Title: Self dual U(1) lattice field theory with a theta-term

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Abstract: Starting from the Villain formulation with an additional constraint we construct a self-dual lattice version of U(1) field theory with a theta-term. An interesting feature is that the self-dual symmetry gives rise to an action that is local but not ultra-local, similar to lattice actions that implement chiral symmetry. We outline how electric and magnetic matter can be coupled in a self-dual way and discuss the emerging symmetry structure with the theta term. We present results from a Monte Carlo simulation of the self-dual system with electric and magnetic matter and explore spontaneous breaking of the self-dual symmetry.

Self-dual U(1) lattice field theory with a θ-term

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Work done in collaboration with Mariia Anosova, Nabil Iqbal and Tin Sulejmanpasic.

T. Sulejmanpasic, C. Gattringer, Nucl. Phys. B943 (2019) [arXiv:1901.02637]
M. Anosova, C. Gattringer, T. Sulemanpasic, JHEP 04 (2022) 120 [arXiv: 2201.09468]
M. Anosova, C. Gattringer, N. Iqbal, T. Sulemanpasic, accepted at JHEP [arXiv: 2203.14774]

Introductory comments and outlook

- Duality and self-duality are powerful tools in quantum field theory.
- So far there are no known lattice constructions of self-dual lattice gauge theories.
- Obvious obstacle: Standard U(1) lattice gauge theory has discretization artifacts in the form of magnetic monopoles, but no matching electric charges.
- We need to remove the monopoles.
- In this project we address (and solve) the problem of a self-dual lattice discretization of U(1) lattice gauge theory with a θ-term.
- Ingredients of the construction are:
  - Use of a generalized Villain action.
  - Removing the monopoles with suitable constraints for the Villain variables.
  - Identification of a non-ultralocal θ-term.
- In a further generalization we couple electrically and magnetically charged matter in a self-dual way and numerically study the spontaneous breaking of self-duality as a function of the matter coupling.

U(1) lattice gauge theory with Villain action

$$Z = \int D[A] \sum_{\{n\}} e^{-\frac{\beta}{2} \sum_{x,\mu<\nu} F_{x,\mu\nu} F_{x,\mu\nu}}$$

$$(dA)_{x,\mu\nu} = A_{x+\hat{\mu},\nu} - A_{x,\nu} - A_{x+\hat{\nu},\mu} + A_{x,\mu}$$

$$F_{x,\mu\nu} = (dA)_{x,\mu\nu} + 2\pi\, n_{x,\mu\nu}$$

$$\sum_{\{n\}} = \prod_{x,\mu<\nu} \sum_{n_{x,\mu\nu}} \; , \quad \int D[A] = \prod_{x,\mu} \int_{-\pi}^{\pi} \frac{dA_{x,\mu}}{2\pi}$$

- Link variables $U_{x,\mu} = e^{iA_{x,\mu}}$ are needed for coupling matter fields.
- They are invariant under $A_{x,\mu} \to A_{x,\mu} + 2\pi\, k_{x,\mu}$
- Exterior derivatives transform as $(dA)_{x,\mu\nu} \to (dA)_{x,\mu\nu} + 2\pi\, (dk)_{x,\mu\nu}$
- Summation over the Villain variables $n_{x,\mu\nu}$ eats up the shifts $(dk)_{x,\mu\nu}$

Villain formulation admits additional constraints that remove monopoles

- Under shifts exterior derivatives transform as $(d\Lambda)_{x,\mu\nu} \to (d\Lambda)_{x,\mu\nu} + 2\pi\,(dk)_{x,\mu\nu}$
- Summation of the Villain variables $n_{x,\mu\nu} \in \mathbb{Z}$ eats up the shifts $(dk)_{x,\mu\nu}$
- Exterior derivative is nilpotent, $d^2 = 0$, and thus $d(dk)_{x,\mu\nu} = 0$
- As a consequence we can impose the closedness constraint $\Rightarrow$ removes monopoles

$$(dn)_{x,\mu\nu\rho} = 0 \qquad \forall \quad \text{cubes} \quad (x,\mu\nu\rho)$$

- The constrained Villain variables still eat up all shifts.
- For completeness: General definition of the exterior derivative d and the divergence ∂

$$(d\,f)_{x,\mu_1\mu_2\ldots\mu_r} = \sum_{j=1}^{r}(-1)^{j+1}\left[f_{x+\hat{\mu}_j,\mu_1\ldots\hat{\mu}_j\ldots\mu_r} - f_{x,\mu_1\ldots\hat{\mu}_j\ldots\mu_r}\right]$$

$$(\partial f)_{x,\mu_1\mu_2\ldots\mu_r} = \sum_{\nu=1}^{d}\left[f_{x,\mu_1\ldots\mu_r\nu} - f_{x-\hat{\nu},\mu_1\ldots\mu_r\nu}\right]$$

Generalized Villain formulation and θ-term

Generalized Villain formulation with constraints

$$Z = \int D[A] \sum_{\{n\}} \prod_{\substack{x \\ \mu<\nu<\rho}} \delta\Big((dn)_{x,\mu\nu\rho} \Big) \, e^{-\frac{\beta}{2} \sum_{x,\mu<\nu} F_{x,\mu\nu} F_{x,\mu\nu}}$$

$$F_{x,\mu\nu} = (dA)_{x,\mu\nu} + 2\pi \, n_{x,\mu\nu}$$

Constraining the Villain variables allows one to define and introduce a θ-term

$$Z = \int D[A] \sum_{\{n\}} \prod_{\substack{x \\ \mu<\nu<\rho}} \delta\Big((dn)_{x,\mu\nu\rho} \Big) \; e^{-S_\beta[F] - i\theta \, Q[F]}$$

With a family of topological charges defined as

$$Q[F] = \frac{1}{32\pi^2} \sum_x F_{x,\mu\nu} \, \epsilon_{\mu\nu\rho\sigma} F_{x-\hat{\rho}-\hat{\sigma}+k\hat{s},\rho\sigma} \quad , \qquad \hat{s} = \hat{1} + \hat{2} + \hat{3} + \hat{4} \, , \; k \in \mathbb{Z}$$

Properties of the family of topological charges

$$Q[F] \;=\; \frac{1}{32\pi^2}\sum_x F_{x,\mu\nu}\,\epsilon_{\mu\nu\rho\sigma}\,F_{x-\hat\rho-\hat\sigma+k\hat{s},\rho\sigma}\,, \qquad \hat{s}=\hat{1}+\hat{2}+\hat{3}+\hat{4}\,,\; k\in\mathbb{Z}$$

$$F_{x,\mu\nu} \;=\; (dA)_{x,\mu\nu} + 2\pi\, n_{x,\mu\nu}$$

Under the condition $(dn)_{x,\mu\nu\rho} \;=\; 0 \quad \forall \quad (x,\mu\nu\rho)$ one can show:

- $Q[F]$ is an integer which is independent of the parameter k
- $Q[F]$ is determined by the harmonics in the Hodge decomposition of $n_{x,\mu\nu}$ and this is indeed a topological term

$$n_{x,\mu\nu} \;=\; (dl)_{x,\mu\nu} \;+\; h_{x,\mu\nu}$$

$$h_{x,\mu\nu} \;=\; \omega_{\mu\nu}\sum_{i=1}^{N_\rho}\sum_{j=1}^{N_\sigma}\delta^{(4)}_{x,i\hat\rho+j\hat\sigma} \quad \text{with} \quad \rho\neq\mu,\nu\,;\;\sigma\neq\mu,\nu\,;\;\rho\neq\sigma$$

- Result:

$$Q[F] \;=\; \frac{1}{8}\,\epsilon_{\mu\nu\rho\sigma}\,\omega_{\mu\nu}\,\omega_{\rho\sigma} \;=\; \omega_{12}\,\omega_{34} - \omega_{13}\,\omega_{24} + \omega_{14}\,\omega_{23} \;\in\; \mathbb{Z}$$

Interpretation of the constraints and Witten effect

- The closedness constraint of the Villain variables corresponds to the absence of monopoles.
- This can be seen by direct evaluation of the monopole charge.
- More interesting ..... Witten effect: In the presence of a θ-term a monopole with magnetic charge $m = 2\pi$ receives an electric charge $q = \theta/2\pi$
- To check this we relax the closedness constraint for a single cube

$$(dn)_{x,\nu\rho\sigma} = \delta^{(4)}_{x,x_0+\hat{2}+\hat{3}+\hat{4}}\, \delta_{\nu,2}\, \delta_{\rho,3}\, \delta_{\sigma,4}$$

- Evaluation of the θ-term then gives

$$i\theta\, Q[F] = i\theta\, Q[n] + i\,\frac{\theta}{2\pi}\, \frac{\Lambda_{x_0,1} + \Lambda_{x_0-\hat{8},1}}{2}$$

Quadratic form and further generalization to obtain self-duality

We write action and θ-term as quadratic form

$$Z = \int D[A] \sum_{\{n\}} \prod_{\substack{x \\ \mu<\nu<\rho}} \delta\Big((dn)_{x,\mu\nu\rho}\Big)\, e^{-\frac{\beta}{2}\sum_{\substack{x,\mu<\nu \\ y\rho<\sigma}} F_{x,\mu\nu} K_{x,\mu\nu|y,\rho\sigma} F_{y,\rho\sigma}} \quad , \quad F_{x,\mu\nu} = (dA + 2\pi n)_{x,\mu\nu}$$

The kernel $K_{x,\mu\nu|y,\rho\sigma}$ generalizes action and θ-term as needed for self-duality

$$K_{x,\mu\nu|y,\rho\sigma} = \sum_z H_{xz}^{-\frac{1}{2}} \Big[\delta_{\mu\rho}\delta_{\nu\sigma}\delta^{(4)}_{z,y} \; + \; i\frac{\gamma}{2}\epsilon_{\mu\nu\rho\sigma}\Big(\delta^{(4)}_{z-\hat\rho-\hat\sigma,y} + \delta^{(4)}_{z+\hat s-\hat\rho-\hat\sigma,y}\Big)\Big]$$

$$K^{-1}_{x,\mu\nu|y,\rho\sigma} = \sum_z H_{xz}^{\frac{1}{2}} \Big[\delta_{\mu\rho}\delta_{\nu\sigma}\delta^{(4)}_{z,y} \; - \; i\frac{\gamma}{2}\epsilon_{\mu\nu\rho\sigma}\Big(\delta^{(4)}_{z-\hat\rho-\hat\sigma,y} + \delta^{(4)}_{z+\hat s-\hat\rho-\hat\sigma,y}\Big)\Big]$$

with

$$H_{xz} = \delta^{(4)}_{x,z} + \frac{\gamma^2}{4}\Big[\delta^{(4)}_{x+\hat s,z} + 2\delta^{(4)}_{x,z} + \delta^{(4)}_{x-\hat s,z}\Big] \quad , \quad \beta = \frac{1}{e^2}\sqrt{1+\left(\frac{e^2\theta}{4\pi^2}\right)^2} \quad , \quad \gamma = \frac{\theta}{4\pi^2}\, e^2$$

Note that K is only local, but not ultra-local. This is reminiscent of chiral lattice fermions.

Write the constraints with cube-based Lagrange multipliers $A^m_{x,\mu\nu\rho}$

Integral representation of Kronecker deltas in constraints: A^m ... "magnetic gauge field"

$$\prod_{\substack{x\\ \mu<\nu<\rho}} \delta\Big((dn)_{x,\mu\nu\rho}\Big) = \prod_{\substack{x\\ \mu<\nu<\rho}} \int_{-\pi}^{\pi} \frac{dA^m_{x,\mu\nu\rho}}{2\pi}\, e^{iA^m_{x,\mu\nu\rho}(dn)_{x,\mu\nu\rho}}$$

$$= \int D[A^m]\, e^{\frac{i}{2\pi}\sum A^m_{x,\mu\nu\rho}(dF)_{x,\mu\nu\rho}} = \int D[A^m]\, e^{\frac{i}{2\pi}\sum_{x,\mu<\nu}(\partial A^m)_{x,\mu\nu}F_{x,\mu\nu}}$$

Partition sum is converted into a Gaussian integral ...

$$Z = \int D[A^m] \int D[A] \sum_{\{n\}} e^{\frac{\beta}{2}\sum_{\substack{x,\mu<\nu\\ y,\rho<\sigma}} (dA + 2\pi n)_{x,\mu\nu} K_{x,\mu\nu|y,\rho\sigma} (dA + 2\pi n)_{y,\rho\sigma} \quad \frac{i}{2\pi}\sum_{x,\mu<\nu}(\partial A^m)_{x,\mu\nu}(dA + 2\pi n)_{x,\mu\nu}}$$

... which can be solved with a generalized Poisson resummation formula $\Rightarrow$

$$Z = C \int D[A^m] \sum_{\{p\}} e^{\frac{\tilde{\beta}}{2}\sum_{\substack{x,\mu<\nu\\ y,\rho<\sigma}} (\partial A^m + 2\pi p)_{x,\mu\nu} K^{-1}_{x,\mu\nu|y,\rho\sigma} (\partial A^m + 2\pi p)_{y,\rho\sigma}} \int D[A]\, e^{-i\sum_{x,\mu<\nu}(dA)_{x,\mu\nu}p_{x,\mu\nu}}$$

$$\tilde{\beta} = \frac{1}{4\pi^2\,\beta}, \qquad p_{x,\mu\nu} \in \mathbb{Z}$$

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$$= \int D[A^m]\, e^{\frac{i}{2\pi}\sum A^m_{x,\mu\nu\rho}(dF)_{x,\mu\nu\rho}} = \int D[A^m]\, e^{\frac{i}{2\pi}\sum_{x,\mu<\nu}(\partial A^m)_{x,\mu\nu}F_{x,\mu\nu}}$$

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$$\tilde{\beta} = \frac{1}{4\pi^2 \beta} \quad , \qquad p_{x,\mu\nu} \in \mathbb{Z}$$

Gauge field integration generates worldsheet constraints

Gauge field integration generates constraints for the plaquette occupation numbers $p_{x,\mu\nu}$

$$\int D[A]\, e^{-i\sum_{x,\mu<\nu}(dA)_{x,\mu\nu}\, p_{x,\mu\nu}} \;=\; \prod_{x,\mu} \int_{-\pi}^{\pi} \frac{dA_{x,\mu}}{2\pi}\, e^{\,i A_{x,\mu}(\partial p)_{x,\mu}} \;=\; \prod_{x,\mu} \delta\Big((\partial p)_{x,\mu}\Big)$$

The generalized zero divergence condition $(\partial p)_{x,\mu} = 0$ forces the total flux $(\partial p)_{x,\mu}$ from all plaquette occupation numbers at a link (x,μ) to vanish $\Rightarrow$ $p_{x,\mu\nu}$ form closed worldsheets.

Partition sum in worldsheet representation

$$Z \;=\; C \int D[\Lambda^m] \sum_{\{p\}} e^{\,-\frac{\beta}{2}\sum_{x,\mu<\nu;\, y,\rho<\sigma} (\partial\Lambda^m + 2\pi p)_{x,\mu\nu}\, K^{-1}_{x,\mu\nu|y,\rho\sigma}\, (\partial\Lambda^m + 2\pi p)_{y,\rho\sigma}} \prod_{x,\mu} \delta\Big((\partial p)_{x,\mu}\Big)$$

$K^{-1}_{x,\mu\nu|y,\rho\sigma} \sim K_{x,\mu\nu|y,\rho\sigma}$ and $p_{x,\mu\nu} \sim n_{x,\mu\nu}$ $\Rightarrow$ structural similarity to original formulation

But $d \leftrightarrow \partial$!!

Switch to the dual lattice

Introduce variables on the dual lattice: $\tilde{p}_{\tilde{x},\mu\nu} \in \mathbb{Z}$ and $\tilde{A}^{m}_{\tilde{x},\mu} \in [-\pi, \pi]$

$$p_{x,\mu\nu} = \sum_{\rho<\sigma} \epsilon_{\mu\nu\rho\sigma}\, \tilde{p}_{\tilde{x}-\hat{\rho}-\hat{\sigma},\rho\sigma} \quad , \qquad A^{m}_{x,\mu\nu\rho} = \sum_{\sigma} \epsilon_{\mu\nu\rho\sigma}\, \tilde{A}^{m}_{\tilde{x}\ \ \hat{\sigma},\sigma}$$

Conversion of discretized differential operators into their duals

$$(\partial\, p)_{x,\mu} = \sum_{\nu<\rho<\sigma} \epsilon_{\mu\nu\rho\sigma}\, (d\, \tilde{p})_{\tilde{x}\ \ \hat{\nu}\ \ \hat{\rho}\ \ \hat{\sigma},\nu\rho\sigma} \; , \quad (\partial\, A^{m} + 2\pi p)_{x,\mu\nu} = \sum_{\rho<\sigma} \epsilon_{\mu\nu\rho\sigma}\, (d\, \tilde{A}^{m} + 2\pi \tilde{p})_{\tilde{x}\ \ \hat{\rho}\ \ \hat{\sigma},\rho\sigma}$$

- Zero divergence condition $\partial\, p = 0$ is converted to closedness condition $d\, \tilde{p} = 0$
- Divergence operator $\partial\, A^{m}$ is converted to exterior derivative $d\, \tilde{A}^{m}$

$\Rightarrow$ Self duality established !!

Final result for the duality transformation

Original form:

$$Z = \int D[A] \sum_{\{n\}} e^{-S_e[F] - i\theta Q[F]} \prod_{\substack{x \\ \mu<\nu<\rho}} \delta\Big((dn)_{x,\mu\nu\rho}\Big) \quad , \quad F_{x,\mu\nu} = (dA + 2\pi n)_{x,\mu\nu}$$

Dual form:

$$Z = C \int D[\widetilde{A}^m] \sum_{\{\widetilde{p}\}} e^{-S_{\widetilde{e}}[\widetilde{F}^m] - i\widetilde{\theta} Q[\widetilde{F}^m]} \prod_{\substack{\widetilde{x} \\ \mu<\nu<\rho}} \delta\Big((d\widetilde{p})_{\widetilde{x},\mu\nu\rho}\Big) \quad , \quad \widetilde{F}^m_{\widetilde{x},\mu\nu} = (d\widetilde{A}^m + 2\pi\widetilde{p})_{\widetilde{x},\mu\nu}$$

with:

$$\frac{1}{\widetilde{e}^{\,2}} = \frac{1}{e^2} f \quad , \quad \widetilde{\theta} = -\theta f \quad , \quad f = \frac{1}{\left(\frac{2\pi}{e^2}\right)^2 + \left(\frac{\theta}{2\pi}\right)^2}$$

Consistency check:

$$\widetilde{\widetilde{e}} = e \quad , \quad \widetilde{\widetilde{\theta}} = \theta \quad , \quad C\widetilde{C} = 1$$

The SL$(2,\mathbb{Z})$ symmetry structure

Self-dual symmetry $\mathcal{S}: \quad e \to \tilde{e}\,, \ \theta \to \tilde{\theta}$

θ-shift symmetry $\mathcal{T}: \quad \theta \to \overline{\theta} \equiv \theta - 2\pi$

- The so-called *modular parameter* is defined as $\tau \equiv i\,\frac{2\pi}{e^2} - \frac{\theta}{2\pi}$
- $\mathcal{S}$ and $\mathcal{T}$ transform τ as follows

$$\mathcal{S}: \quad \tau \to \tilde{\tau} = -\frac{1}{\tau}$$
$$\mathcal{T}: \quad \tau \to \tau = \tau + 1$$

- General SL$(2,\mathbb{Z})$ transformation M of τ

$$\tau \to M\tau \equiv \frac{a\tau + b}{c\tau + d} \quad \text{where} \quad M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{with} \quad a,b,c,d \in \mathbb{Z} \quad \text{and} \quad ad - bc = 1$$

- $\mathcal{S}$ and $\mathcal{T}$ correspond to

$$\mathcal{S}: \quad M_{\mathcal{S}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad , \quad \mathcal{T}: \quad M_{\mathcal{T}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

- $M_{\mathcal{S}}$ and $M_{\mathcal{T}}$ generate all of SL$(2,\mathbb{Z})$.

$\Rightarrow$ families of equivalent theories related by different transformations M

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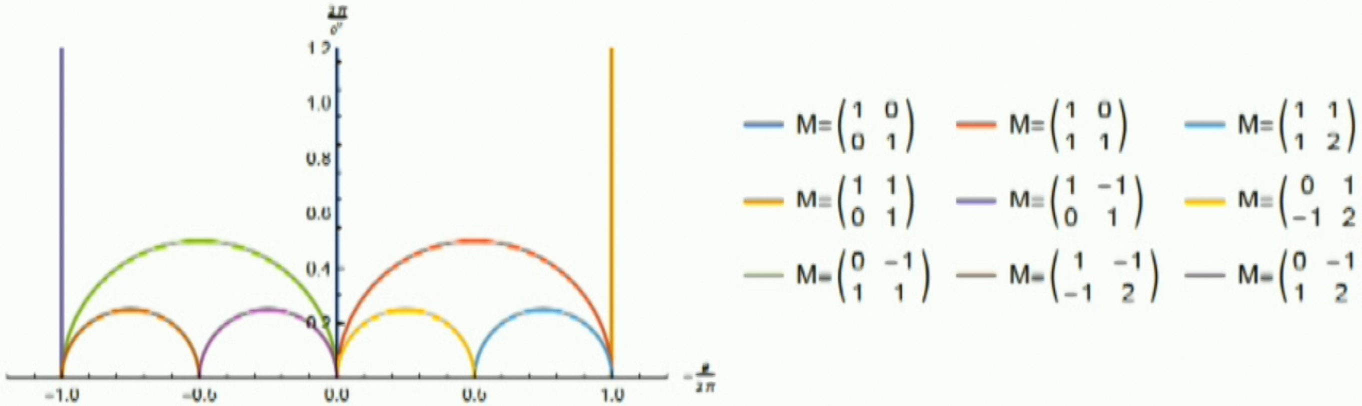
Mappings between families of theories



- The family of theories on the vertical axis has no sign problem and can be simulated.
- Different choices of M map this familiy to other families with $\theta \neq 0$.
- Identification of the $\mathrm{SL}(2,\mathbb{Z})$ symmetry nicely extends to theories with dyonic matter.

Summary

- Within the Villain formulation, imposing the closedness condition on the Villain variables removes monopoles and self-duality becomes possible.
- Based on the closedness constraint one may define a family of integer-valued topological charges and couple a θ-term.
- Generalizing action and topological charge to a non-ultra local form one may construct self-dual U(1) lattice gauge theory with an θ-term.
- The $\mathrm{SL}(2,\mathbb{Z})$ symmetry relates theories without sign problem to theories at $\theta \neq 0$.
- Electrically and magnetically charged matter fields can be coupled in a self-dual way.
- The self-dual symmetry may be broken spontaneously as a function of the matter field couplings.