Title: Towards deriving a gravity dual to complexity

Speakers: Michal Heller

Series: Quantum Fields and Strings

Date: May 24, 2022 - 2:00 PM

URL: https://pirsa.org/22050060

Abstract: Holographic complexity proposals are interesting because, on one hand, they express universal properties of black hole interiors and, on the other, they go beyond the area-centric view on quantum gravity. Our best take on complexity in quantum field theory is based on assigning a cost to circuits generated by time-dependent Hamiltonians and its minimization. I will discuss recent progress on understanding how the relevant circuits are realized in holography and what are possible gravity duals to good costs. Based on 2112.12158, 2203.08842 and work in progress.

Zoom Link: https://pitp.zoom.us/j/96487059058?pwd=ZHk1ZHpXMEZoT0UycUZqTEdIb1ZpQT09

Towards deriving a gravity dual to complexity

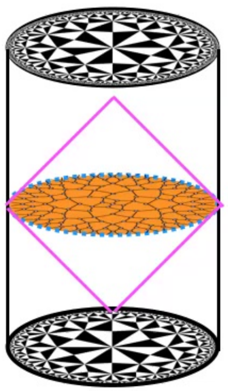
Michal P. Heller

Ghent University

2203.08842 (and 2101.01185) with Chandra, de Boer, Flory, Hörtner, Rolph
2112.12158 with Erdmenger, Flory, Gerbershagen, Weigel

(Zoo of) holographic complexity proposals

1402.5674 by Susskind, 1509.07876 by Brown et al., 1610.02038 by Couch et al., … , 2111.02429 by Belin et al.



$\mathcal{C}_V \sim$ volume of $\begin{matrix}\text{max (Lorentzian)}\\ \text{min (Euclidean)}\end{matrix}$ volume time slice

$\mathcal{C}_A \sim$ bulk action in the Wheeler - de Witt patch

$\mathcal{C}_{V\,2.0} \sim$ bulk volume of the Wheeler - de Witt patch

$\mathcal{C}_{anything} \sim$ covariantly defined bulk volumes using a whole class of functionals

…

1/23

Complexity in quantum field theory

The approach that naturally applies to QFTs comes from quant-ph/0502070 by Nielsen :

different costs

$$|T\rangle \sim U|R\rangle$$

with

$$U = \mathcal{P}e^{-i\int_0^1 d\tau\, Q(\tau)}$$

$$Q(\tau) = \sum_I O_I\, \epsilon^I(\tau)$$

$$\mathcal{C}_{L_1} \sim \min\left[\int_0^1 \mathrm{d}\tau \sum_I \Pi_I\, |\epsilon^I(\tau)|\right]$$

$$\mathcal{C}_{L_2} \sim \min\left[\int_0^1 \mathrm{d}\tau \sqrt{\sum_{I,J} \Pi_{IJ}\, \epsilon^I(\tau)\, \epsilon^J(\tau)}\right]$$

$$\mathcal{C}_{FS} \sim \min\left[\int_0^1 \mathrm{d}\tau \sqrt{\langle Q^2\rangle - |\langle Q\rangle|^2}\right]$$

Implementations of these ideas in free QFTs reproduce some of the static properties of holographic complexity, but crucially rely on Gaussianity

1707.08570 by Jefferson & Myers, 1707.08582 with Chapman, Marrochio, Pastawski, …

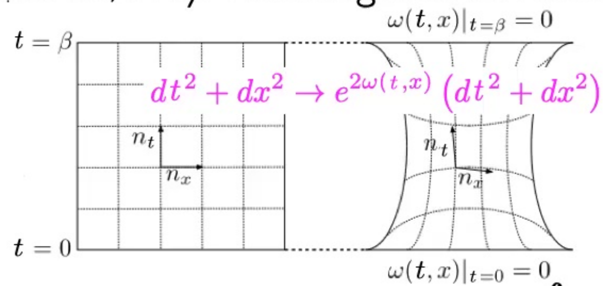
2/23

Path integral optimization

1703.00456 by Caputa, Kundu, Miyaji, Takayanagi, Watanabe, …

In conformal field theories, Weyl-rescalings act as redundancies

$\rho_\beta \sim e^{-\beta H}$:



Path integral optimization : cost ~ Liouville action $\int dtdx \left\{ \frac{1}{\delta^2} e^{2\omega} + (\partial\omega)^2 \right\}$

I always liked this idea because it phrases the problem in terms of a QFT sources, which should allow to map it to gravity via $Z_{QFT}[J] = Z_{grav}[J]$

3/23

Cost of conformal transformations

1807.04422 by Caputa, Magan;

2004.03619 by Erdmenger et al.; 2005.02415 and 2007.11555 with Mario Flory

The stress tensor sector of 1+1D CFTs offer a soluble example of cost and complexity problem, which is universal and, therefore, should map to gravity

Such circuits are realized by unitaries of the form $U = \mathcal{P} e^{-i \int_0^1 \mathrm{d}\tau \, Q(\tau)}$

with $Q(\tau) = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} T(\sigma) \, \dot{f}(\tau, F(\tau,\sigma))$ and $f(\tau, F(\tau,\sigma)) = \sigma$

right- or left-moving component of $T_{\mu\nu}$ $\qquad \epsilon(\tau,\sigma) = -\frac{\dot{F}(\tau,\sigma)}{F'(\tau,\sigma)}$

and can be thought of as a gradual diffeomorphism of a circle

$\sigma \to f(\sigma)$ via $f(\tau,\sigma)$ with $f(\tau=0,\sigma) = \sigma$ and $f(\tau=1,\sigma) = f(\sigma)$

4/23

The logic of my talk

perspective: bulk is a circuit

fixed bulk geometry
giving rise to many circuits

specially-crafted bulk
geometry dual to a particular circuit
(holding $|R\rangle$ and $|T\rangle$ fixed)

$\downarrow$ *choice*

bulk cost

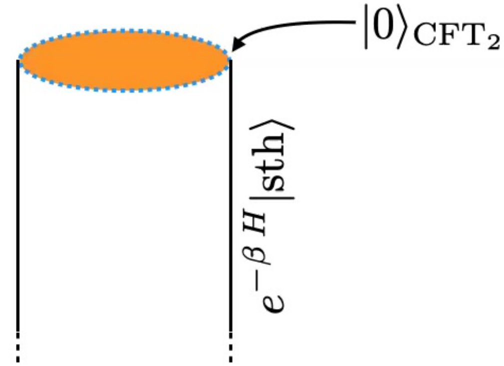
$\downarrow$ *minimization*

(possibly new)
holographic complexities

Spacetime as a circuit

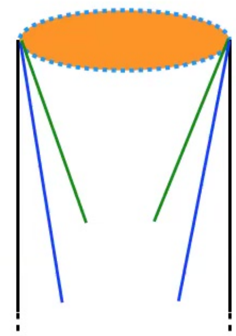
From now for a while we are Euclidean

One can think of half of the $EAdS_3$ as a state preparation of an unnormalized vacuum using Euclidean time evolution on the asymptotic boundary:



One can then speculate that curving out other boundaries reaching  provides an alternative way of preparing $|0\rangle_{\mathrm{CFT}_2}$

1808.09072 by Takayanagi



Seen from this perspective,  = H_2 defines one example of a circuit that perhaps optimizes some cost function

6/23

Beyond path integral optimization

It is know from the outset that path integral optimization produces an H_2 as an optimum of the Liouville action $\int dt dx \left\{ \frac{1}{\delta^2} e^{2\omega} + (\partial\omega)^2 \right\}$ in the limit $\beta \to \infty$

While superficially one may claim it is  and call it a day, some caution is required:

- there infinitely many H_2's in $EAdS_3$ geometry
- the Liouville action should be seen as two leading terms in a derivative expansion both as a change in the path integral measure and a cost function
- optimizing the Liouville action sets variations of ω to occurs on the cut-off scale δ, which violates the assumption underlying the gradient expansion

1904.02713 with Camargo, Jefferson, Knaute

This provides a strong motivation to go beyond the Liouville action and CFT_2

7/23

The role of $T\bar{T}$ deformations

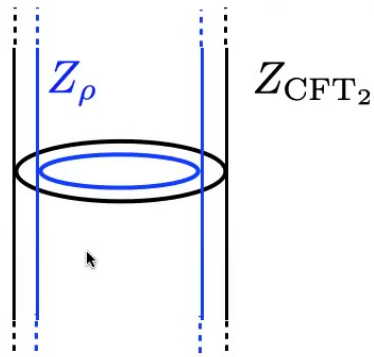
2101.01185 with Chandra, de Boer, Flory, Hörtner, Rolph

We know by now that a gravity partition function with the asymptotic region in EAdS_3 cut out corresponds to a $T\bar{T}$ deformation of the boundary CFT

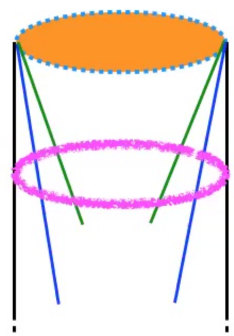
1611.03470 by McGough, Mezei, Verlinde

$$\frac{\delta}{\delta\rho}\log Z_\rho \sim \int d^2\xi\,\sqrt{g}\,\left(\pi_{\mu\nu}\pi^{\mu\nu} - (\pi^\rho_\rho)^2\right)$$

Brown-York stress tensor at a fixed cut-off ρ



Picking a bulk time slice in our procedure implies that the circuits we are after create states in $T\bar{T}$-deformed theories as intermediate states



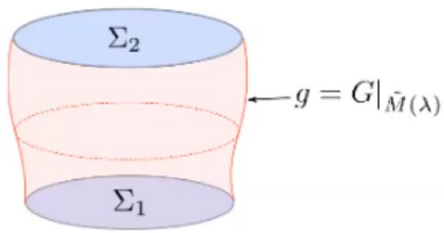
8/23

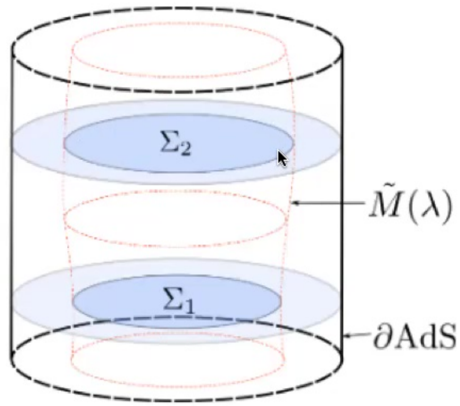
The key idea

2203.08842 with Chandra, de Boer, Flory, Hörtner, Rolph

Michal P. Heller [he/hi...

We want to prepare the state $|\Sigma_2\rangle$ from $|\Sigma_1\rangle$ using path integral on $\tilde{M}(\lambda)$





Each $\tilde{M}(\lambda)$ will be assigned a functional (gravitational representation of a cost);
For an intuition, a candidate functional is the bulk volume in $\Sigma_1 - \tilde{M} - \Sigma_2$

Optimal path integral minimizes the cost, which gives rise to complexity

9/23

What are permitted bulk costs?

2203.08842 with Chandra, de Boer, Flory, Hörtner, Rolph

We want to see what bulk functionals act as reasonable costs in a dual QFT; we do not insist on knowing what exactly they count. Our key criteria are:

- the cost is non-negative
- if we do not do anything, the cost is zero
- the cost is additive
- the cost is covariant (sounds necessary, but can be relaxed to include time foliation dependence as in 1904.02713 with Camargo, Jefferson, Knaute)

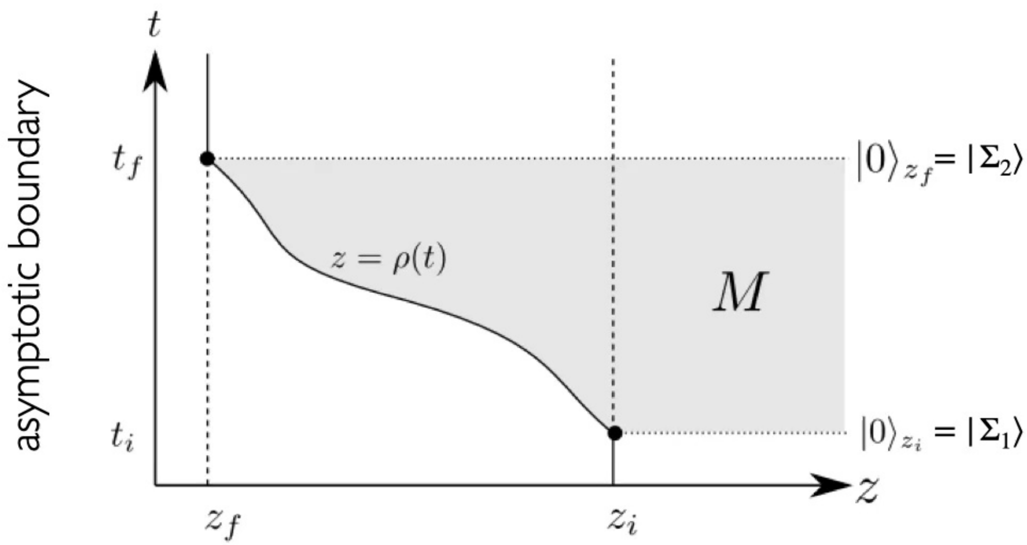
10/23

Cost = gravity action (Euclidean): setup

2101.01185 with Chandra, de Boer, Flory, Hörtner, Rolph

We consider the Poincaré patch of EAdS3 $ds^2 = \frac{dz^2 + dt^2 + dx^2}{z^2}$

We look at circuits interpolating between vacua of theories with 2 cut-offs:

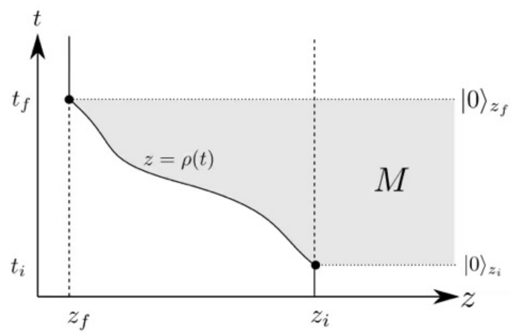


Such circuits involve Euclidean time evolution and coarse-graining

Cost = gravity action (Euclidean): evaluation

2101.01185 with Chandra, de Boer, Flory, Hörtner, Rolph

The cost of this task is given by the bulk action evaluated in M:



$$\longrightarrow \quad I = \frac{1}{\kappa}\int_M d^3x\,\sqrt{G}\,(R+2) + \frac{2}{\kappa}\int_{\partial M} d^2x\,\sqrt{g}K + I_c$$

joint terms, but no volume counter term at ∂M

The answer takes the form

$\int dx$

$$I = \frac{2V_x}{\kappa}\int_{t_i}^{t_f} dt\,\frac{\rho\ddot{\rho} + (1+\dot{\rho}^2)}{\rho^2(1+\dot{\rho}^2)} + I_c = \frac{2V_x}{\kappa}\boxed{\int_{t_i}^{t_f} dt\left(\frac{1}{\rho^2} + \frac{\dot{\rho}\arctan\dot{\rho}}{\rho^2}\right)} + \frac{\pi V_x}{\kappa}\left(\frac{1}{z_f} + \frac{1}{z_i}\right)$$

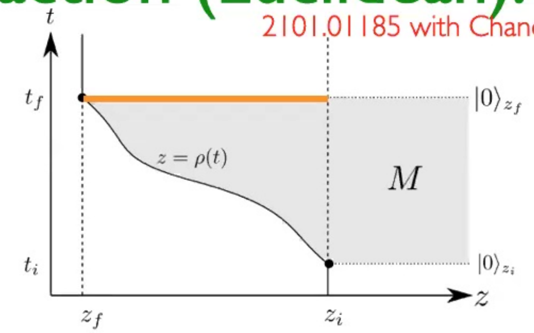
and one can view it as an all order in δ generalization of the Liouville action

$$\boxed{\int_{t_i}^{t_f} dt\left(\frac{1}{\rho^2} + \frac{\dot{\rho}\arctan\dot{\rho}}{\rho^2}\right)}\Bigg|_{\rho=\delta\,e^{-\omega}} \sim \int dtdx\left\{\frac{1}{\delta^2}e^{2\omega} + (\partial\omega)^2 + \mathcal{O}\left(\delta^2\,(\partial\omega)^4\right)\right\}$$

12/23

Cost = gravity action (Euclidean): optimization

2101.01185 with Chandra, de Boer, Flory, Hörtner, Rolph



We propose a 2-stage optimization procedure:

- fix z_i , z_f , $\Delta t \equiv t_f - t_i$ and minimize the action with respect to $\rho(t)$

$$\frac{\delta I}{\delta \rho} = \frac{\rho\ddot{\rho} + 1 + \dot{\rho}^2}{\rho^3(1+\dot{\rho}^2)^2} = 0 \longrightarrow \rho(t) = \sqrt{\mathcal{R}^2 - (t-t_0)^2}$$

- minimize the evaluated action $I = \frac{2V_x}{\kappa}\left(\frac{1}{z_f}\arctan\frac{z_i^2 - z_f^2 + \Delta t^2}{2z_f\Delta t} - \frac{1}{z_i}\arctan\frac{z_i^2 - z_f^2 - \Delta t^2}{2z_i\Delta t}\right)$ with respect to Δt

The resulting circuit aligns along the $\mathcal{C}_V$ slice and has no time evolution with

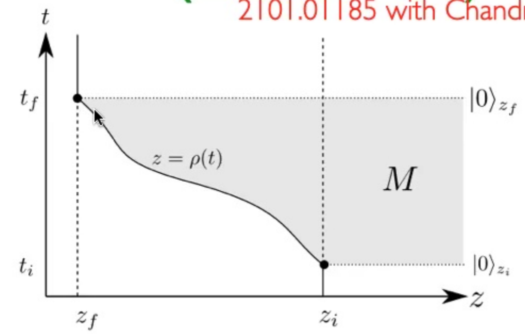
$$I_{min} = \frac{c\pi V_x}{24}\left(\frac{1}{z_f} - \frac{1}{z_i}\right)$$

fixes normalization in $\mathcal{C}_V$

13/23

Cost = gravity action (Euclidean): counting?

2101.01185 with Chandra, de Boer, Flory, Hörtner, Rolph



It is natural to view the circuit defined by $\rho(t)$ as

$$|0\rangle_{z_f} = P\exp\left[-\int_{t_i}^{t_f} dt\,(H_{\rho(t)} + \dot{\rho}\,[T\bar{T}]_{\rho(t)})\right]|0\rangle_{z_i}$$

Euclidean time evolution — coarse graining

One can introduce straightforward cost functions like

$$\#(H_{\rho(t)}) \sim (\text{some function of } \rho) \times dt$$
$$\#([T\bar{T}]_{\rho(t)}) \sim (\text{some function of } \rho) \times |\dot{\rho}|\,dt$$

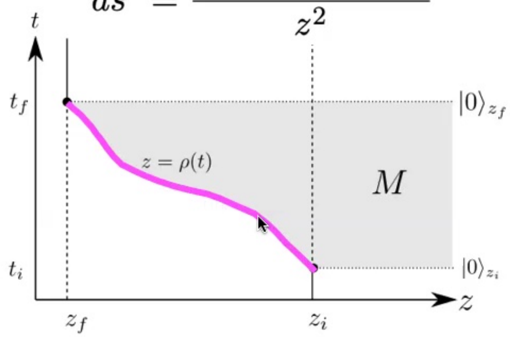
leading to, for example to $\mathcal{C}_{L_1} = \min_{\rho(t),\Delta t} \int_{t_i}^{t_f} (dt + |\dot{\rho}|dt) = |z_f - z_i|$

However, explaining in this way the $\dot{\rho}\,\arctan\dot{\rho}$ term in the action is hard

Cost = cut-off surface volume (Euclidean)

2203.08842 with Chandra, de Boer, Flory, Hörtner, Rolph

$$ds^2 = \frac{dz^2 + dt^2 + dx^2}{z^2}$$



$\longrightarrow$ cost ~ **volume** ~ $\int_{\tilde{M}} dx d\tau \frac{\sqrt{1+\dot{\rho}^2}}{\rho^2}$

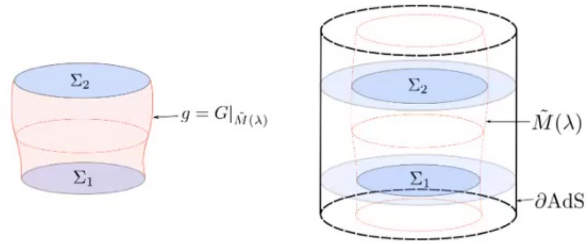
It is a much more reasonable quantity from the counting point of view, for example it can be thought as a total number of tensors in a spacetime TN

15/23

Cost = action (Lorentzian)?

2203.08842 with Chandra, de Boer, Flory, Hörtner, Rolph

In the Lorentzian case we want the circuit to live on a timelike boundary with null boundaries seen as a limiting case



$$ds^2 = \frac{dz^2 - dt^2 + dx^2}{z^2} \longrightarrow \text{action} \sim \int dx \int_{t_i}^{t_f} dt \left(\frac{1 - \boxed{\dot{\rho}\,\mathrm{arctanh}\,\dot{\rho}}}{\boxed{\rho^2}} \right)$$

The null limit is subtle c.f. 2104.00010 by Boruch, Caputa, Ge and Takayanagi

16/23

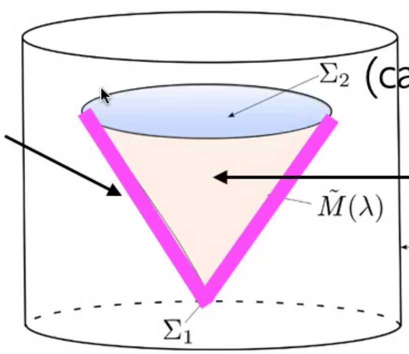
Cost = bulk+cut-off surface volumes (Lorentzian)

2203.08842 with Chandra, de Boer, Flory, Hörtner, Rolph

While the Lorentzian action is not a good cost, there is another natural candidate that satisfy all our criteria:

cost ~ bulk + cut off boundary volumes

Its optimization subject to a cut-off boundary being timelike (or null as a limiting case) gives as a result half of CV2.0 proposal:



Cf. geometric interpretation of MERA 1812.00529 by Milsted & Vidal

Summary of possibilities we explored

2203.08842 with Chandra, de Boer, Flory, Hörtner, Rolph

Cost equals...	Bulk signature	Satisfies physical properties of cost?					Reduces to which state complexity proposal?
		Zero cost ⇔ trivial path integral	Additivity	Symmetry	Covariance	Non-negativity	
Codim-1 boundary volume	Euclidean	✓	✓	✓	✓	✓	CV[0]
Codim-1 boundary volume	Lorentzian	✗	✓	✓	✓	✓	N/A
Codim-0 bulk volume	Euclidean	✓[1]	✓	✓	✓	✓	CV[0]
Codim-0 bulk volume	Lorentzian	✓[1]	✓	✓	✓	✓	CV2.0[3]
Codim-0 gravitational action	Euclidean	✓[1]	✓	✓	✓	✓[2]	CV[0,4]
Codim-0 gravitational action	Lorentzian	✓[1]	✓	✓	✓	✗	N/A

18/23

Comparison to complexity = anything

2203.08842 with Chandra, de Boer, Flory, Hörtner, Rolph

In complexity = anything one seeks for covariant bulk functionals satisfying the key properties shared by CV, CA and CV2.0

2111.02429 + work in progress by Belin, Myers, Ruan, Sárosi, Speranza

Such covariant functionals are obtained in two steps:

- defining a codimension-1 or -0 object, possibly using minimization of a certain covariant functional
- evaluate possibly another functional on this object to define complexity

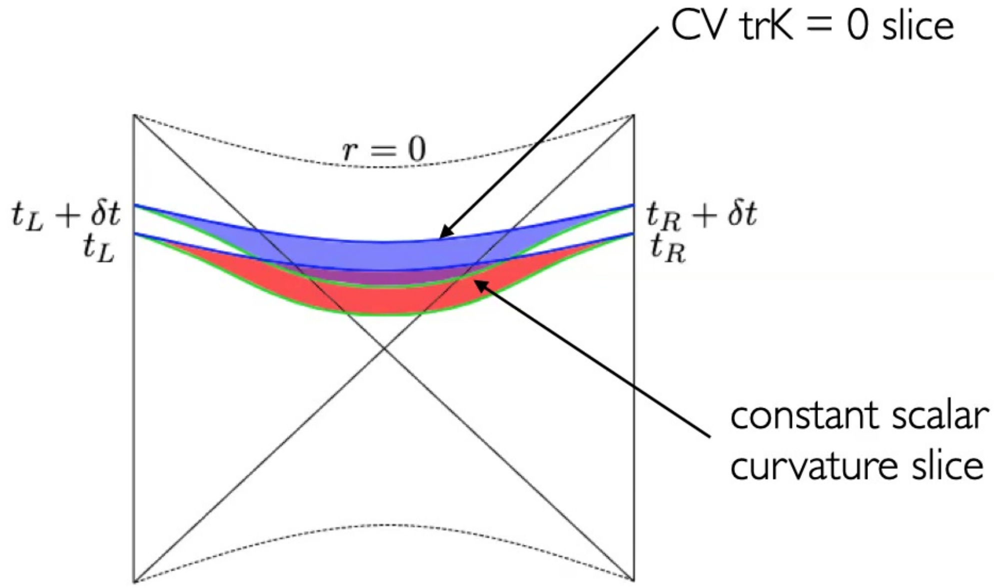
In our approach, the value of the complexity, the functional from minimization of which it arises and the bulk object it gives rise to are deeply related

Similarly, to complexity = anything, there will be many good bulk cost functionals which will give rise to many holographic notions of complexity

Spin-off: possible new notion of codim-0 complexity

2203.08842 with Chandra, de Boer, Flory, Hörtner, Rolph

BTZ black hole:



The action or spacetime volume evaluated between such slices grows linearly

This suggests many possible codim-0 complexity = anything proposals

20/23

Towards a precise bulk dual to a circuit ("1 bulk = 1 circuit")

2112.12158 with Erdmenger, Flory, Gerbershagen, Weigel

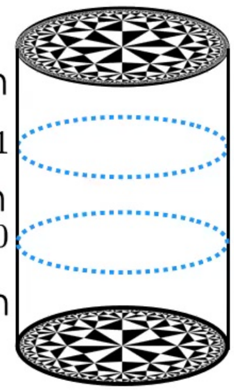
Towards a precise bulk dual to a circuit

2112.12158 with Erdmenger, Flory, Gerbershagen, Weigel

Local conformal transformations lead to a transformation of the stress tensor

$$\langle T \rangle \to \frac{1}{(\partial_\sigma f)^2}\left(\langle T \rangle - \frac{c}{12}\{f,\sigma\}\right) \text{ with } \langle \bar{T} \rangle \text{ and } \langle T^{\mu}{}_{\mu} \rangle \text{ staying the same}$$

The key idea: embed this kind of circuit on the boundary of AdS_3



The gravity dual is obtained by using the exact Fefferman-Graham expansion

Excellent testbed for holographic complexity ideas (work in progress), since both the bulk is known and the circuit is 100% under control

The logic of my talk

perspective: bulk is a circuit

fixed bulk geometry
giving rise to many circuits

specially-crafted bulk
geometry dual to a particular circuit
(holding $|R\rangle$ and $|T\rangle$ fixed)

✔ 2112.12158 with Erdmenger, Flory, Gerbershagen, Weigel

2203.08842 (and 2101.01185) with
Chandra, de Boer, Flory, Hörtner & Rolph

Cost equals...	Bulk signature	Satisfies physical properties of cost?					Reduces to which state complexity proposal?
		Zero cost ⇔ trivial path integral	Additivity	Symmetry	Covariance	Non-negativity	
Codim-1 boundary volume	Euclidean	✓	✓	✓	✓	✓	CV[0]
Codim-1 boundary volume	Lorentzian	✗	✓	✓	✓	✓	N/A
Codim-0 bulk volume	Euclidean	✓[1]	✓	✓	✓	✓	CV[0]
Codim-0 bulk volume	Lorentzian	✓[1]	✓	✓	✓	✓	CV2.0[3]
Codim-0 gravitational action	Euclidean	✓[1]	✓	✓	✓	✓[2]	CV[0,4]
Codim-0 gravitational action	Lorentzian	✓[1]	✓	✓	✓	✗	N/A

choice

bulk cost

work in progress

minimization

(possibly new)
holographic complexities

22/23

Outlook

Attempts to prove RT/HRT has not only put the discussion of entanglement in gravity on a firm footing, but also provided many new insights and tools

It is natural to expect the same from deriving holographic complexity

Today I discussed two approaches to defining a gravity dual to a circuit (which in holography always acts on reference / auxiliary states):

fixed bulk, varying cut-off

+ trivial bulk
+ cost and complexity manifest
+- circuits involve double traces

variable bulk, CFT

+ circuits under full control
+ costs assigned to each spacetime
+- complexity = cost for one member*

When it comes to deriving a holographic complexity, we are still not there yet, but these and related works certainly makes this goal much closer

23/23