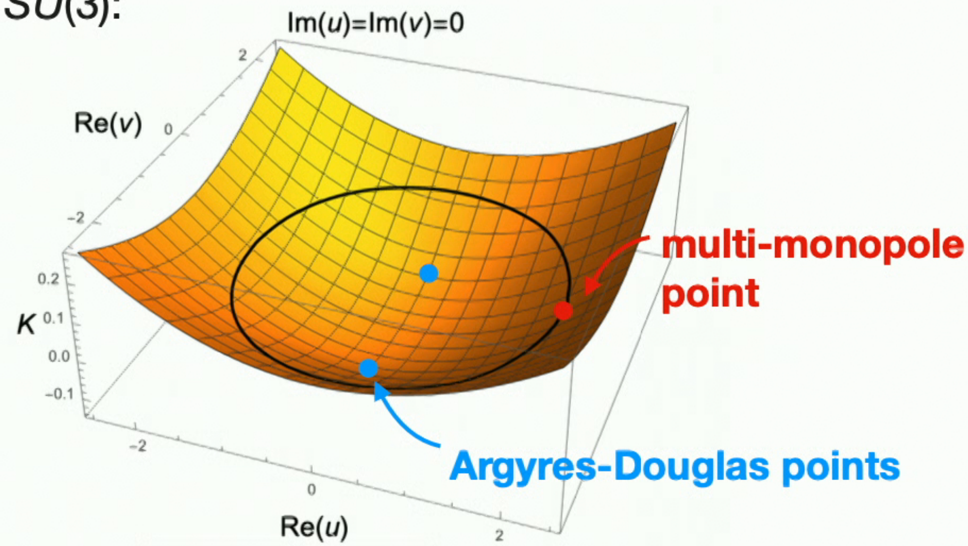
- For *SU*(2), *K* is convex with a single **global minimum** at the origin of moduli space.



see [Luty, Rattazzi '99] [Cordova, Dumitrescu '18]

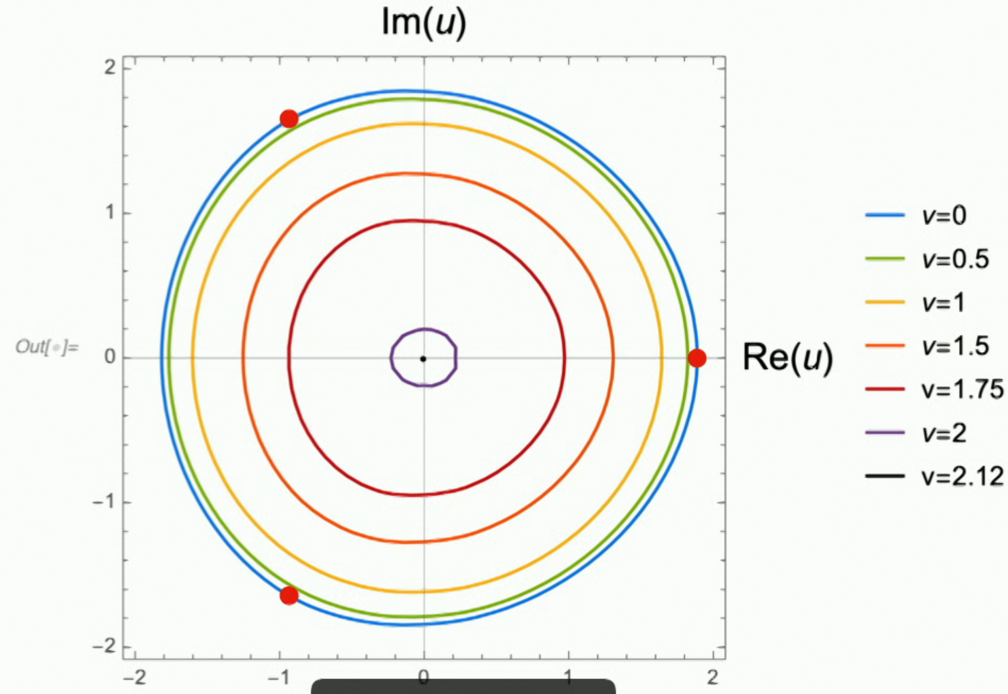
Results for higher N

- The symmetric point is a stationary point. It lies at $K\big|_{u_n=0} \sim -N^2$.
- There is a region around the origin for which K is negative. The multi-monopole points lie on the boundary of this region.
- For instance, in $SU(3)$:



The structure of the Kähler potential for $SU(3)$

- One can visualize the boundary of the region for which K is negative by **numerically** plotting K=0 contours on slices.



Comments on stationary points

- K is negative at an arbitrary stationary point.
- It is natural to conjecture that K is everywhere convex, such that the origin is the unique minimum.
    - We have numerical evidence for N=3, but in general have only proven this on special slices.

III) An application to computing candidate walls of marginal stability in $SU(3)$

The condition of marginal stability

- The periods determine the masses of heavy BPS particles,

$$Z[\mu] = \sum_{I=1}^{N-1} \left(g_I a_{DI} + q_I a_I \right) , \quad M[\mu] = \left| Z[\mu] \right|$$

- A BPS state can decay into other BPS states when

$$M[\mu] = \sum_i M[\mu_i] = M[\sum_i \mu_i]$$

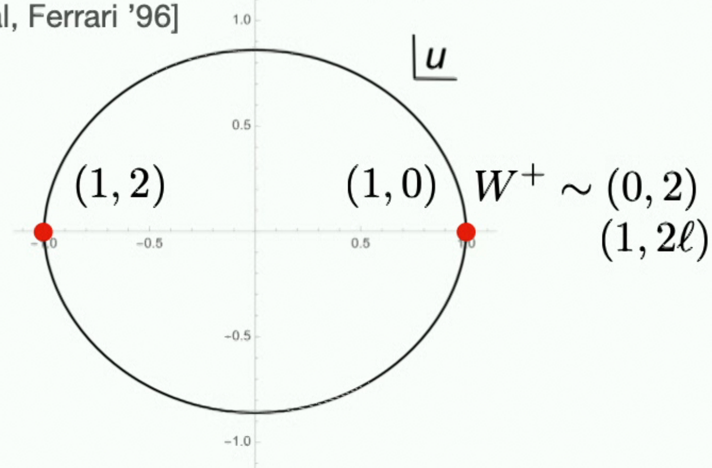
$$\Rightarrow \arg(Z[\mu]) = \arg(Z[\mu_i])$$

- For a given charge assignment, this condition carves out a real codimension-1 subspace of the moduli space.

Marginal stability in *SU*(2)

- For *SU*(2) this yields the curve $\mathrm{Im}(a_D/a) = 0$, which divides $\mathcal{M}$,

  [Seiberg, Witten '94][Bilal, Ferrari '96]



- This curve coincides with the *K*=0 curve in the *u*-plane,

$$\mathrm{Im}(a_D/a) = 0 \quad \Leftrightarrow \quad K \sim \mathrm{Im}(a\bar{a}_D) = 0$$

The strong coupling chamber of *SU*(*N*)

- In the generalization to larger *N* there are many different curves of marginal stability, and more exotic phenomena.

e.g. [Galakhov, Longhi, Mainiero, Moore, Neitzke '13]

- In the strong coupling chamber there are *N* towers of *N*-1 mutually local dyons μ_{km},

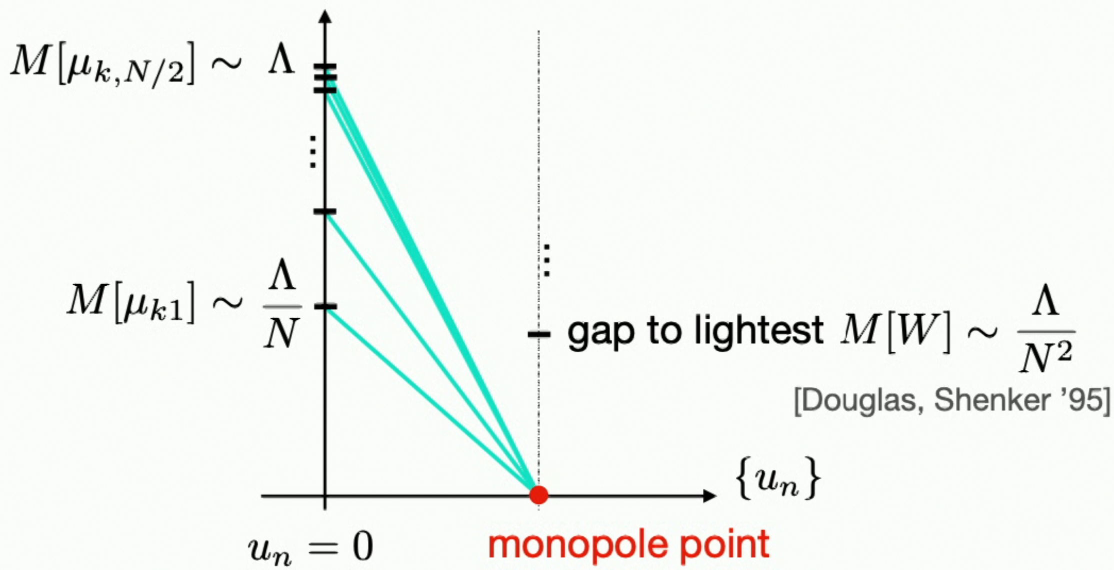
$$\begin{array}{c} \mu_{0m} \\ \mu_{1m} \\ \vdots \\ \mu_{N-1,m} \end{array} \qquad m = 1, \ldots, N-1$$

[Lerche '00][Alim, Cecotti, Cordova, Espahbodi, Rastogi, Vafa '11]...

The strong coupling chamber of *SU*(*N*)

- One member of each tower becomes massless at each of the *N* multi-monopole points.

$$u_n = 0: \quad M[\mu_{km}] = c(N)\Lambda \sin\frac{\pi m}{N}\ , \quad m = 1, \ldots, N-1$$

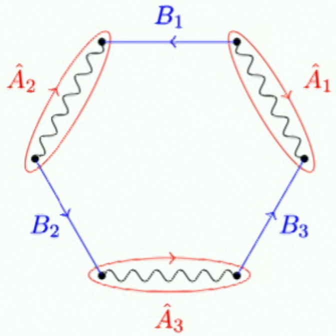


Candidate walls of marginal stability in *SU*(3)

- With our chosen basis, the strong coupling states in *SU*(3) have central charges

$$Z = g_1 a_{D1} + g_2 a_{D2} + q_1 a_1 + q_2 a_2$$

BPS state	$(g_1, g_2;\ q_1, q_2)$
μ_{01}	$(-1, 0;\ -1, 0)$
μ_{02}	$(0, -1;\ 0, 1)$
μ_{11}	$(-1, -1;\ 1, 0)$
μ_{12}	$(0, 1;\ -1, 1)$
μ_{21}	$(1, 1;\ 0, 1)$
μ_{22}	$(-1, 0;\ 1, -1)$



A wall around the origin on the slice *v*=0

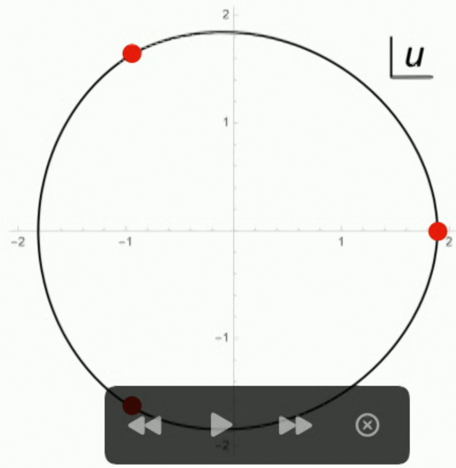
- By inspection, the periods on this slice are related as,

$$a_2(u) = -a_{D1}(u) \ , \qquad a_{D2}(u) = a_1(u)$$

leading to a simplification of the central charges,

$$Z[\mu] = (g_1 - q_2)a_{D1}(u) + (g_2 + q_1)a_1(u)$$

- As such, the problem becomes identical to *SU*(2), and the *K*=0 slice coincides with a wall of marginal stability.

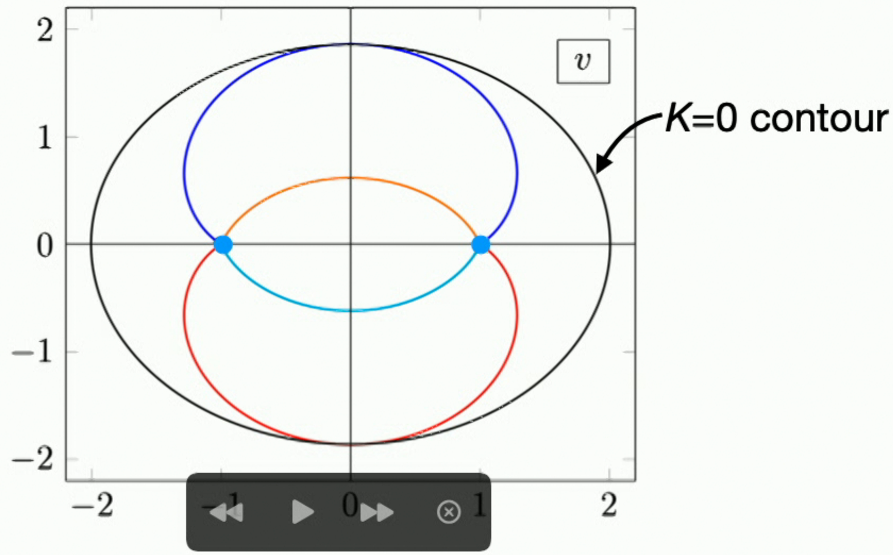


The u=0 slice

- On this slice the periods are related as

$$a_2(v) = \epsilon a_1(v) \ , \quad a_{D2}(v) = \epsilon^2 a_{D1}(v) \ , \quad \epsilon = e^{\frac{i\pi}{3}}$$

- We find candidate lines of marginal stability emanating from the Argyres-Douglas points, along which pairs of the BPS states satisfy the threshold to form marginally stable bound states.



IV) An application to soft-SUSY breaking

$\mathcal{N} = 2$ super Yang-Mills with $G = SU(N)$

UV $\quad V_{\cancel{\text{SUSY}}} \sim M^2 \text{Tr}\bar{\phi}\phi$

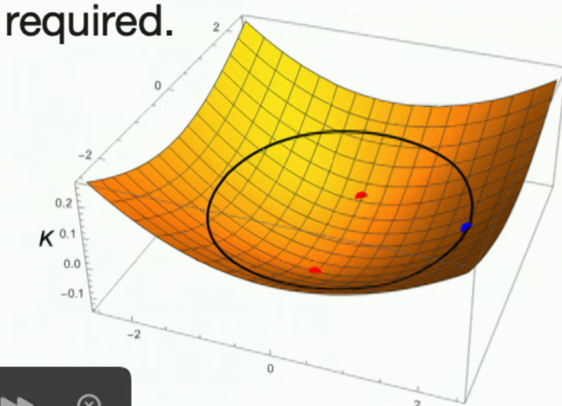
IR

$N_f = 2$ adjoint QCD

SUSY-breaking at small *M*

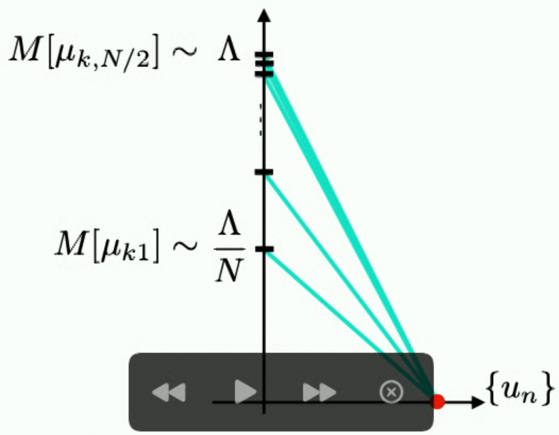
[D'Hoker, Dumitrescu, Gerchkovitz, **EN**-to appear]
(*N*=2: [Cordova, Dumitrescu '18])

- At small *M*, *K* determines the potential $V_{\cancel{\rm SUSY}} \sim M^2 K$
- Our evidence then suggests that the potential has a single stable minimum at the origin of moduli space.
- The corresponding vacuum in in the Coulomb phase with unbroken chiral symmetry.
- If this theory *does* confine and break chiral symmetry breaking, a phase transition at larger *M* is required.



Not the end of the story?

- A clue for what happens when *M* is increased comes from the BPS spectrum in the strong coupling chamber.
- Naively, one expects that when $M \sim M[\mu_{k1}]$, the minimum at the origin is no longer reliable.
- The BPS states that we encounter at larger *M* become massless at the multi-monopole points; by expanding near these points, we might access this new physics.



A “dual” description of the large-*M* physics

- At these points, the light fields are captured by an effective $U(1)_D^{N-1}$ abelian Higgs model, with:

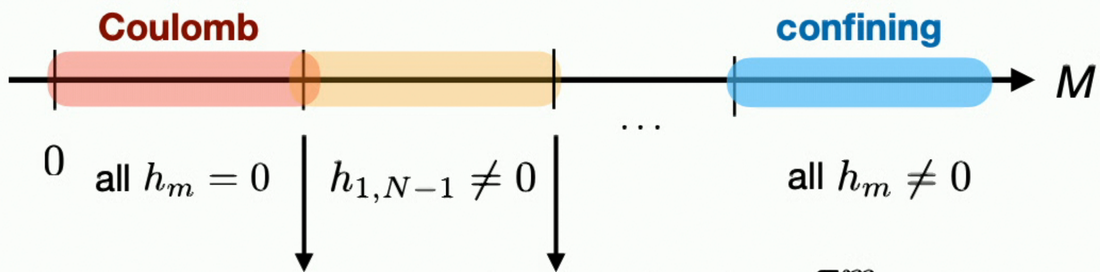
$$\varphi_m \sim a_{Dm} \text{ abelian vector multiplet scalars}$$

$$h_m^{i=1,2} \text{ hypermultiplet scalars (the monopoles)}$$

- Doing so, $V_{\cancel{\text{SUSY}}} \sim M^2 \mathrm{Tr}\bar{\phi}\phi \to M^2 K$ upon integrating out the light monopoles $\Rightarrow$ *K* determines the couplings in the potential.

Key features of the phase diagram as a function of *M*

[D'Hoker, Dumitrescu, Gerchkovitz, **EN**-to appear]



stable below $M_{*m} \lesssim M[\mu_{km}] \sim \Lambda \sin \frac{\pi m}{N}$

- Increasing *M*, the monopoles become activated pair-wise in a series of **first order** phase transitions.

- The monopole VEVs Higgs the $U(1)_D$ gauge groups. In the large-*M* phase the $\mathbb{Z}_N^{(1)}$ 1-form symmetry is unbroken, and the theory is confining.

Summary

- Revisiting the Seiberg-Witten solution to $\mathcal{N}=2$ *SU*(*N*) super Yang-Mills in 4d, we derive a useful Taylor series expansion for the periods in the strong coupling chamber of moduli space.
- Using this and other results, we compute and analyze the Kähler potential for general *N*, providing evidence for the conjecture that the minimum is the unique minimum.
- These expansions are readily applied to an analysis of candidate walls of marginal stability on slices of the *SU*(3) moduli space.
- The *SU*(*N*) Kähler potential plays a distinguished role in a soft-supersymmetry breaking analysis of the low energy structure of adjoint QCD.

Thank you!