

Title: Self dual U(1) lattice field theory with a theta-term

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Abstract: Starting from the Villain formulation with an additional constraint we construct a self-dual lattice version of U(1) field theory with a theta-term. An interesting feature is that the self-dual symmetry gives rise to an action that is local but not ultra-local, similar to lattice actions that implement chiral symmetry. We outline how electric and magnetic matter can be coupled in a self-dual way and discuss the emerging symmetry structure with the theta term. We present results from a Monte Carlo simulation of the self-dual system with electric and magnetic matter and explore spontaneous breaking of the self-dual symmetry.

Self-dual $U(1)$ lattice field theory with a θ -term

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Work done in collaboration with [Mariia Anosova](#), [Nabil Iqbal](#) and [Tin Sulejmanpasic](#).

[T. Sulejmanpasic, C. Gattringer, Nucl. Phys. B943 \(2019\) \[arXiv:1901.02637\]](#)

[M. Anosova, C. Gattringer, T. Sulemanpasic, JHEP 04 \(2022\) 120 \[arXiv: 2201.09468\]](#)

[M. Anosova, C. Gattringer, N. Iqbal, T. Sulemanpasic, accepted at JHEP \[arXiv: 2203.14774\]](#)

Introductory comments and outlook

- Duality and self-duality are powerful tools in quantum field theory.
- So far there are no known lattice constructions of self-dual lattice gauge theories.
- Obvious obstacle: Standard $U(1)$ lattice gauge theory has discretization artifacts in the form of magnetic monopoles, but no matching electric charges.
- We need to remove the monopoles.
- In this project we address (and solve) the problem of a self-dual lattice discretization of $U(1)$ lattice gauge theory with a θ -term.
- Ingredients of the construction are:
 - Use of a generalized Villain action.
 - Removing the monopoles with suitable constraints for the Villain variables.
 - Identification of a non-ultralocal θ -term.
- In a further generalization we couple electrically and magnetically charged matter in a self-dual way and numerically study the spontaneous breaking of self-duality as a function of the matter coupling.

U(1) lattice gauge theory with Villain action

$$Z = \int \mathcal{D}[\Lambda] \sum_{\{n\}} e^{-\frac{\beta}{2} \sum_{x,\mu < \nu} F_{x,\mu\nu} F_{x,\mu\nu}}$$

$$(d\Lambda)_{x,\mu\nu} = \Lambda_{x+\hat{\nu},\mu} - \Lambda_{x,\nu} - \Lambda_{x+\hat{\mu},\nu} + \Lambda_{x,\mu}$$

$$F_{x,\mu\nu} = (d\Lambda)_{x,\mu\nu} + 2\pi n_{x,\mu\nu}$$

$$\sum_{\{n\}} = \prod_{x,\mu < \nu} \sum_{n_{x,\mu\nu}}, \quad \int \mathcal{D}[A] = \prod_{x,\mu} \int_{-\pi}^{\pi} \frac{d\Lambda_{x,\mu}}{2\pi}$$

- Link variables $U_{x,\mu} = e^{i\Lambda_{x,\mu}}$ are needed for coupling matter fields.
- They are invariant under $\Lambda_{x,\mu} \rightarrow \Lambda_{x,\mu} + 2\pi k_{x,\mu}$
- Exterior derivatives transform as $(dA)_{x,\mu\nu} \rightarrow (dA)_{x,\mu\nu} + 2\pi (dk)_{x,\mu\nu}$
- Summation over the Villain variables $n_{x,\mu\nu}$ eats up the shifts $(dk)_{x,\mu\nu}$

Villain formulation admits additional constraints that remove monopoles

- Under shifts exterior derivatives transform as $(d\Lambda)_{x,\mu\nu} \rightarrow (d\Lambda)_{x,\mu\nu} + 2\pi (dk)_{x,\mu\nu}$
- Summation of the Villain variables $n_{x,\mu\nu} \in \mathbb{Z}$ eats up the shifts $(dk)_{x,\mu\nu}$
- Exterior derivative is nilpotent, $d^2 = 0$, and thus $d(dk)_{x,\mu\nu} = 0$
 \circlearrowleft
- As a consequence we can impose the **closedness constraint** \Rightarrow removes monopoles

$$(dn)_{x,\mu\nu\rho} = 0 \quad \forall \text{ cubes } (x, \mu\nu\rho)$$

- The constrained Villain variables still eat up all shifts.
- For completeness: General definition of the exterior derivative d and the divergence ∂

$$(df)_{x, \mu_1 \mu_2 \dots \mu_r} = \sum_{j=1}^r (-1)^{j+1} \left[f_{x+\hat{\mu}_j, \mu_1 \dots \hat{\mu}_j \dots \mu_r} - f_{x, \mu_1 \dots \hat{\mu}_j \dots \mu_r} \right]$$

$$(\partial f)_{x, \mu_1 \mu_2 \dots \mu_r} = \sum_{\nu=1}^d \left[f_{x, \mu_1 \dots \mu_r \nu} - f_{x, \hat{\nu}, \mu_1 \dots \mu_r \nu} \right]$$

Generalized Villain formulation and θ -term

Generalized Villain formulation with constraints

$$Z = \int \mathcal{D}[\Lambda] \sum_{\{n\}} \prod_{\mu < \nu < \rho} \delta\left(\frac{(dn)_{x,\mu\nu\rho}}{\hbar}\right) e^{-\frac{\beta}{2} \sum_{x,\mu < \nu} F_{x,\mu\nu} F_{x,\mu\nu}}$$

$$F_{x,\mu\nu} = (d\Lambda)_{x,\mu\nu} + 2\pi n_{x,\mu\nu}$$

Constraining the Villain variables allows one to define and introduce a θ -term

$$Z = \int \mathcal{D}[\Lambda] \sum_{\{n\}} \prod_{\mu < \nu < \rho} \delta\left(\frac{(dn)_{x,\mu\nu\rho}}{\hbar}\right) e^{-S_\beta[F] - i\theta Q[F]}$$

With a family of topological charges defined as

$$Q[F] = \frac{1}{32\pi^2} \sum_x F_{x,\mu\nu} \epsilon_{\mu\nu\rho\sigma} F_{x-\hat{\rho},\hat{\sigma}} \quad , \quad \hat{s} = \hat{1} + \hat{2} + \hat{3} + \hat{4} \quad , \quad k \in \mathbb{Z}$$

Properties of the family of topological charges

$$Q[F] = \frac{1}{32\pi^2} \sum_x F_{x,\mu\nu} \epsilon_{\mu\nu\rho\sigma} F_{x,\rho\sigma} - k \delta_{,\rho\sigma}, \quad \hat{s} = \hat{1} + \hat{2} + \hat{3} + \hat{4}, \quad k \in \mathbb{Z}$$

$$F_{x,\mu\nu} = (dA)_{x,\mu\nu} + 2\pi n_{x,\mu\nu}$$

Under the condition $(dn)_{x,\mu\nu\rho} = 0 \quad \forall \quad (x, \mu\nu\rho)$ one can show:

- $Q[F]$ is an integer which is independent of the parameter k
- $Q[F]$ is determined by the harmonics in the Hodge decomposition of $n_{x,\mu\nu}$ and this is indeed a topological term

$$n_{x,\mu\nu} = (dl)_{x,\mu\nu} + h_{x,\mu\nu}$$

$$h_{x,\mu\nu} = \omega_{\mu\nu} \sum_{i=1}^{N_\rho} \sum_{j=1}^{N_\sigma} \delta_{x, i\hat{\rho} | j\hat{\sigma}}^{(A)} \quad \text{with} \quad \rho \neq \mu, \nu; \sigma \neq \mu, \nu; \rho \neq \sigma$$

- **Result:**

$$Q[F] = \frac{1}{8} \epsilon_{\mu\nu\rho\sigma} \omega_{\mu\nu} \omega_{\rho\sigma} = \omega_{12}^2 \omega_{34} - \omega_{13} \omega_{24} + \omega_{14} \omega_{23} \in \mathbb{Z}$$

Interpretation of the constraints and Witten effect

- The closedness constraint of the Villain variables corresponds to the **absence of monopoles**.
- This can be seen by direct evaluation of the monopole charge.
- More interesting **Witten effect**: In the presence of a θ -term a monopole with magnetic charge $m = 2\pi$ receives an electric charge $q = \theta/2\pi$

- To check this we relax the closedness constraint for a single cube

$$(dn)_{x,\nu\rho\sigma} = \delta_{x,x_0|2|3|4}^{(1)} \delta_{\nu,2} \delta_{\rho,3} \delta_{\sigma,4}$$

Q.

- Evaluation of the θ -term then gives

$$i\theta Q[F] = i\theta Q[m] + i \frac{\theta}{2\pi} \frac{\Lambda_{x_0,1} + \Lambda_{x_0-\hat{1},1}}{2}$$

Quadratic form and further generalization to obtain self-duality

We write action and θ -term as quadratic form

$$Z = \int D[A] \sum_{\{n\}} \prod_x \delta((dn)_{x,\mu\nu\rho}) e^{-\frac{\theta}{2} \sum_{\substack{\mu < \nu < \rho \\ y\rho < \sigma}} F_{x,\mu\nu} K_{x,\mu\nu|y,\rho\sigma} F_{y,\rho\sigma}} \quad , \quad F_{x,\mu\nu} = (dA + 2\pi n)_{x,\mu\nu}$$

The kernel $K_{x,\mu\nu|y,\rho\sigma}$ generalizes action and θ -term as needed for self-duality

$$K_{x,\mu\nu|y,\rho\sigma} = \sum_z H_{xz}^{-\frac{1}{2}} \left[\delta_{\mu\rho} \delta_{\nu\sigma} \delta_{z,y}^{(4)} + i \frac{\gamma}{2} \epsilon_{\mu\nu\rho\sigma} \left(\delta_{z-\hat{\rho}-\hat{\sigma},y}^{(4)} + \delta_{z+\hat{\rho}-\hat{\sigma},y}^{(4)} \right) \right]$$

$$K_{x,\mu\nu|y,\rho\sigma}^{-1} = \sum_z H_{xz}^{\frac{1}{2}} \left[\delta_{\mu\rho} \delta_{\nu\sigma} \delta_{z,y}^{(4)} - i \frac{\gamma}{2} \epsilon_{\mu\nu\rho\sigma} \left(\delta_{z-\hat{\rho}-\hat{\sigma},y}^{(4)} + \delta_{z+\hat{\rho}-\hat{\sigma},y}^{(4)} \right) \right]$$

with

$$H_{xz} = \delta_{x,z}^{(4)} + \frac{\gamma^2}{4} \left[\delta_{x+\hat{\rho},z}^{(4)} + 2\delta_{x,z}^{(4)} + \delta_{x-\hat{\rho},z}^{(4)} \right] \quad , \quad \beta = \frac{1}{e^2} \sqrt{1 + \left(\frac{e^2 \theta}{4\pi^2} \right)^2} \quad , \quad \gamma = \frac{\theta}{4\pi^2} e^2$$

Note that K is only local, but not ultra-local. This is reminiscent of chiral lattice fermions.

Write the constraints with cube-based Lagrange multipliers $A_{x,\mu\nu\rho}^m$

Integral representation of Kronecker deltas in constraints: $A^m \dots$ "magnetic gauge field"

$$\prod_{\substack{x \\ \mu < \nu < \rho}} \delta((dn)_{x,\mu\nu\rho}) = \prod_{\substack{x \\ \mu < \nu < \rho}} \int_{-\pi}^{\pi} \frac{dA_{x,\mu\nu\rho}^m}{2\pi} e^{iA_{x,\mu\nu\rho}^m (dn)_{x,\mu\nu\rho}}$$

$$\stackrel{\text{Q.}}{=} \int D[A^m] e^{i \sum_{x,\mu<\nu<\rho} A_{x,\mu\nu\rho}^m (dF)_{x,\mu\nu\rho}} = \int D[A^m] e^{i \sum_{x,\mu<\nu} (\partial A^m)_{x,\mu\nu} F_{x,\mu\nu}}$$

Partition sum is converted into a Gaussian integral ...

$$Z = \int D[A^m] \int D[A] \sum_{\{n\}} e^{i \sum_{\substack{x,\mu<\nu \\ y,\rho<\sigma}} (dA | 2\pi n)_{x,\mu\nu} K_{x,\mu\nu|y,\rho\sigma} (dA | 2\pi n)_{y,\rho\sigma} - i \sum_{x,\mu<\nu} (\partial A^m)_{x,\mu\nu} (dA | 2\pi n)_{x,\mu\nu}}$$

... which can be solved with a generalized Poisson resummation formula \Rightarrow

$$Z = C \int D[A^m] \sum_{\{p\}} e^{i \sum_{\substack{x,\mu<\nu \\ y,\rho<\sigma}} (\partial A^m | 2\pi p)_{x,\mu\nu} K_{x,\mu\nu|y,\rho\sigma}^{-1} (\partial A^m | 2\pi p)_{y,\rho\sigma}} \int D[A] e^{-i \sum_{x,\mu<\nu} (dA)_{x,\mu\nu} p_{x,\mu\nu}}$$

$$\vec{\beta} = \frac{1}{4\pi^2 \beta} \quad , \quad p_{x,\mu\nu} \in \mathbb{Z}$$

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$$= \int D[A^m] e^{i \sum_{x,\mu < \nu} A_{x,\mu\nu\rho}^m (dF)_{x,\mu\nu\rho}} = \int D[A^m] e^{i \sum_{x,\mu < \nu} (\partial A^m)_{x,\mu\nu} F_{x,\mu\nu}}$$

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$$\tilde{\beta} = \frac{1}{4\pi^2 \beta} \quad , \quad p_{x,\mu\nu} \in \mathbb{Z}$$

Gauge field integration generates worldsheet constraints

Gauge field integration generates constraints for the plaquette occupation numbers $p_{x,\mu\nu}$

$$\int \mathcal{D}[A] e^{-i \sum_{x,\mu < \nu} (dA)_{x,\mu\nu} p_{x,\mu\nu}} = \prod_{x,\mu} \int_{-\pi}^{\pi} \frac{dA_{x,\mu}}{2\pi} e^{i A_{x,\mu} (\partial p)_{x,\mu}} = \prod_{x,\mu} \delta((\partial p)_{x,\mu})$$

The generalized zero divergence condition $(\partial p)_{x,\mu} = 0$ forces the total flux $(\partial p)_{x,\mu}$ from all plaquette occupation numbers at a link (x, μ) to vanish $\Rightarrow p_{x,\mu\nu}$ form closed worldsheets.

Partition sum in worldsheet representation

$$Z = C \int \mathcal{D}[\Lambda^m] \sum_{\{p\}} e^{i \sum_{x,\mu < \nu} (\partial \Lambda^m + 2\pi p)_{x,\mu\nu} K_{x,\mu\nu|y,\rho\sigma}^{-1} (\partial \Lambda^m + 2\pi p)_{y,\rho\sigma}} \prod_{x,\mu} \delta((\partial p)_{x,\mu})$$

$K_{x,\mu\nu|y,\rho\sigma}^{-1} \sim K_{x,\mu\nu|y,\rho\sigma}$ and $p_{x,\mu\nu} \sim n_{x,\mu\nu} \Rightarrow$ structural similarity to original formulation

But $d \leftrightarrow \partial$!!

Switch to the dual lattice

Introduce variables on the dual lattice: $\tilde{p}_{\tilde{x},\mu\nu} \in \mathbb{Z}$ and $\tilde{\Lambda}_{\tilde{x},\mu}^m \in [-\pi, \pi]$

$$p_{x,\mu\nu} = \sum_{\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} \tilde{p}_{\tilde{x}=\hat{\rho}-\hat{\sigma},\rho\sigma} \quad , \quad A_{x,\mu\nu\rho}^m = \sum_{\sigma} \epsilon_{\mu\nu\rho\sigma} \tilde{A}_{\tilde{x}=\hat{\sigma},\sigma}^m$$

Conversion of discretized differential operators into their duals

$$(\partial p)_{x,\mu} = \sum_{\nu < \rho < \sigma} \epsilon_{\mu\nu\rho\sigma} (d\tilde{p})_{\tilde{x}=\hat{\nu}-\hat{\rho}-\hat{\sigma},\nu\rho\sigma} \quad , \quad (\partial \Lambda^m + 2\pi p)_{x,\mu\nu} = \sum_{\rho < \sigma} \epsilon_{\mu\nu\rho\sigma} (d\tilde{\Lambda}^m + 2\pi\tilde{p})_{\tilde{x}=\hat{\rho}-\hat{\sigma},\rho\sigma}$$

- Zero divergence condition $\partial p = 0$ is converted to closedness condition $d\tilde{p} = 0$
- Divergence operator ∂A^m is converted to exterior derivative $d\tilde{A}^m$

\Rightarrow Self duality established !!

Final result for the duality transformation

Original form:

$$Z = \int D[\Lambda] \sum_{\{n\}} e^{-S_c[F] - i\theta Q[F]} \prod_{\substack{\mu < \nu < \rho \\ \mu < \nu < \rho}} \delta\left((dn)_{\bar{x},\mu\nu\rho}\right) , \quad F_{\bar{x},\mu\nu} = (d\Lambda + 2\pi n)_{\bar{x},\mu\nu}$$

Dual form:

$$Z = C \int D[\tilde{A}^m] \sum_{\{\tilde{n}\}} e^{S_{\tilde{c}}[\tilde{I}^m] + i\tilde{\theta} Q[\tilde{I}^m]} \prod_{\substack{\tilde{x} \\ \mu < \nu < \rho}} \delta\left((d\tilde{p})_{\tilde{x},\mu\nu\rho}\right) , \quad \tilde{I}_{\tilde{x},\mu\nu}^m = (d\tilde{A}^m + 2\pi\tilde{p})_{\tilde{x},\mu\nu}$$

with:

$$\frac{1}{\tilde{c}^2} = \frac{1}{c^2} f , \quad \tilde{\theta} = -\theta f , \quad f = \frac{1}{\left(\frac{2\pi}{c^2}\right)^2 + \left(\frac{\theta}{2\pi}\right)^2}$$

Consistency check:

$$\tilde{c} = c , \quad \tilde{\theta} = \theta , \quad C\tilde{C} = 1$$

The $SL(2, \mathbb{Z})$ symmetry structure

Self-dual symmetry $\mathcal{S}: \epsilon \rightarrow \bar{\epsilon}, \theta \rightarrow \tilde{\theta}$

□

θ -shift symmetry $\mathcal{T}: \theta \rightarrow \bar{\theta} \equiv \theta - 2\pi$

- The so-called *modular parameter* is defined as $\tau \equiv i \frac{2\pi}{e^2} - \frac{\theta}{2\pi}$

- \mathcal{S} and \mathcal{T} transform τ as follows $\mathcal{S}: \tau \rightarrow \tilde{\tau} = -\frac{1}{\tau}$
 $\mathcal{T}: \tau \rightarrow \tau = \tau + 1$

- General $SL(2, \mathbb{Z})$ transformation M of τ

$$\tau \rightarrow M\tau \equiv \frac{a\tau + b}{c\tau + d} \quad \text{where } M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ with } a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1$$

- \mathcal{S} and \mathcal{T} correspond to

$$\mathcal{S}: M_{\mathcal{S}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathcal{T}: M_{\mathcal{T}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

- $M_{\mathcal{S}}$ and $M_{\mathcal{T}}$ generate all of $SL(2, \mathbb{Z})$.

\Rightarrow families of equivalent theories related by different transformations M

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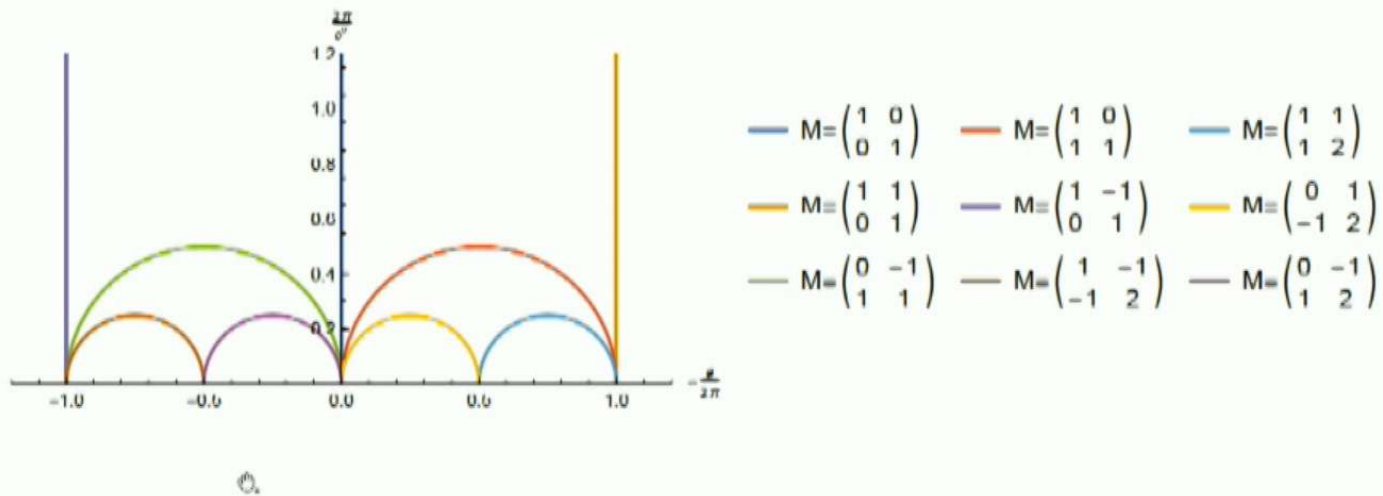
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Mappings between families of theories



- The family of theories on the vertical axis has no sign problem and can be simulated.
- Different choices of M map this family to other families with $\theta \neq 0$.
- Identification of the $SL(2, \mathbb{Z})$ symmetry nicely extends to theories with dyonic matter.

Summary

- Within the Villain formulation, imposing the closedness condition on the Villain variables removes monopoles and self-duality becomes possible.
- Based on the closedness constraint one may define a family of integer-valued topological charges and couple a θ -term.
- Generalizing action and topological charge to a non-ultra local form one may construct self-dual U(1) lattice gauge theory with an θ -term.
- The $SL(2, \mathbb{Z})$ symmetry relates theories without sign problem to theories at $\theta \neq 0$.
- Electrically and magnetically charged matter fields can be coupled in a self-dual way.
- The self-dual symmetry may be broken spontaneously as a function of the matter field couplings.