Title: Symplectic formulation of the covariant phase space with boundaries

Speakers: Juan Margalef-Bentabol

Series: Quantum Fields and Strings

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Abstract: There are two different standard ways of endowing a physical theory with a symplectic structure: the canonical and the covariant. The former is derived from the well-known symplectic structure of a certain cotangent bundle. The latter is based on the variational calculus. Including a boundary in the canonical formalism poses no problem, however, in the covariant formalism things break apart. In this talk, I will briefly introduce both formalisms without boundary and explain in detail a new framework that allows us to include boundaries in a straightforward way. To show it in action, time permitting, I will apply it to several theories of gravity. Finally, I will briefly show a new result where we proved that, actually, the canonical and covariant formalisms are equivalent in full generality.

Zoom Link: https://pitp.zoom.us/j/97641588140?pwd=dFl0SFdHOG5BYko2djNVWk11UkhSUT09

Symplectic formulation of the covariant phase space with boundaries

Juan Margalef Bentabol

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CAN vs CPS

CAN	CPS
Spacetime $M \rightsquigarrow \mathbb{R} \times \Sigma$	It deals with fields over the whole space-time.

CAN

- Dynamical perspective.
- Universal structures.
- Numerical approach.
- Quantization schemes.

- Aparent loss of symmetries.

CPS

- Explicit symmetries.
- Null infinity.
- Conserved quantities.
- Higher-derivative $\equiv$ 1st-order.

- No canonical structures.

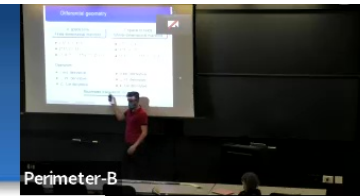
Differential geometry

M space-time Finite dimensional manifold	$\mathcal{F}$ space of fields Infinite-dimensional manifold
• TM, $V \in \mathfrak{X}(M)$.	• $T\mathcal{F}$, $\mathbb{V} \in \mathfrak{X}(\mathcal{F})$.
• T^*M, $\Omega^1(M)$.	• $T^*\mathcal{F}$, $\Omega^1(\mathcal{F})$.
• $(TM)^{\otimes r} \otimes (T^*M)^{\otimes s}$, $\mathfrak{T}(M)$.	• $(T\mathcal{F})^{\otimes r} \otimes (T^*\mathcal{F})^{\otimes s}$, $\mathfrak{T}(\mathcal{F})$.

Operators:

• d ext. derivative.	• $\mathbb{d}$ ext. derivative.
• ι_V int. derivative.	• $\mathbb{\iota}_{\mathbb{V}}$ int. derivative.
• $\mathcal{L}_V$ Lie derivative.	• $\mathbb{L}_{\mathbb{V}}$ Lie derivative.

Differential geometry



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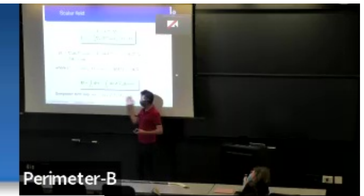
Bicomplex framework: $M \times \mathcal{F}$

Scalar field



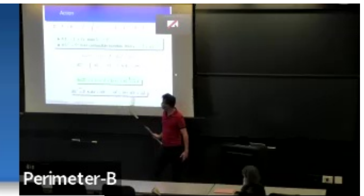
$$\mathcal{F} = C^\infty(M)$$
$$L(\phi) = \frac{1}{2}\nabla_\alpha\phi\nabla^\alpha\phi\,\mathrm{vol}_g \in \Omega^n(M)$$

$$\mathbb{d}L = \nabla_\alpha \mathbb{d}\phi\nabla^\alpha\phi\,\mathrm{vol}_g = \nabla_\alpha(\mathbb{d}\phi\nabla^\alpha\phi)\mathrm{vol}_g - \mathbb{d}\phi\nabla_\alpha\nabla^\alpha\phi\,\mathrm{vol}_g =$$
$$= E\mathbb{d}\phi + \mathrm{d}\Theta$$

where $E = -\Box\phi\,\mathrm{vol}_g$, $\Theta = \iota_W\mathrm{vol}_g$, and $W^\alpha = \mathbb{d}\phi\nabla^\alpha\phi$.

$$\boldsymbol{\Omega} = \int_\Sigma \mathbb{d}\Theta = \int_\Sigma \mathbb{d}\phi \wedge \nabla_{\vec{n}}\mathbb{d}\phi\,\mathrm{vol}_\Sigma$$

Symplectic form over $\mathrm{Sol} = \{\phi \in C^\infty(M)\ /\ \Box\phi = 0\}$



Action

$$\mathbb{S}^L : \mathcal{F} \to \mathbb{R} \qquad \mathbb{S}^L = \int_M L \qquad L \in \Omega^{(n,0)}(M \times \mathcal{F})$$

- If $L' = L + \mathrm{d}Y$, then $\mathbb{S}^{L'} = \mathbb{S}^L$.
- If $\mathbb{S}^{L'} = \mathbb{S}^L$ over contractible bundles, then $L' = L + \mathrm{d}Y$.

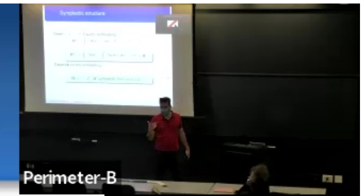
$$\mathrm{Sol}(\mathbb{S}^L) = \{\phi \in \mathcal{F} \ / \ \mathbb{d}_\phi \mathbb{S}^L = 0\}$$

$$\mathbb{d}\mathbb{S}^L = \int_M \mathbb{d}L \qquad \mathbb{d}L = E \wedge \mathbb{d}\phi + \mathrm{d}\Theta$$

$$\mathrm{Sol}(\mathbb{S}^L) = \{\phi \in \mathcal{F} \ / \ E(\phi) = 0\} \xrightarrow{\jmath_{\mathrm{CPS}}} \mathcal{F}$$

$$\mathbb{d}L' = E \wedge \mathbb{d}\phi + \mathrm{d}\Theta' \qquad \Theta' = \Theta + \mathbb{d}Y + \mathrm{d}Z$$

Symplectic structure



Given $\imath : \Sigma \hookrightarrow M$ Cauchy embedding:

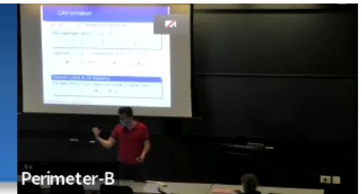
$$\boldsymbol{\Omega}^{\imath} := \int_{\Sigma} \imath^{*} \mathbb{d}\Theta \qquad \mathbb{d}\Theta \in \Omega^{(n-1,2)}(M \times \mathcal{F})$$

$$(\boldsymbol{\Omega}')^{\imath} = \int_{\Sigma} \imath^{*} \mathbb{d}\Theta' = \int_{\Sigma} \imath^{*} \mathbb{d}(\Theta + \mathbb{d}Y + \mathrm{d}Z) = \boldsymbol{\Omega}^{\imath}$$

Depends on the embedding $\imath$.

$$\boldsymbol{\Omega}_{\mathrm{CPS}} := \jmath^{*}_{\mathrm{CPS}} \boldsymbol{\Omega}^{\imath} \text{ symplectic form on } \mathrm{Sol}(\mathbb{S}^{L}).$$

Part IV: CAN formalism in a nutshell



CAN formalism

$\mathcal{Q} = \mathcal{Q}^1 \times \cdots \times \mathcal{Q}^M$ configuration space over Σ

CAN Lagrangian-action $\mathcal{L} : T\mathcal{Q} \to \mathbb{R}$

$$\mathcal{L} = \int_\Sigma \hat{\mathrm{L}} \qquad\qquad \hat{\mathrm{L}} \in \Omega^{(n-1,0)}(\Sigma \times T\mathcal{Q})$$

Lagrangian $(\mathcal{L}, T\mathcal{Q}) \;\to\;$ Hamiltonian $(\mathcal{H}, \mathrm{PCM} \overset{\jmath_{\mathrm{CAN}}}{\hookrightarrow} T^*\mathcal{Q})$

$$\boldsymbol{\Omega}_{T^*\mathcal{Q}} := \mathbb{d}q^I \wedge \mathbb{d}p_I \quad\longrightarrow\quad \boldsymbol{\Omega}_{\mathrm{CAN}} := \overset{*}{\jmath}_{\mathrm{CAN}} \boldsymbol{\Omega}_{T^*\mathcal{Q}}$$

Theorem (J.M-B, E.J.S Villaseñor)

For every theory of any order such that $\boldsymbol{\Omega}_{\mathrm{CAN}}$ exists, then

$$\boldsymbol{\Omega}_{\mathrm{CPS}} = \boldsymbol{\Omega}_{\mathrm{CAN}}$$

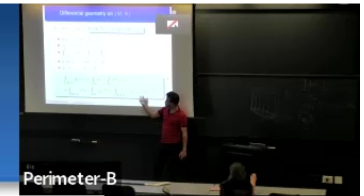
Part V: Adding boundaries

Differential geometry on (M, N)

$N \subset \partial M \xhookrightarrow{\jmath} M$ $\qquad$ $\Omega^r(M,N) := \Omega^r(M) \times \Omega^{r-1}(N)$

- $\underline{\mathrm{d}}(\alpha,\beta) := (\mathrm{d}\alpha, \jmath^*\alpha - \mathrm{d}\beta)$
- $(\alpha_1,\beta_1)\underline{\triangle}(\alpha_2,\beta_2) := \cdots$
- $\displaystyle\int_{(M,N)}(a,b) := \int_M a - \int_N b$
- $\underline{\partial}(M,N) := (\partial M \setminus N, \partial N)$

$$\int_{(M,N)} \underline{\mathrm{d}}(\alpha,\beta) = \int_M \mathrm{d}\alpha - \int_N (\jmath^*\alpha - \mathrm{d}\beta) =$$
$$= \int_{\partial M \setminus N} \jmath^*\alpha - \int_{\partial N} (\jmath|_\partial)^*\beta = \int_{(\partial M\setminus N, \partial N)} (\jmath^*\alpha, (\jmath|_\partial)^*\beta)$$



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- $\displaystyle\int_{(M,N)} (a, b) := \int_M a - \int_N b$
- $\underline{\partial}(M,N) := (\partial M \setminus N, \partial N)$
- $\underline{F}^*(\alpha, \beta) := (F^*\alpha, (F|_\partial)^*\beta)$.

$$\int_{(M,N)} \underline{\mathrm{d}}(\alpha, \beta) = \int_M \mathrm{d}\alpha - \int_N (\jmath^*\alpha - \mathrm{d}\beta) =$$
$$= \int_{\partial M \setminus N} \jmath^*\alpha - \int_{\partial N} (\jmath|_\partial)^*\beta = \int_{\underline{\partial}(M,N)} \underline{\jmath}^*(\alpha, \beta)$$

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$$\int_{(M,N)} \underline{\mathrm{d}}(\alpha,\beta) = \int_{\underline{\partial}(M,N)} \underline{\jmath}^*(\alpha,\beta)$$

- $\underline{\iota}_V(\alpha,\beta) := \cdots \qquad \underline{\mathcal{L}}_V(\alpha,\beta) := \cdots$

Differential geometry on (M, N)

$$\underline{\mathrm{d}}(\alpha, \beta) = (\mathrm{d}\alpha, \jmath^*\alpha - \mathrm{d}\beta) \qquad \underline{\partial}(M, N) := (\partial M \setminus N, \partial N)$$

$$\int_{(M,N)} \underline{\mathrm{d}}(\alpha, \beta) = \int_{\underline{\partial}(M,N)} \underline{\jmath}^*(\alpha, \beta)$$

(M, N) behaves as a manifold with boundary $\underline{\partial}(M, N)$

$(M, \partial M)$ behaves like a manifold **without boundary**!
$\underline{\partial}(M, \partial M) = \emptyset$

Relative bicomplex framework
$(M, N) \times \mathcal{F} := (M \times \mathcal{F}, N \times \mathcal{F})$

Action

$$\mathbb{S}^{(L,\bar{\ell})} = \int_{(M,\partial M)} (L, \bar{\ell}) \qquad\qquad L \in \Omega^{(n,0)}(M \times \mathcal{F})$$

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- If $L' = L + \mathrm{d}Y$ then $\mathbb{S}^{L'} = \mathbb{S}^{L}$

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$$\mathbb{S}^{(L,\overline{\ell})} = \int_{(M,\partial M)} (L,\overline{\ell}) \qquad\qquad (L,\overline{\ell}) \in \Omega^{(n,0)}((M,\partial M)\times\mathcal{F})$$

- If $(L',\overline{\ell}') = (L,\overline{\ell}) + \underline{\mathrm{d}}(Y,\overline{y})$, then $\mathbb{S}^{(L',\overline{\ell}')} = \mathbb{S}^{(L,\overline{\ell})}$

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- If $\mathbb{S}^{L'} = \mathbb{S}^{L}$ over c.b., then $L' = L + \mathrm{d}Y$

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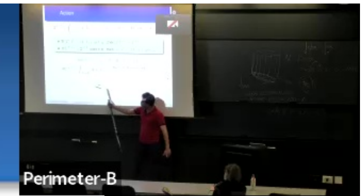
$$\mathrm{Sol}(\mathbb{S}^{(L,\bar{\ell})}) = \{\phi \in \mathcal{F} \;/\; \mathbb{d}_\phi \mathbb{S}^{(L,\bar{\ell})} = 0\}$$

$$\mathbb{d}\mathbb{S}^{L} = \int_M \mathbb{d}L \qquad \mathbb{d}L = E \wedge \mathbb{d}\phi + d\Theta$$

Action



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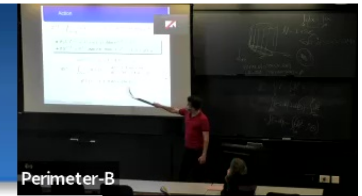
Action



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$$\underline{\mathbb{d}}(L,\bar{\ell}) = (E,\bar{b}) \wedge \mathbb{d}\phi + \underline{d}(\Theta,\bar{\theta})$$

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$$\mathrm{Sol}(\mathbb{S}^{(L,\bar{\ell})}) = \{\phi\in\mathcal{F}\ /\ (E,\bar{b})(\phi) = 0\} \xrightarrow{\jmath_{\mathrm{CPS}}} \mathcal{F}$$

Symplectic structure

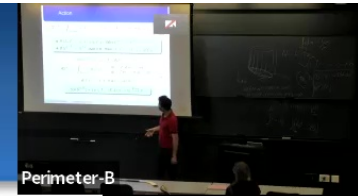
Given $\imath : \Sigma \hookrightarrow M$ Cauchy embedding:

$$\boldsymbol{\Omega}^{\imath} := \int_{(\Sigma,\partial\Sigma)} \underline{\imath}^* \underline{\mathbb{d}}(\Theta, \overline{\theta}) = \int_{\Sigma} \imath^* \mathbb{d}\Theta - \int_{\partial\Sigma} (\imath|_{\partial})^* \mathbb{d}\overline{\theta}$$

$$(\boldsymbol{\Omega}')^{\imath} = \int_{(\Sigma,\partial\Sigma)} \underline{\imath}^* \underline{\mathbb{d}}(\Theta', \overline{\theta}') = \int_{(\Sigma,\partial\Sigma)} \underline{\imath}^* \underline{\mathbb{d}}((\Theta, \overline{\theta}) + \underline{\mathbb{d}}(Y, \overline{y}) + \underline{d}(Z, \overline{z})) = \boldsymbol{\Omega}^{\imath}$$

Depends on the embedding $\imath$.

$\boldsymbol{\Omega}_{\mathrm{CPS}} := \jmath^*_{\mathrm{CPS}} \boldsymbol{\Omega}^{\imath}$ symplectic form on $\mathrm{Sol}(\mathbb{S}^{(L,\overline{\ell})})$.



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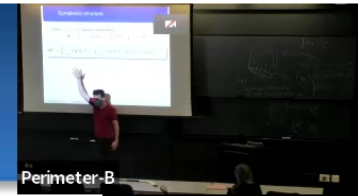
Symplectic structure



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Symplectic structure

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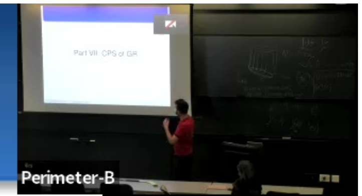
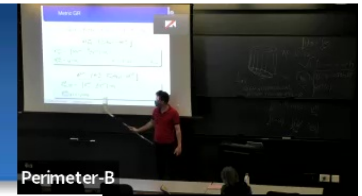
$$\boldsymbol{\Omega}_{\mathrm{CPS}} = \boldsymbol{\Omega}_{\mathrm{CAN}}$$

Conclusions

1. We have developed a new formalism to derive the CPS presymplectic form canonically associated with an action.
2. The bicomplex formalism is essential.
3. We have proved that the CPS and CAN presymplectic structures are equivalent.
4. Also true for higher-order theories and with boundaries.
5. We have successfully applied to several theories of gravity solving some open questions.



Part VII: CPS of GR



Metric GR

$$L^{(m)}_{\mathrm{EH}}(g) = R_g \mathrm{vol}_g \qquad \overline{\ell}^{(m)}_{\mathrm{GHY}}(g) = -2\overline{g}^{\overline{\alpha}\overline{\beta}}\overline{K}_{\overline{\alpha}\overline{\beta}}\mathrm{vol}_{\overline{g}}$$

$$\mathbb{d} L^{(m)}_{\mathrm{EH}} = E^{\alpha\beta}_{(m)} \mathbb{d} g_{\alpha\beta} + d\Theta^{\mathrm{GR}}_{(m)}$$

$$E^{\alpha\beta}_{(m)} = -\left(\mathrm{Ric}^{\alpha\beta} - \frac{R_g}{2} g^{\alpha\beta}\right)\mathrm{vol}_g$$

$$\Theta^{\mathrm{GR}}_{(m)} = \iota_{\vec{W}}\mathrm{vol}_g \qquad W_\alpha = \nabla^\beta \mathbb{d} g_{\alpha\beta} - g^{\mu\nu}\nabla_\alpha \mathbb{d} g_{\mu\nu}$$

$$\mathbb{d}\overline{\ell}^{(m)} - \jmath^*\Theta^{\mathrm{GR}}_{(m)} = \overline{b}^{\overline{\alpha}\overline{\beta}}_{(m)} \mathbb{d}\overline{g}_{\overline{\alpha}\overline{\beta}} - d\overline{\theta}^{\mathrm{GR}}_{(m)}$$

$$\overline{b}^{\overline{\alpha}\overline{\beta}}_{(m)}(g) = -\left(\overline{K}^{\overline{\alpha}\overline{\beta}} - \overline{K}\overline{g}^{\overline{\alpha}\overline{\beta}}\right)\mathrm{vol}_{\overline{g}}$$

$$\overline{\theta}^{\mathrm{GR}}_{(m)}(g) = \iota_{\vec{V}}\mathrm{vol}_{\overline{g}}$$

Metric GR

$$L^{(m)}_{\text{EH}}(g) = R_g \text{vol}_g \qquad \overline{\ell}^{(m)}_{\text{GHY}}(g) = -2\overline{g}^{\overline{\alpha}\overline{\beta}}\overline{K}_{\overline{\alpha}\overline{\beta}}\text{vol}_{\overline{g}}$$

$$\boxed{\mathbb{d} L^{(m)}_{\text{EH}} = E^{\alpha\beta}_{(m)} \mathbb{d} g_{\alpha\beta} + d\Theta^{\text{GR}}_{(m)}}$$

$$E^{\alpha\beta}_{(m)} = -\left(\text{Ric}^{\alpha\beta} - \frac{R_g}{2} g^{\alpha\beta}\right)\text{vol}_g$$

$$\Theta^{\text{GR}}_{(m)} = \iota_{\vec{W}} \text{vol}_g \qquad W_\alpha = \nabla^\beta \mathbb{d} g_{\alpha\beta} - g^{\mu\nu}\nabla_\alpha \mathbb{d} g_{\mu\nu}$$

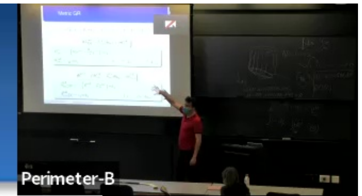
$$\boxed{\mathbb{d}\overline{\ell}^{(m)} - \jmath^* \Theta^{\text{GR}}_{(m)} = \overline{b}^{\overline{\alpha}\overline{\beta}}_{(m)} \mathbb{d}\overline{g}_{\overline{\alpha}\overline{\beta}} - d\overline{\theta}^{\text{GR}}_{(m)}}$$

$$\overline{b}^{\overline{\alpha}\overline{\beta}}_{(m)}(g) = -\left(\overline{K}^{\overline{\alpha}\overline{\beta}} - \overline{K}\overline{g}^{\overline{\alpha}\overline{\beta}}\right)\text{vol}_{\overline{g}}$$

$$\overline{\theta}^{\text{GR}}_{(m)}(g) = \iota_{\vec{V}} \text{vol}_{\overline{g}} \qquad \overline{V}_{\overline{\alpha}} = -\jmath^{\alpha}_{\overline{\alpha}} \nu^\beta \mathbb{d} g_{\alpha\beta}$$

Tetrad GR

$$\begin{aligned} \Phi_{\mathrm{GR}} : \quad \text{Tetrads} &\longrightarrow \quad \text{Metrics} \\ e^I_\alpha &\longmapsto g_{\alpha\beta} = \eta_{IJ} e^I_\alpha e^J_\beta \end{aligned}$$



Metric GR

$$L^{(m)}_{\mathrm{EH}}(g) = R_g \mathrm{vol}_g \qquad \bar{\ell}^{(m)}_{\mathrm{GHY}}(g) = -2\bar{g}^{\bar{\alpha}\bar{\beta}}\bar{K}_{\bar{\alpha}\bar{\beta}}\mathrm{vol}_{\bar{g}}$$

$$\mathbb{d} L^{(m)}_{\mathrm{EH}} = E^{\alpha\beta}_{(m)} \mathbb{d} g_{\alpha\beta} + d\Theta^{\mathrm{GR}}_{(m)}$$

$$E^{\alpha\beta}_{(m)} = -\left(\mathrm{Ric}^{\alpha\beta} - \frac{R_g}{2} g^{\alpha\beta}\right)\mathrm{vol}_g$$

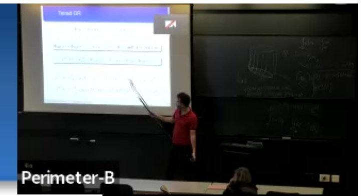
$$\Theta^{\mathrm{GR}}_{(m)} = \iota_{\vec{W}}\mathrm{vol}_g \qquad W_\alpha = \nabla^\beta \mathbb{d} g_{\alpha\beta} - g^{\mu\nu}\nabla_\alpha \mathbb{d} g_{\mu\nu}$$

$$\mathbb{d}\bar{\ell}^{(m)} - \jmath^*\Theta^{\mathrm{GR}}_{(m)} = \bar{b}^{\bar{\alpha}\bar{\beta}}_{(m)}\mathbb{d}\bar{g}_{\bar{\alpha}\bar{\beta}} - d\bar{\theta}^{\mathrm{GR}}_{(m)}$$

$$\bar{b}^{\bar{\alpha}\bar{\beta}}_{(m)}(g) = -\left(\bar{K}^{\bar{\alpha}\bar{\beta}} - \bar{K}\bar{g}^{\bar{\alpha}\bar{\beta}}\right)\mathrm{vol}_{\bar{g}}$$

$$\bar{\theta}^{\mathrm{GR}}_{(m)}(g) = \iota_{\vec{V}}\mathrm{vol}_{\bar{g}} \qquad \bar{V}_{\bar{\alpha}} = -\jmath^{\alpha}_{\bar{\alpha}}\nu^\beta \mathbb{d} g_{\alpha\beta}$$

Tetrad GR



$$\begin{array}{rccc} \Phi_{\mathrm{GR}}: & \text{Tetrads} & \longrightarrow & \text{Metrics} \\ & e^I_\alpha & \longmapsto & g_{\alpha\beta} = \eta_{IJ} e^I_\alpha e^J_\beta \end{array}$$

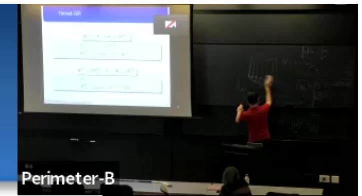
$$\Phi_{\mathrm{GR}}(e) = \Phi_{\mathrm{GR}}(e') \qquad \longleftrightarrow \qquad e'_I = \Psi_I{}^J e_J \text{ with } \Psi \in SO(1,3)$$

$$L^{(t)}(e) := L^{(m)}_{\mathrm{EH}} \circ \Phi_{\mathrm{GR}}(e) \qquad \overline{\ell}^{(t)}(e) := \overline{\ell}^{(m)}_{\mathrm{GHY}} \circ \Phi_{\mathrm{GR}}(e)$$

$$L^{(t)}(e) = \frac{1}{2} \varepsilon_{IJKL} F^{IJ} \wedge e^K \wedge e^L \qquad F^{IJ} = \mathrm{d}\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ}$$

$$\overline{\ell}^{(t)}(e) = -\frac{1}{2} \varepsilon_{IJKL} \left(2N^I \mathrm{d}N^J - \overline{\omega}^{IJ} \right) \overline{e}^K \wedge \overline{e}^L \qquad N^I = e^I_\alpha \nu^\alpha$$

Tetrad GR



$$\mathbb{d}L^{(t)} = E_I^{(t)} \wedge \mathbb{d}e^I + \mathrm{d}\Theta^{\mathrm{GR}}_{(t)}$$

$$E_I^{(t)}(e) = -\varepsilon_{IJKL}F^{JK} \wedge e^L$$

$$\Theta^{\mathrm{GR}}_{(t)} = \frac{1}{2}\varepsilon_{IJKL}e^I \wedge e^J \wedge \mathbb{d}\omega^{KL}$$

$$\mathbb{d}\overline{\ell}^{(t)} - \jmath^*\Theta^{\mathrm{GR}}_{(t)} = \overline{b}_I^{(t)} \wedge \mathbb{d}\overline{e}^I - \mathrm{d}\overline{\theta}^{\mathrm{GR}}_{(t)}$$

$$\overline{b}_I^{(t)} = \cdots$$

$$\overline{\theta}^{\mathrm{GR}}_{(t)} = \varepsilon_{IJKL}\overline{e}^I \wedge \overline{e}^J \wedge N^K \mathbb{d}N^L$$

Metric vs Tetrad

$$\mathbb{S}^{(t)} = \mathbb{S}^{(m)} \circ \Phi_{\mathrm{GR}} \;\to\; \mathbb{d}\mathbb{S}^{(t)} = \mathbb{d}\mathbb{S}^{(m)} \cdot \mathbb{d}\Phi_{\mathrm{GR}} \;\to\; \mathrm{Sol}_{\mathrm{GR}}^{(t)} = \Phi_{\mathrm{GR}}^{-1}\mathrm{Sol}_{\mathrm{GR}}^{(m)}$$

$$\Theta_{(m)}^{\mathrm{GR}} = \Theta_{(t)}^{\mathrm{GR}} + \mathrm{d}\alpha \qquad \overline{\theta}_{(m)}^{\mathrm{GR}} = \overline{\theta}_{(t)}^{\mathrm{GR}} + \jmath^*\alpha$$

$$\left(\Theta_{(m)}^{\mathrm{GR}}, \overline{\theta}_{(m)}^{\mathrm{GR}}\right) = \left(\Theta_{(t)}^{\mathrm{GR}}, \overline{\theta}_{(t)}^{\mathrm{GR}}\right) + \underline{\mathrm{d}}(\alpha, 0)$$

$$\boldsymbol{\Omega}_{(t)}^{\mathrm{GR}} = \Phi_{\mathrm{GR}}^{*}\boldsymbol{\Omega}_{(m)}^{\mathrm{GR}}$$

Conclusions

1. We prove the CPS equivalence of metric-GR and tetrad-GR (same for Palatini and Holst).
2. The problem is cohomological (relative version), so we need cohomological methods.

Conclusions

1. We prove the CPS equivalence of metric-GR and tetrad-GR (same for Palatini and Holst).
2. The problem is cohomological (relative version), so we need cohomological methods.
3. Lagrangian boundary terms are essential.
4. Symplectic boundary terms are required.

Thanks for your attention!

- J. M-B, E.J.S. Villaseñor
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