Title: Quantum Criticality in the 2+1d Thirring Model

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Collection: Quantum Criticality: Gauge Fields and Matter

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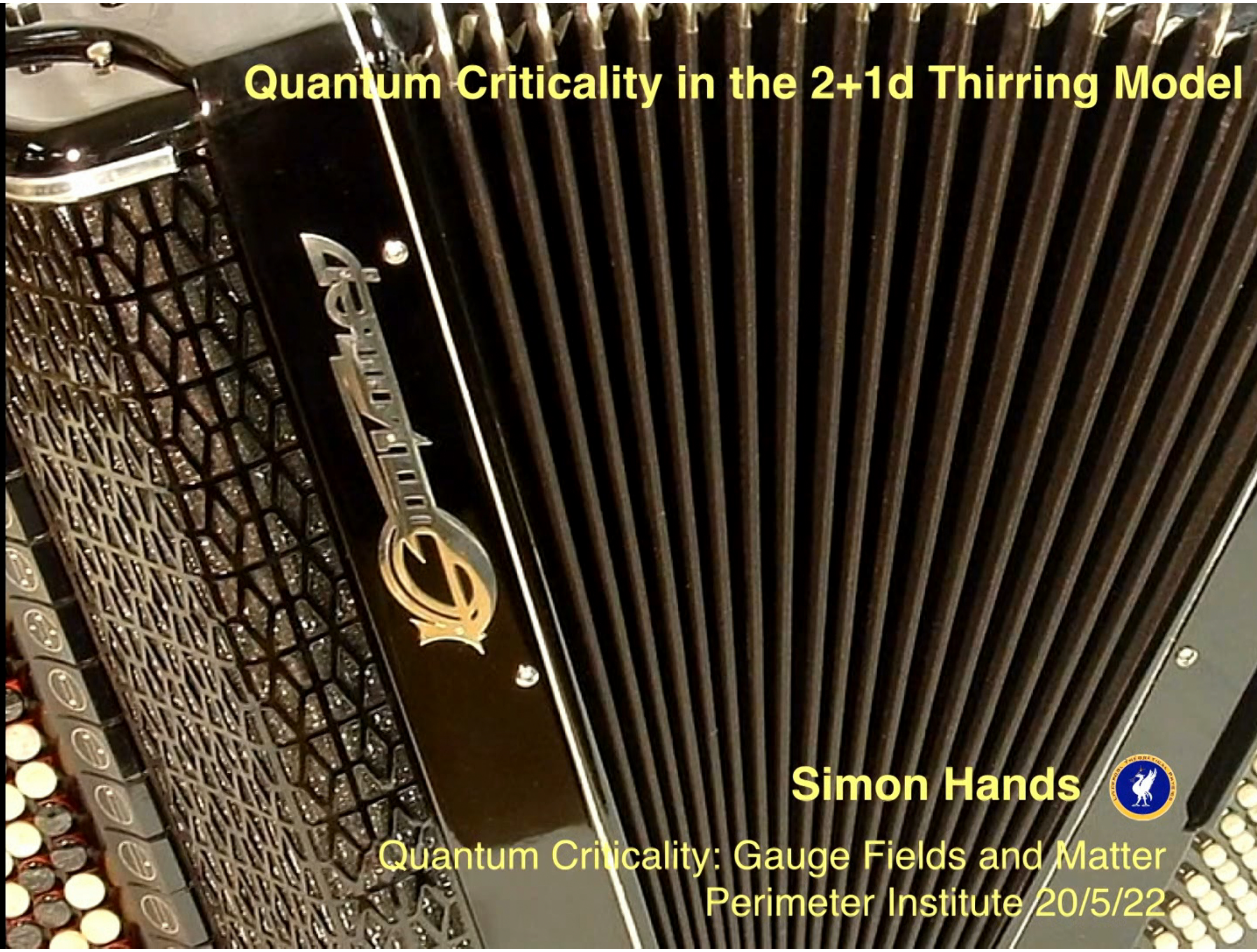
Abstract: The Thirring Model is a covariant quantum field theory of interacting fermions, sharing many features in common with effective theories of two-dimensional electronic systems with linear dispersion such as graphene.

For a small number of flavors and sufficiently strong interactions the ground state may be disrupted by condensation of particle- hole pairs leading to a quantum critical point. With no small dimensionless parameters in play in this regime the Thirring model is plausibly the simplest theory of fermions requiring a numerical solution.

I will review what is currently known focussing on recent results and challenges from simulations employing Domain Wall Fermions, a formulation drawn from state-of-the-art lattice QCD, to faithfully capture the underlying symmetries at the critical point.

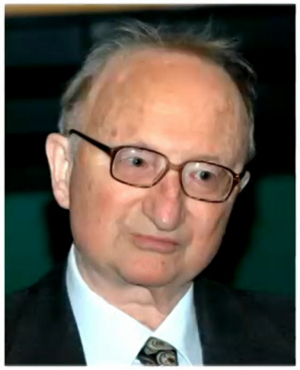

Quantum Criticality in the 2+1d Thirring Model
Simon Hands
Quantum Criticality: Gauge Fields and Matter
Perimeter Institute 20/5/22

Quantum field theories of particle physics are (for the most part) weakly-interacting at short distances: we have a reasonable idea of how to calculate with them.

But this is not the only possibility...

Inspired by condensed-matter systems eg. **graphene**, I will

- discuss QFTs of relativistic fermions in 2+1d exhibiting **critical behaviour,** signalling a new strongly-interacting QFT
- argue that to characterise the **Quantum Critical Point** it is crucial to capture relevant global symmetries accurately
- present results showing that different lattice discretisations tell very different stories...
- present new results for the model's spectrum



The Thirring Model in 2+1d

$$\mathcal{L} = \bar{\psi}_i(\partial\!\!\!/ + m)\psi_i + \frac{g^2}{2N}(\bar{\psi}_i\gamma_\mu\psi_i)^2$$

Covariant quantum field theory of
N flavors of interacting fermion in 2+1 dimensions.
Fermions are spinor fields $\psi, \bar{\psi}$ acted on by 4x4 Dirac matrices γ_μ

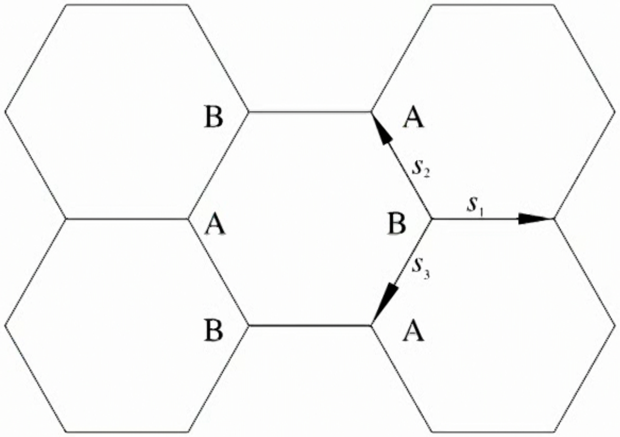
Interaction between conserved currents:
like charges ***repel***, opposite charges ***attract***

$$\partial\!\!\!/ \equiv \partial_\mu\gamma_\mu \quad \mu = 0, 1, 2 \qquad i = 1, \ldots, N$$

$$\mathrm{tr}(\gamma_\mu\gamma_\mu) = 4 \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \qquad \gamma_5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3$$

$$\mu, \nu = 0,1,2,3$$

Many applications of "Flatland fermions" in condensed matter physics

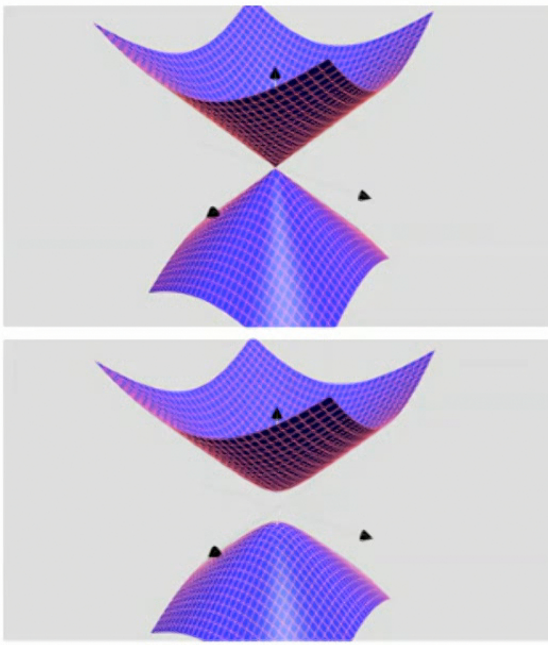


- Nodal fermions in *d*-wave superconductors
- Spin liquids in Heisenberg AFM
- surface states of topological insulators
- graphene

For sufficiently large g^2, or sufficiently small N, the Fock vacuum is conceivably disrupted by a particle-hole **bilinear condensate**

$$\langle \bar{\psi}\psi \rangle \equiv \frac{\partial \ln Z}{\partial m} \neq 0$$

resulting in a dynamically-generated mass gap at the Dirac point



Cf. chiral symmetry breaking in QCD

Hypothesis:
the semimetal-insulator transition at $g_c^2(N)$ defines a **Quantum Critical Point**: whose universal properties characterise low-energy excitations of graphene

D.T. Son, Phys. Rev. B**75** (2007) 235423

Corresponds to a new strongly-interacting QFT...
...a priori there are no small dimensionless parameters 😮

Continuum Symmetries in $d = 2 + 1$

$$\mathcal{S} = \int d^3x\ \bar{\Psi}(\gamma_\mu \partial_\mu)\Psi\ +\ m\bar{\Psi}\Psi$$

For m=0 S is invariant under global U(2N) symmetry generated by

(i) $\Psi \mapsto e^{i\alpha}\Psi\,; \qquad \bar{\Psi} \mapsto \bar{\Psi}e^{-i\alpha},$ (ii) $\Psi \mapsto e^{i\alpha\gamma_5}\Psi\,; \quad \bar{\Psi} \mapsto \bar{\Psi}e^{i\alpha\gamma_5}$

(iii) $\Psi \mapsto e^{\alpha\gamma_3\gamma_5}\Psi\,; \quad \bar{\Psi} \mapsto \bar{\Psi}e^{-\alpha\gamma_3\gamma_5}$ (iv) $\Psi \mapsto e^{i\alpha\gamma_3}\Psi\,; \quad \bar{\Psi} \mapsto \bar{\Psi}e^{i\alpha\gamma_3}$

For $m \neq 0$, γ_3 (iv) and γ_5 (ii) rotations are no longer symmetries

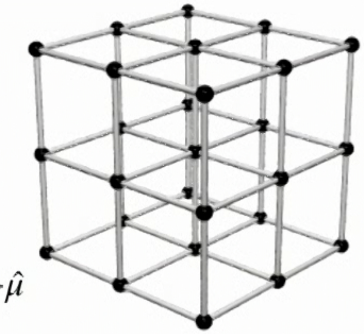
$\Rightarrow$ U(2N) → U(N)⊗U(N)

Mass term $m\bar{\Psi}\Psi$ is **hermitian** & invariant under **parity** $x_\mu \mapsto -x_\mu$

Two physically equivalent **antihermitian** *twisted* or *Kekulé* mass terms: $im_3\bar{\Psi}\gamma_3\Psi; \quad im_5\bar{\Psi}\gamma_5\Psi$

The *Haldane* mass $m_{35}\bar{\Psi}\gamma_3\gamma_5\Psi$ is *not* parity-invariant

Numerical Lattice Approach



Early work used staggered fermions

$$S_{latt} = \frac{1}{2}\sum_{x,\mu i} \bar{\chi}^i_x \eta_{\mu x}(1 + iA_{\mu x})\chi^i_{x+\hat{\mu}} - \bar{\chi}^i_x \eta_{\mu x}(1 - iA_{\mu x-\hat{\mu}})\chi^i_{x-\hat{\mu}}$$

non-unitary link fields

$$+m\sum_{xi} \bar{\chi}^i_x \chi^i_x + \frac{N}{4g^2}\sum_{x\mu} A^2_{\mu x}$$

vector auxiliary $A_{\mu x}$ defined on **link** *between* x and $x+\mu$

symmetry breaking resulting from gap generation: $U(N)\otimes U(N) \rightarrow U(N)$

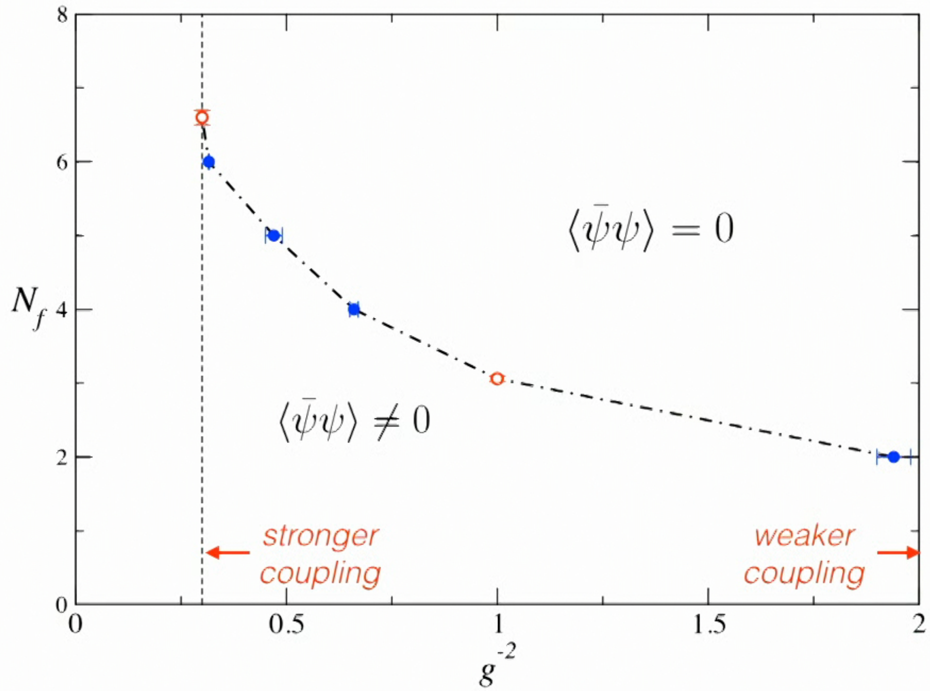
In **weak coupling** continuum limit

$U(2N_f)$ symmetry is recovered, with $N_f = 2N$

this is an instance of "lattice fermion doubling"

no such expectation **in general** at a **QCP**

Phase diagram of the Staggered Thirring Model



Find symmetry broken phase (gapped, insulating)
for small N, large g^2

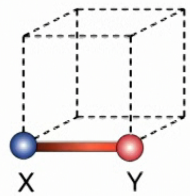
Christofi, SJH, Strouthos PR**D75** (2007) 101701

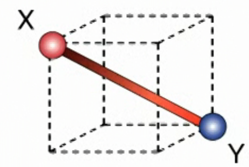
Staggered Fermion Bag Algorithm with minimal $N_f = 2$ (N=1)

Chandrasekharan & Li, PRL **108** (2012) 140404; PR**D88** (2013) 021701

Thirring Model: $\nu=0.85(1)$, $\eta=0.65(1)$, $\eta_\psi=0.37(1)$ $(N_f < N_{fc} \approx 7)$

U(1) GN Model: $\nu=0.849(8)$, $\eta=0.633(8)$, $\eta_\psi=0.373(3)$ $(N_f \to \infty$: $\nu=\eta=1)$



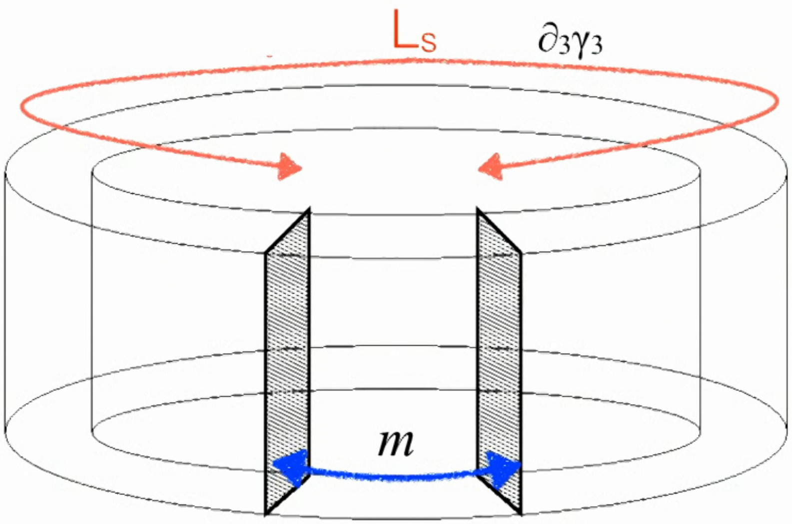


Interactions between staggered fields $\chi, \bar{\chi}$ spread over elementary cubes.
Only difference between Thirring & GN is body-diagonal term

Staggered fermions not reproducing expected distinction between Thirring and GN QCPs

… so we need "better" lattice fermions?

Domain Wall Fermions



$$\mathscr{L} = \bar{\Psi}(x,s) D_{DWF} \Psi(y,s')$$

Fermions propagate freely along a fictitious third direction of extent L_s with open boundaries

Basic idea as $L_s \to \infty$:

- zero-modes of D_{DWF} localised on walls are ± eigenmodes of γ_3
- Modes propagating in bulk can be decoupled (with cunning)

"Physical" fields in 2+1d target space

$$\psi(x) = P_- \Psi(x,1) + P_+ \Psi(x, L_s);$$
$$\bar{\psi}(x) = \bar{\Psi}(x, L_s) P_- + \bar{\Psi}(x,1) P_+;$$

with projectors $P_\pm = \frac{1}{2}(1 \pm \gamma_3)$

Bottom Up View…

in DWF approach we simulate
2+1+1d fermions

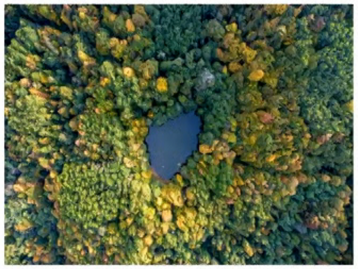
Desiderata…

- Modes localised on walls carry U(2N)-invariant physics
- Fermion doublers don't contribute to normalisable modes
- Bulk modes can be made to decouple

Claim…

It appears to work for….

- carefully-chosen "domain wall height" M
- smooth gauge field background



Top Down View…

write the fermion bilinear: $\bar{\psi}M\psi = \bar{\psi}(D+m)\psi$

Then U(2N) symmetry can be re-expressed:

$$\{\gamma_3, D\} = \{\gamma_5, D\} = [\gamma_3\gamma_5, D] = 0$$

There is no regular lattice discretisation respecting these relations, while simultaneously describing unitary, local dynamics of a single Dirac fermion species

Nielsen & Ninomiya, 1981

The closest we can get is articulated by the **Ginsparg-Wilson** relations:

$$\{\gamma_3, D\} = 2D\gamma_3 D \qquad \{\gamma_5, D\} = 2D\gamma_5 D \qquad [\gamma_3\gamma_5, D] = 0$$

RHS is $O(aD)$, so U(2N) recovered in long-wavelength limit if D local

By construction GW is satisfied by the 2+1d *overlap* operator

$$D_{ov} = \frac{1}{2}\left[(1+m_h) + (1-m_h)\frac{\mathscr{A}}{\sqrt{\mathscr{A}^\dagger \mathscr{A}}}\right]$$

with, eg. $\mathscr{A} = [2 + D_W - M]^{-1}[D_W - M]$ D_W local; $Ma = O(1)$

Shamir kernel $\gamma_3 \mathscr{A} \gamma_3 = \gamma_5 \mathscr{A} \gamma_5 = \mathscr{A}^\dagger$

locality of D_{ov} not manifest but confirmed numerically

SJH, Mesiti, Worthy PRD **102** (2020) 094502

DWF provide a regularisation of overlap with a *local* kernel in 2+1+1d 🙂

ie. $\frac{\det D_{\mathrm{DWF}}(m_i)}{\det D_{\mathrm{DWF}}(m_h = 1)} = \det D_{L_s}(m_i)$ with $\lim_{L_s \to \infty} D_{L_s} = D_{ov}$

SJH PLB **754** (2016) 264

Formulational issues

By analogy with QCD, formulate auxiliary field $A_\mu(x)$ throughout bulk and 3-static ie. $\partial_3 A_\mu = 0$:

$\Leftrightarrow$ A_μ couples to conserved DWF fermion current

$$\mathcal{S} = \bar{\Psi}\mathcal{D}\Psi = \bar{\Psi}D_W\Psi + \bar{\Psi}D_3\Psi + m_i S_i \quad \text{with} \quad \begin{aligned} D_W &= \gamma_\mu D_\mu - (\hat{D}^2 + M); \\ D_3 &= \gamma_3\partial_3 - \hat{\partial}_3^2, \end{aligned}$$

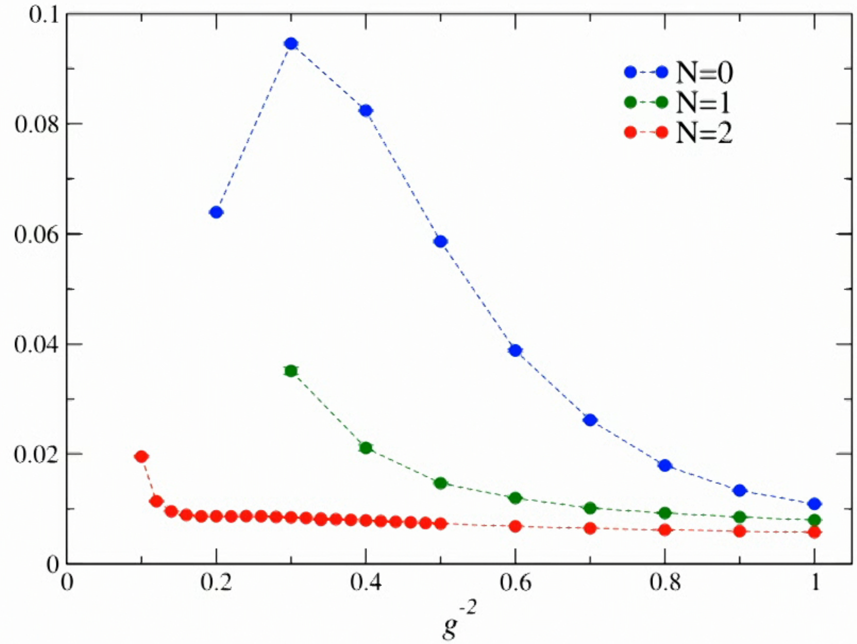
NB $D_\mu \propto (1 + iA_\mu)$, not e^{iA_μ},

ie. links are ***non-compact*** and ***non-unitary***

$[\partial_3, D_\mu] = [\partial_3, \hat{D}^2] = 0$ but $[\partial_3, \hat{\partial}_3^2] \neq 0$ on walls

obstruction to proving $\det\mathscr{D} > 0$

RHMC with measure $\sqrt{\det(\mathscr{D}^\dagger\mathscr{D})}$ for $N = 1$

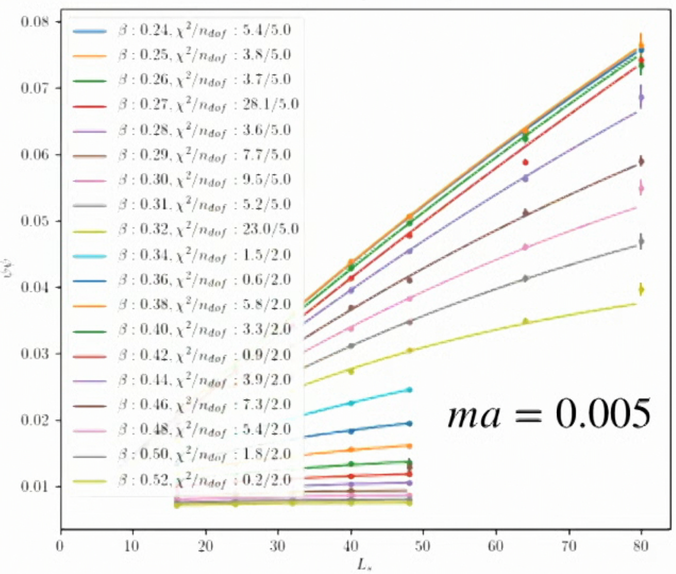


Exploratory Results with fixed $L_s = 16$

Bilinear condensate $\langle i\bar{\psi}\gamma_3\psi\rangle$ for $N = 0,1,2\ldots$
hierarchy consistent with $1 < N_c < 2$

But to achieve U(2) symmetry we need $L_s \to \infty\ldots$

Stress-testing DWF…

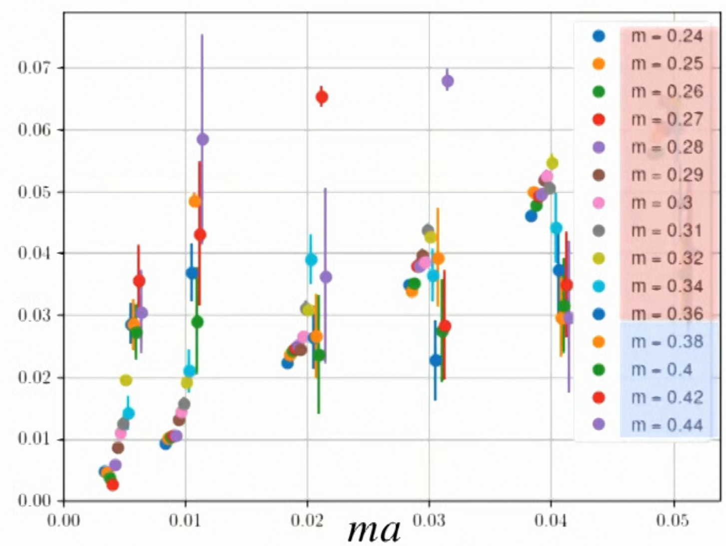


$$\langle\bar{\psi}\psi\rangle_\infty - \langle\bar{\psi}\psi\rangle_{L_s} = A(\beta, m)e^{-\Delta(\beta,m)L_s}$$

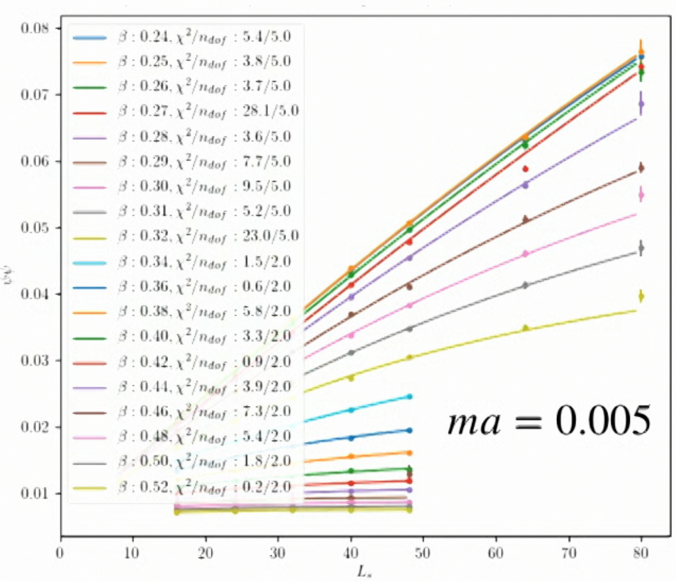
Have $L_s = 8, 16, \ldots, 80$

$L_s \to \infty$ not yet under control
at lightest masses, strongest couplings

Decay constant $\Delta(\beta, m)$:



Stress-testing DWF...



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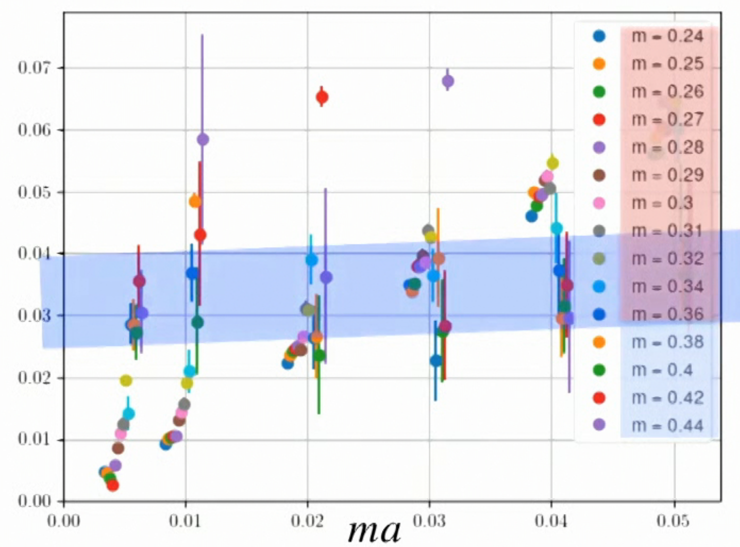
Have $L_s = 8,16,\ldots,80$

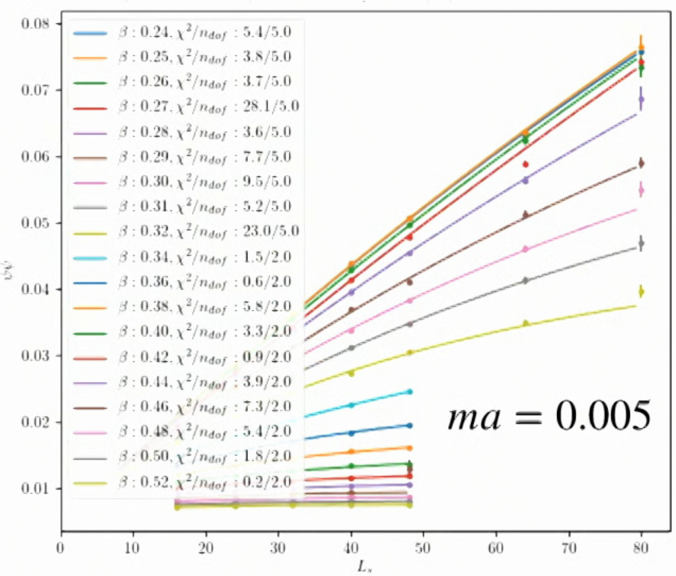
$L_s \to \infty$ not yet under control at lightest masses, strongest couplings

Decay constant $\Delta(\beta, m)$:

$\sim \propto m^0$ at weak coupling



Stress-testing DWF…



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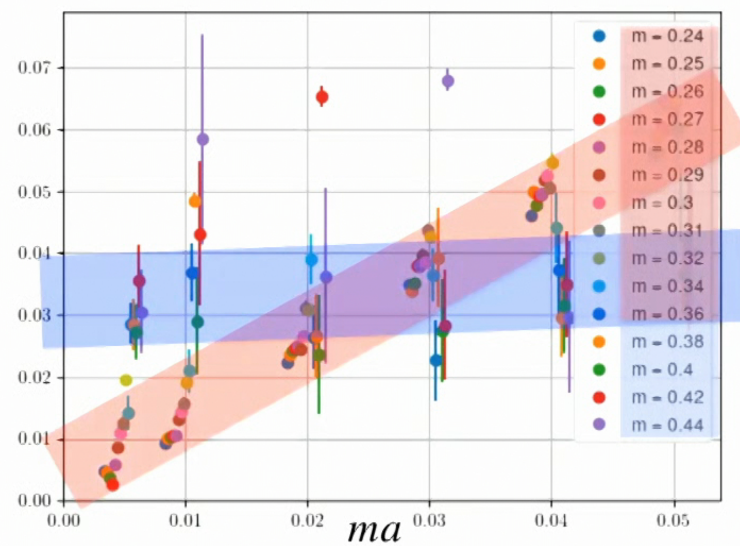
$L_s \to \infty$ not yet under control at lightest masses, strongest couplings

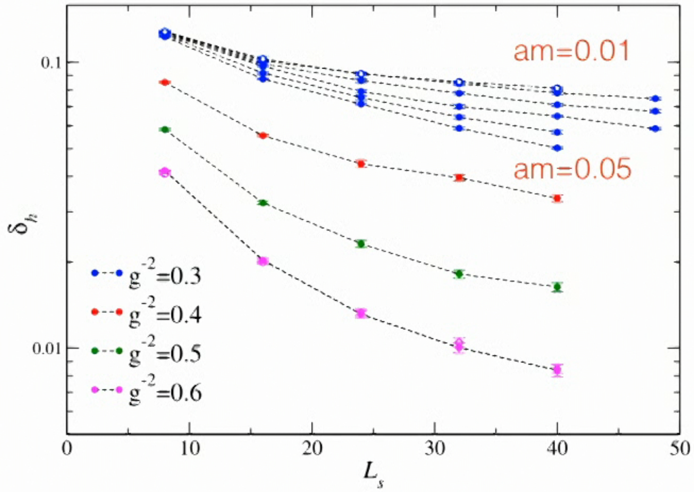
Decay constant $\Delta(\beta, m)$:

$\sim \propto m^0$ at weak coupling

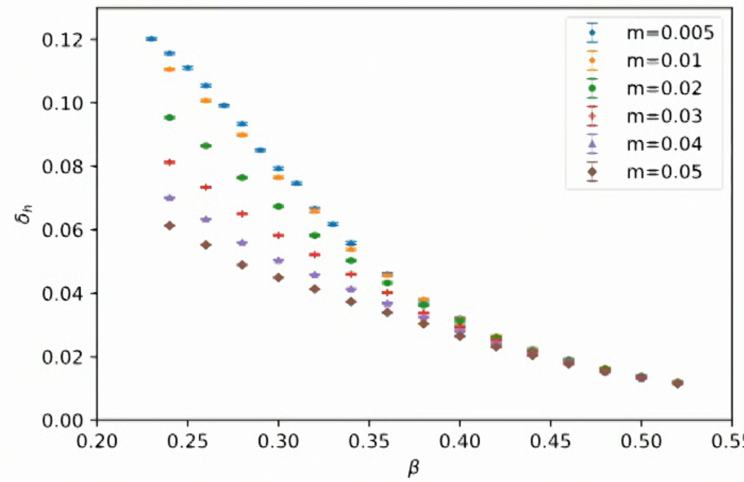
$\sim \propto m$ at strong coupling



U(2) symmetry restoration requires residual $\delta_h \to 0$



on $12^3 \times L_s \to \infty$

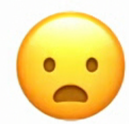


as a function of $\beta \equiv ag^{-2}$ on $16^3 \times 48$

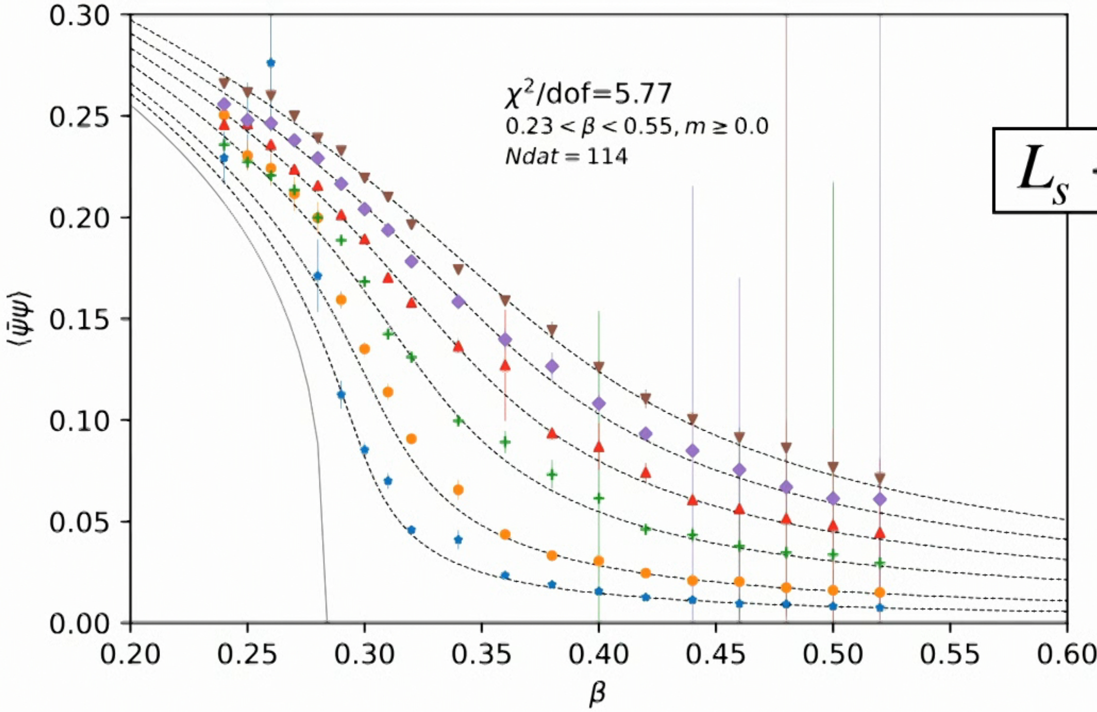
Qualitatively different at strong and weak coupling, and *slow*…

$16^3 \times L_s = 48$, am=0.005, ag^{-2}=0.3:

RHMC Hamiltonian step requires ~20k solver iterations (recall non-unitary links)



Latest data extrapolated with $L_s = 64,80$ and straddling β_c



$L_s \rightarrow \infty$

$$m = A(g^{-2} - g_c^{-2})\langle\bar{\psi}\psi\rangle^{\delta-1/\beta} + B\langle\bar{\psi}\psi\rangle^{\delta}$$

Fit to renormalisation group-inspired equation of state suggests QCP exists for $N=1$

SJH, Mesiti, Worthy arXiv:2110.03944

Preliminary **EoS fit in $L_s \to \infty$ limit** $1 < N_c < 2$

Critical parameters $\begin{cases} \beta_c \equiv g_c^{-2} = 0.283(1) \\ \delta = 4.17(5) \quad \beta = 0.320(5) \end{cases}$

hyperscaling ⇒ $\nu = 0.55(1) \quad \eta = 0.16(1)$

Cf: old result for $N = 1$ staggered fermion
⇔ $N = 1$ **Kähler-Dirac fermion**

≠ $N_f = 2$ Dirac fermions!

Del Debbio, SJH, Mehegan
NP**B502** (1997) 269
Christofi, SJH, Strouthos
PR**D75** (2007) 101701
SJH Symmetry **13** (2021) 8

$3 < N_c < 4$ $\delta = 2.75(9) \quad \beta = 0.57(2)$
$\nu = 0.71(3) \quad \eta = 0.60(4)$

Dirac and Kähler-Dirac fermions have distinct QCPs

First results for the Thirring Model spectrum

- need spectral quantities to identify correlation length & perform scale setting
- elucidate physical content of both symmetric and broken phases near QCP

For mass term $im_3\bar{\psi}\gamma_3\psi$,

parity operation: $\psi \mapsto \gamma_3\psi; \quad \bar{\psi} \mapsto \bar{\psi}\gamma_3$

We have examined:

- Spin-0 meson sector

  local sources, requires 2 inversions $D^{-1}(m_3),\ D^{-1}(-m_3)$

- Fermion quasiparticle

  wall source

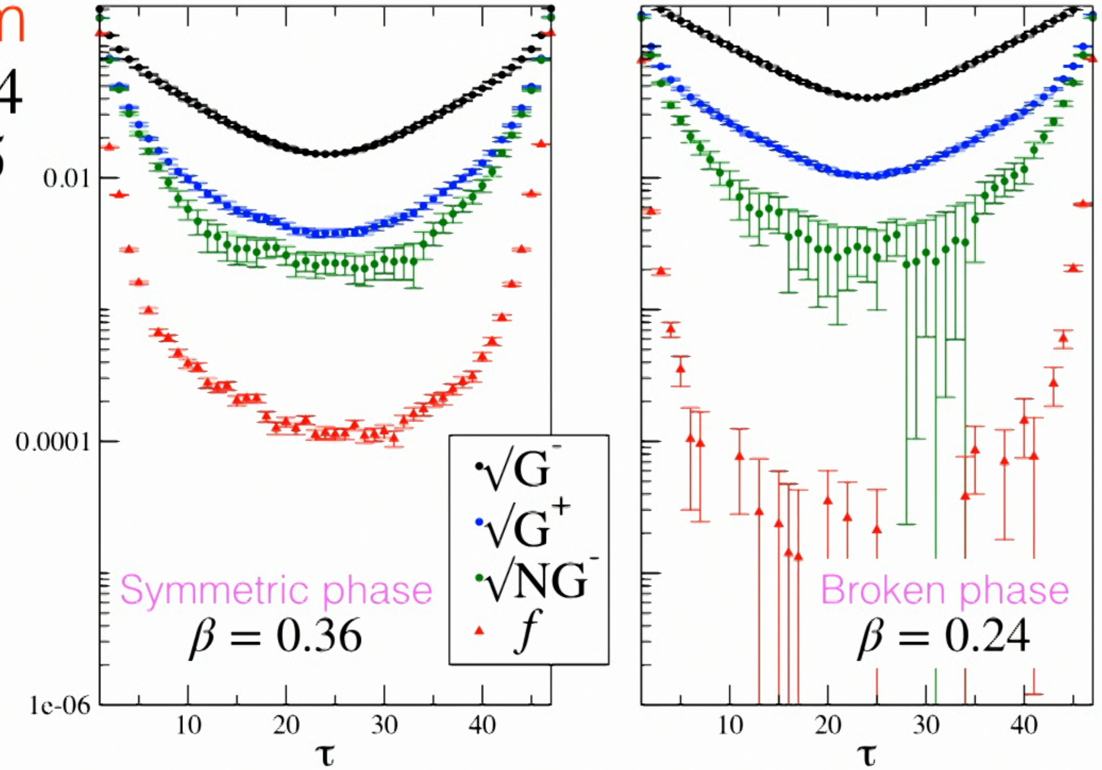
For U(2) symmetry broken by $\langle\bar{\psi}\gamma_3\psi\rangle \neq 0$:

- Goldstones $G^- \ \bar{\psi}\gamma_5\psi, \quad G^+ \ \bar{\psi}\psi$
- non-Goldstones $NG^- \ \bar{\psi}\gamma_3\gamma_5\psi, \ NG^+ \ \bar{\psi}\gamma_3\psi$

Results from

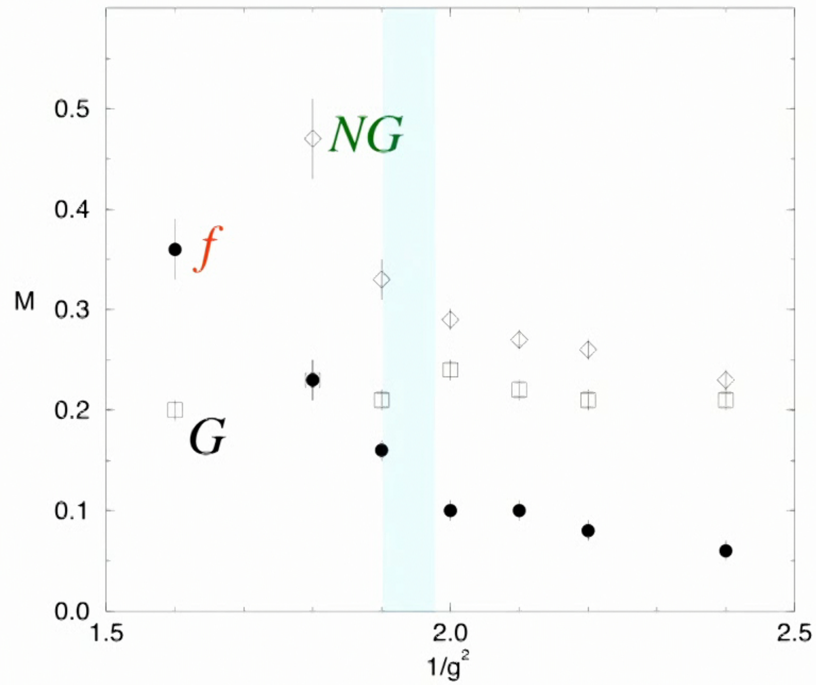
$16^2 \times 48 \times 64$

$m_3 a = 0.005$



- decay over first O(5) time slices dominated by unphysical branch cut
- symmetric phase meson states not obviously pure exponential
- NG^- correlator is difference of two large numbers and hence very noisy
- quasiparticle propagator with $m_3 \neq 0$ not symmetric under $\tau \mapsto -\tau$
- extremely noisy in broken phase

Plus ça change?



Cf. old results with staggered fermions on 16^3, $ma = 0.01$

Del Debbio, SJH & Mehegan, NPB**502** (1997) 269

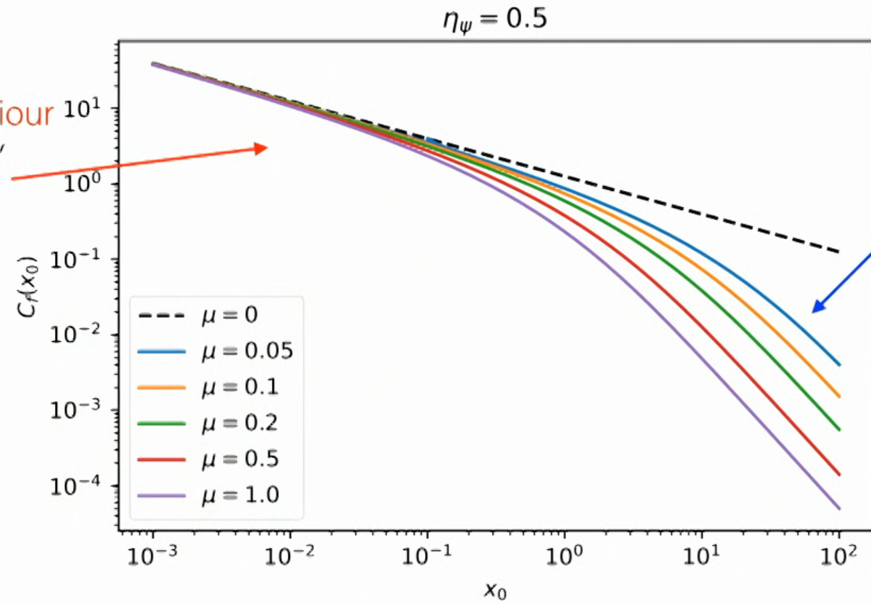
Fermion propagator at QCP

$$C_f(\vec{p}) \propto \frac{\hat{p}\,.\,\vec{\gamma}}{|\,\vec{p}\,|^{1-\eta_\psi}} \quad \Leftrightarrow \quad C_f(\vec{x}) \propto \frac{\hat{x}\,.\,\vec{\gamma}}{|\,\vec{x}\,|^{2+\eta_\psi}}$$

In symmetric phase $g^2 < g_c^2$ **postulate** $C_f(x_0) \propto \int dp_0 \frac{\cos p_0 x_0}{(p_0+\mu)^{1-\eta_\psi}}$

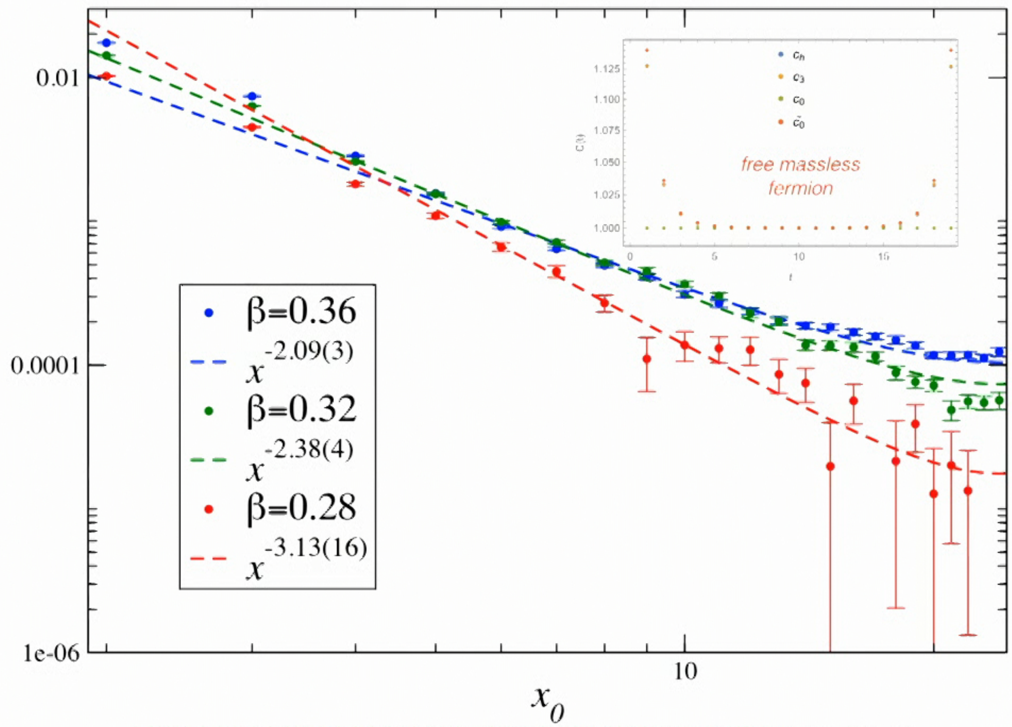
μ is an inverse correlation length,
but ***not*** related to a pole on the physical sheet

Conformal behaviour
$C_f(x_0) \sim x_0^{-\eta_\psi}$
for $\mu x_0 \ll 1$



Asymptotic behaviour
$C_f(x_0) \sim x_0^{-2}$
for $\mu x_0 \gg 1$

Work in Progress on 16^2x48x64 with m_3=0



Data consistent with algebraic decay of $C_f(x_0)$

$C_f(x_0) \sim x_0^{-2}$ at weak coupling,

inconsistent with flat behaviour of free massless fermions…

….but also inconsistent with $0 < \eta_\psi < 1$ closer to QCP

More work/thought needed

Are there "better" DWF formulations?

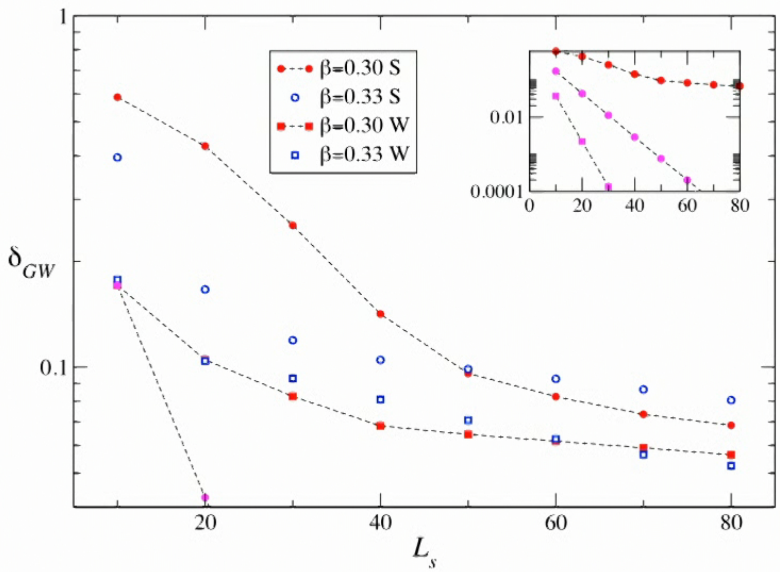
We have been using ***Shamir kernel*** throughout $\mathscr{A} = (2 + D_W)^{-1} D_W$

$$D^S_{0,DW} = \begin{pmatrix} D^+_W & -P_- & 0 & 0 \\ -P_+ & D^+_W & -P_- & 0 \\ 0 & -P_+ & D^+_W & -P_- \\ 0 & 0 & -P_+ & D^+_W \end{pmatrix}$$

Have recently experimented with ***Wilson kernel*** DWF $\mathscr{A} = D_W$

$$D^W_{0,DW} = \begin{pmatrix} D^+_W & D^-_W P_- & 0 & 0 \\ D^-_W P_+ & D^+_W & D^-_W P_- & 0 \\ 0 & D^-_W P_+ & D^+_W & D^-_W P_- \\ 0 & 0 & D^-_W P_+ & D^+_W \end{pmatrix}$$

with $D^{\pm}_W[A_\mu] = \gamma_\mu D_\mu - \hat{D}^2 - M \pm 1$



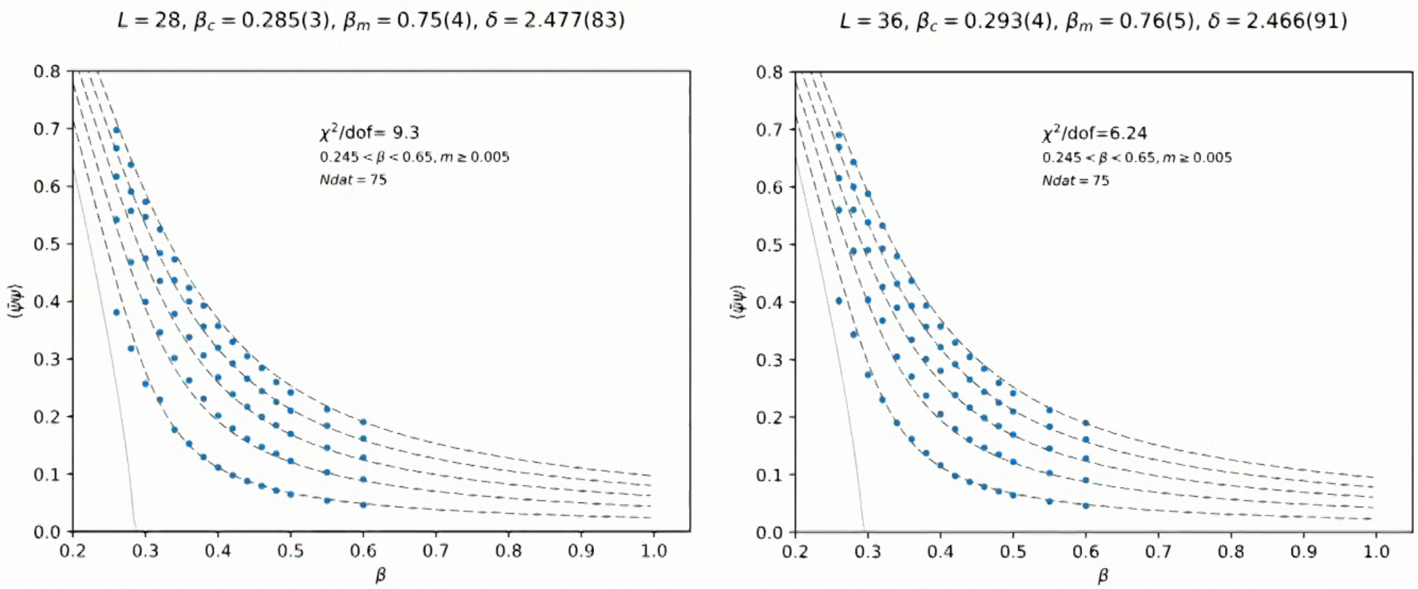
Significant improvement eg. in recovery of GW symmetry as $L_s \to \infty$

$$\delta_{GW} = ||(\gamma_3 D + D\gamma_3 - 2D\gamma_3 D)\phi||_\infty$$

SJH, Mesiti, Worthy PR**D102** 094502

Worthy, SJH 2112.02988

Equation of State with Wilson kernel DWF (preliminary)



First results reveal U(2) symmetry-breaking transition at very similar $\beta_c \sim 0.28 - 0.29$, with apparent convergence by $L_s \simeq 36$

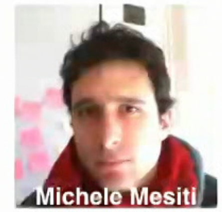
Discrepancy with Shamir exponent estimates $\beta_m \simeq 0.32$, $\delta \simeq 4.2$

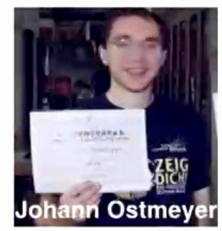
More work/thought needed

Summary

- new QCP for Dirac fermions in $2{+}1\mathrm{d}$ with $1 < N_c < 2$
- not everyone agrees: $N_c \approx 0.8$ with SLAC fermions

  Lenz, Wellegehausen & Wipf, PRD**100** (2019) 054501
- Dirac / Kähler-Dirac fermions support distinct QCPs
- meson propagators confirm Goldstone/non-Goldstone nature of bound states?
- critical fermion propagators access exponent η_ψ ?
- Wilson kernel DWF smarter way to access $L_s \to \infty$?





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