Title: Cosmology (2021/2022)

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Abstract: This class is an introduction to cosmology. We'll cover expansion history of the universe, thermal history, dark matter models, and as much cosmological perturbation theory as time permits.

LAST TIME

$$S = \int d^4x \sqrt{-g}\left[\frac{M_{pl}^2}{2}R - \frac{1}{2}(\nabla^\mu\phi)(\nabla_\mu\phi) - V(\phi)\right]$$

UNITA

BACKGROUND $\left(\bar{\phi}(t),\ a(t)\right)$

$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} + V'(\bar{\phi}) = 0$$

$$H^2 = \frac{1}{3M_{pl}^2}\underbrace{\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)}_{\rho(\phi)}$$

LINEAR PERTURBATIONS

$$\phi(x,t) = \bar{\phi}(t) + \delta\phi(x,t)$$

$$g_{\mu\nu}(x,t) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(x,t)$$

Perimeter-A

UNITARY GAUGE

$$\left.\begin{aligned} \delta\phi &= 0 \\ \partial_j \gamma_{ij} &= 0 \end{aligned}\right\} \quad (\star) \qquad \varepsilon^\mu = (\varepsilon^0, \varepsilon^i)$$

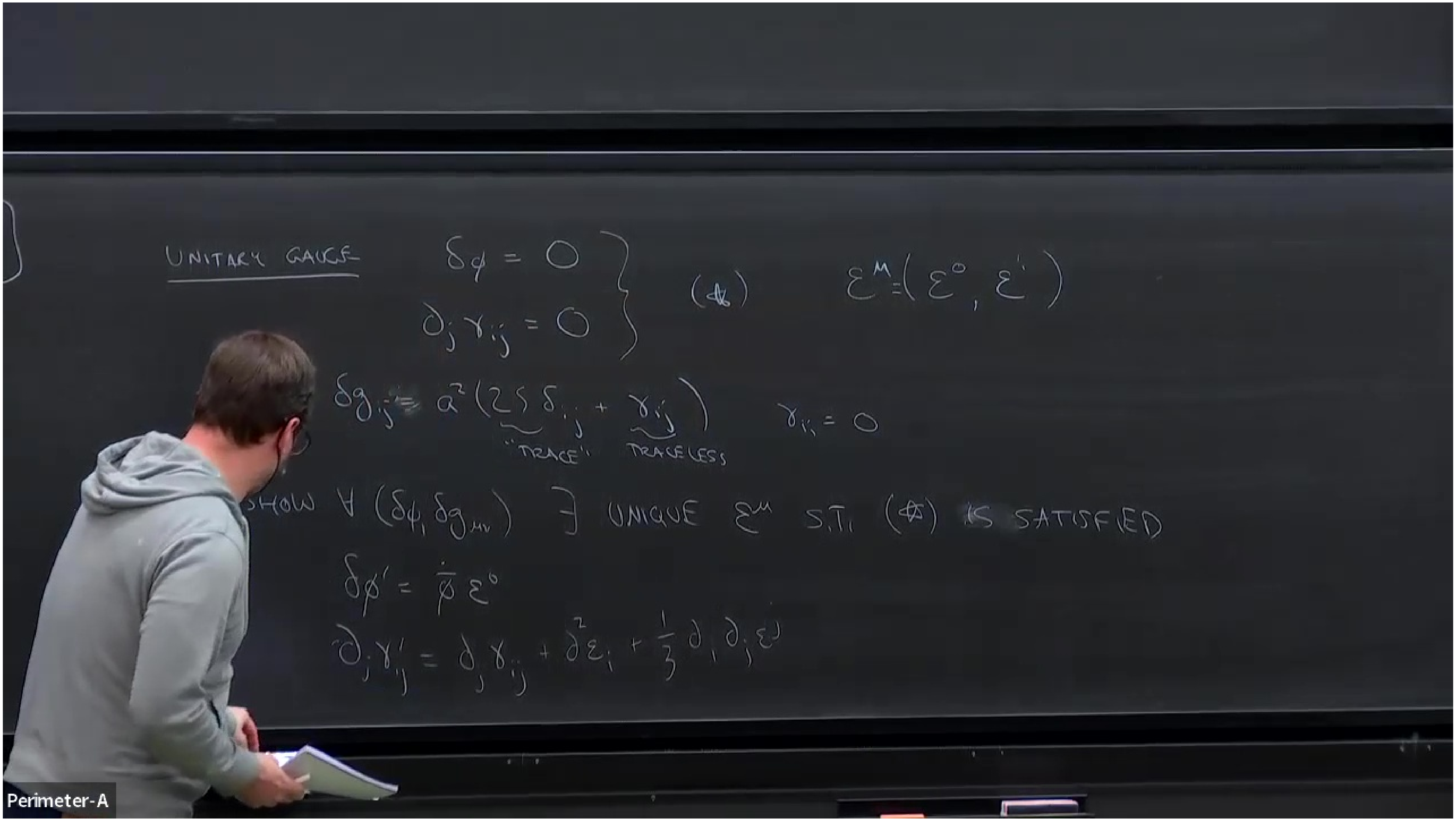
WHERE $\delta g_{ij} = a^2 \big(\underbrace{2\zeta\,\delta_{ij}}_{\text{"TRACE"}} + \underbrace{\gamma_{ij}}_{\text{TRACELESS}} \big) \qquad \gamma_{ii} = 0$

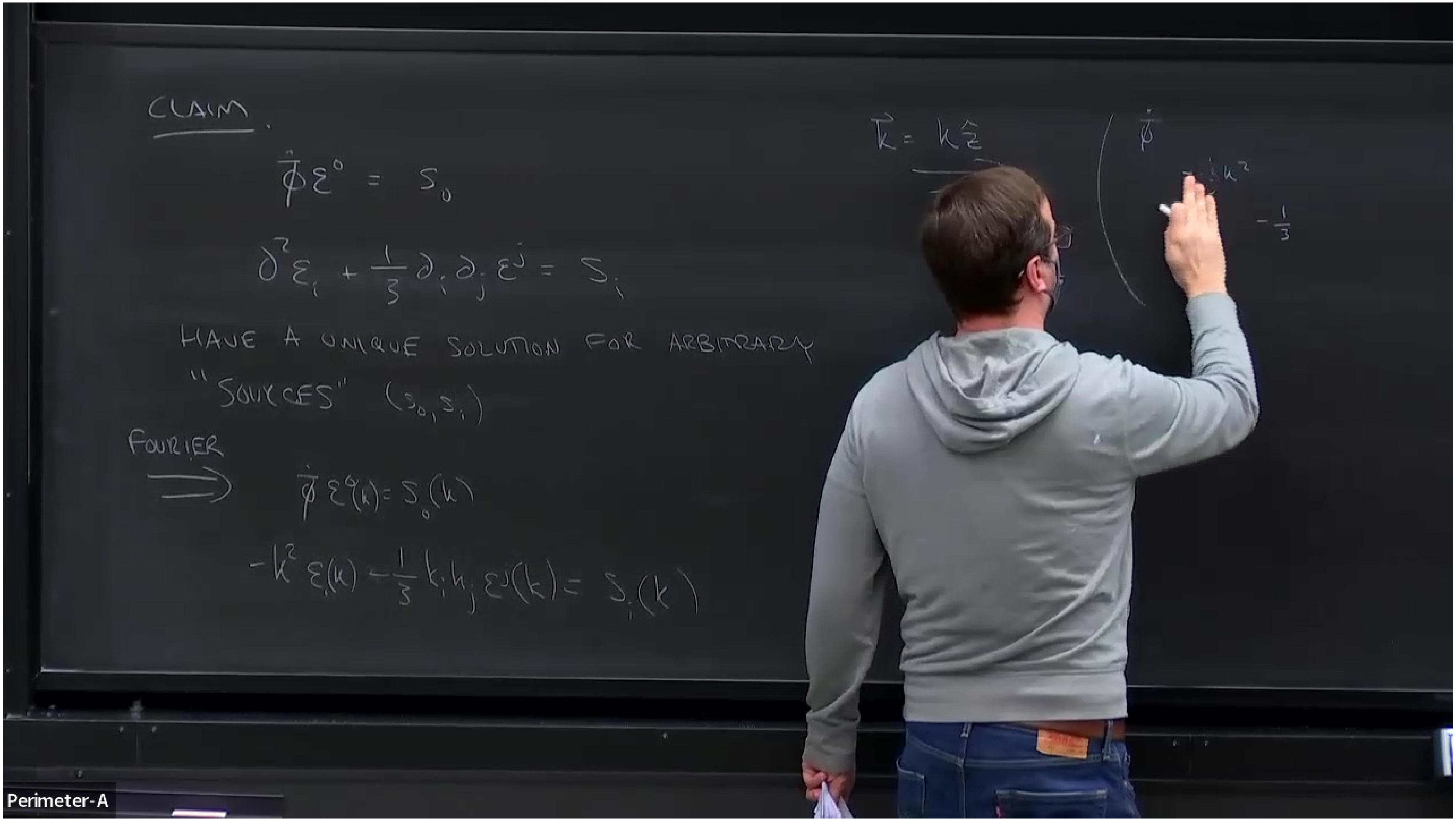
LET'S SHOW $\forall\ (\delta\phi, \delta g_{\mu\nu})\ \exists$ UNIQUE ε^μ S.T. $(\star)$ IS SATISFIED

$$\delta\phi' = \dot{\phi}\,\varepsilon^0$$

$$\partial_j \gamma'_{ij} =$$

Perimeter-A


UNITARY GAUGE
$\delta\phi = 0$
$\partial_j \gamma_{ij} = 0$
(*)
$\varepsilon^\mu = (\varepsilon^0, \varepsilon^i)$
$\delta g_{ij} = a^2 (2\zeta\delta_{ij} + \gamma_{ij})$
$\gamma_{ii} = 0$
"TRACE"
TRACELESS
SHOW $\forall\ (\delta\phi, \delta g_{\mu\nu})\ \exists$ UNIQUE ε^μ S.T. (*) IS SATISFIED
$\delta\phi' = \dot{\phi}\,\varepsilon^0$
$\partial_j \gamma'_{ij} = \partial_j \gamma_{ij} + \partial^2 \varepsilon_i + \frac{1}{3}\partial_i \partial_j \varepsilon^j$
Perimeter-A


CLAIM.
$\dot{\phi}\,\varepsilon^0 = S_0$
$\partial^2 \varepsilon_i + \frac{1}{3}\partial_i \partial_j \varepsilon^j = S_i$
HAVE A UNIQUE SOLUTION FOR ARBITRARY
"SOURCES" (S_0, S_i)
FOURIER
$\Longrightarrow$ $\dot{\phi}\,\varepsilon^0(k) = S_0(k)$
$-k^2 \varepsilon_i(k) - \frac{1}{3} k_i k_j \varepsilon^j(k) = S_i(k)$
$\vec{k} = k\hat{z}$
$\dot{\phi}$
$-k^2$
$-\frac{1}{3}$
Perimeter-A

"TRACE" TRACELESS

LET'S SHOW $\forall\ (\delta\phi, \delta g_{\mu\nu})\ \exists$ UNIQUE ε^μ S.T. (✲) IS SATISFIED

$$x'^\mu = x^\mu - \varepsilon^\mu$$

$$\delta\phi' = \dot{\bar\phi}\,\varepsilon^0$$

$$\partial_j \gamma'_{ij} = \partial_j \gamma_{ij} + \partial^2 \varepsilon_i + \frac{1}{3}\partial_i \partial_j \varepsilon^j$$

$\vec{k} = k\hat{z} \Longrightarrow$

$$\underbrace{\begin{pmatrix} \dot{\bar\phi} & & & \\ & -k^2 & & \\ & & -k^2 & \\ & & & -\frac{4}{3}k^2 \end{pmatrix}}_{\text{INVERTIBLE}} \begin{pmatrix} \varepsilon^0(k) \\ \varepsilon^1(k) \\ \varepsilon^2(k) \\ \varepsilon^3(k) \end{pmatrix} = \begin{pmatrix} S_0(k) \\ S_1(k) \\ S_2(k) \\ S_3(k) \end{pmatrix}$$

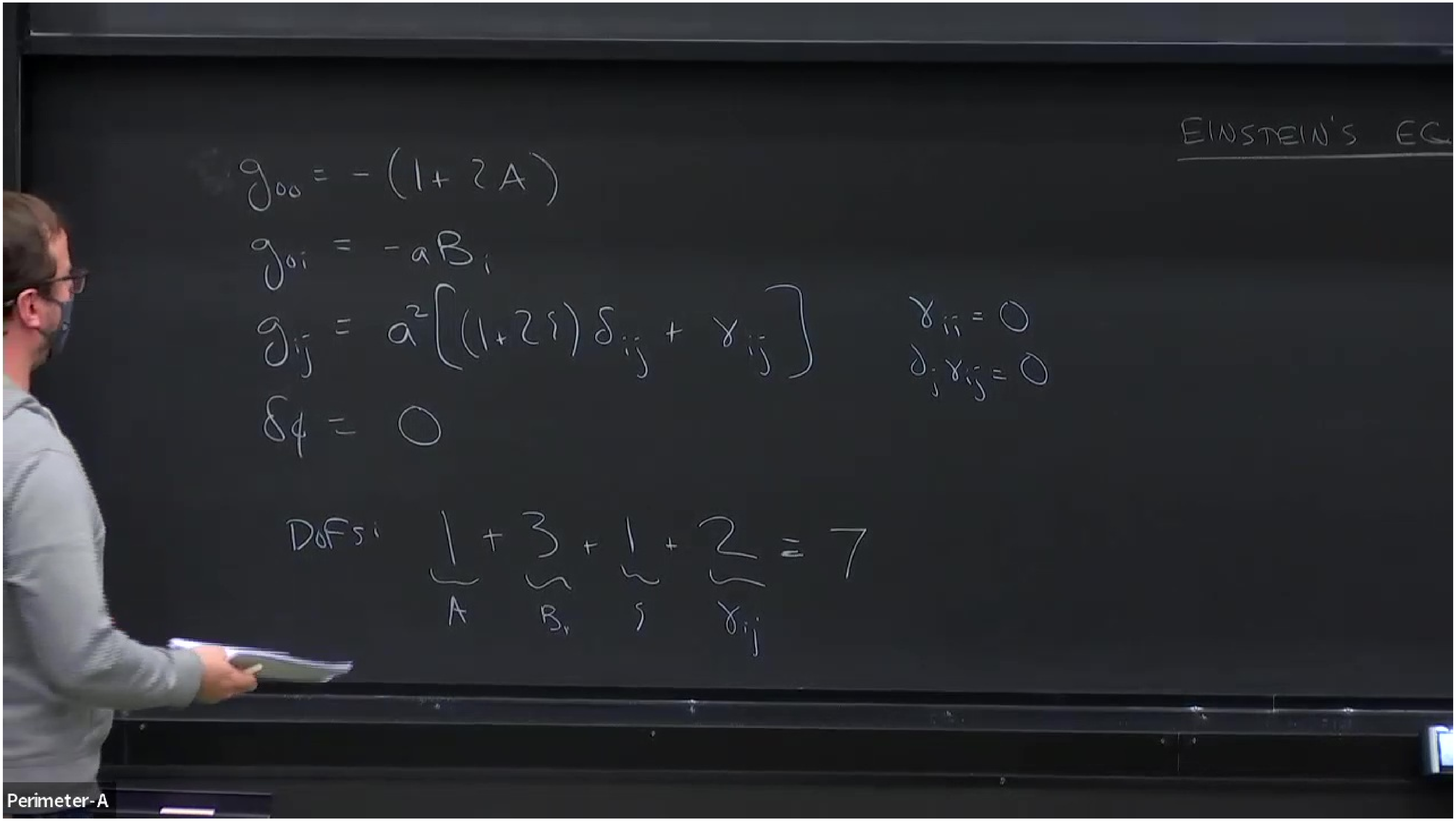
"SPATIALLY FLAT GAUGE"

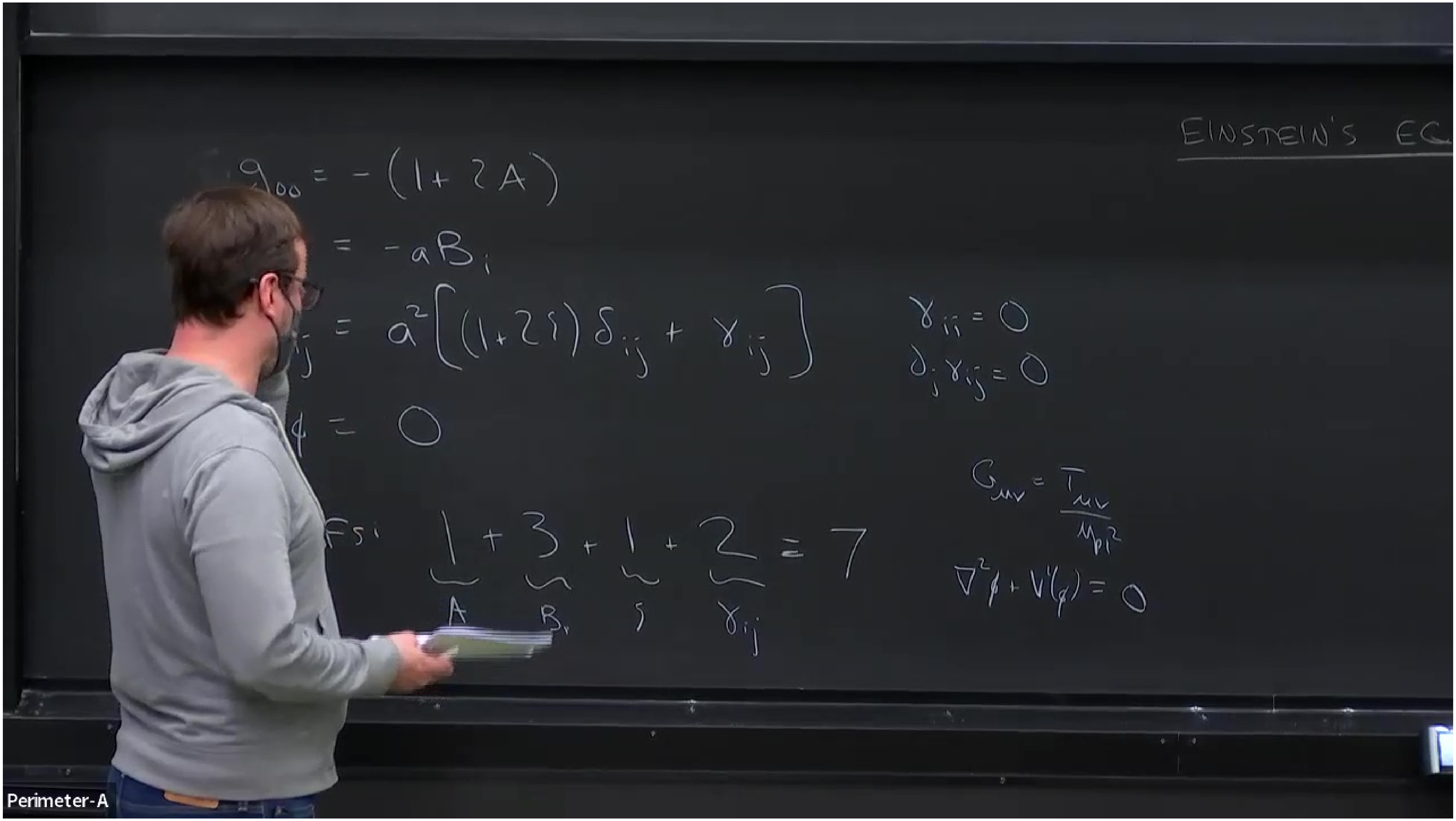
$$\delta g_{ij} = a^2 \gamma_{ij} \qquad \gamma_{ii} = \partial_j \gamma_{ij} = 0$$

$$\delta\phi \neq 0$$

$$\zeta_{\text{UNITARY}} = H(\Delta t)$$

$$(\delta\phi)_{\text{SP FLAT}} = -\dot{\bar\phi}\,(\Delta t)$$


EINSTEIN'S EQ
$g_{00} = -(1+2A)$
$g_{0i} = -aB_i$
$g_{ij} = a^2\left[(1+2\zeta)\delta_{ij} + \gamma_{ij}\right]$
$\delta\phi = 0$
$\gamma_{ii} = 0$
$\partial_j \gamma_{ij} = 0$
DoFs: $\underbrace{1}_{A} + \underbrace{3}_{B_i} + \underbrace{1}_{\zeta} + \underbrace{2}_{\gamma_{ij}} = 7$
Perimeter-A


EINSTEIN'S EQ
$g_{00} = -(1+2A)$
$= -aB_i$
$= a^2\left[(1+2\zeta)\delta_{ij} + \gamma_{ij}\right]$
$\phi = 0$
$\gamma_{ii} = 0$
$\partial_i \gamma_{ij} = 0$
$G_{\mu\nu} = \frac{T_{\mu\nu}}{M_{pl}^2}$
$\nabla^2\phi + V'(\phi) = 0$
$\underbrace{1}_{A} + \underbrace{3}_{B_i} + \underbrace{1}_{\zeta} + \underbrace{2}_{\gamma_{ij}} = 7$
Perimeter-A

EINSTEIN'S EQS

$$A = \frac{\dot{\zeta}}{H}$$

$$B_i = \partial_i \left[\frac{\zeta}{aH} - \frac{\dot{\phi}^2}{2 M_{pl}^2 H^2} \partial^{-2} \dot{\zeta} \right]$$

$$\frac{\partial}{\partial t}\left[\frac{\dot{\phi}^2 a^3}{H^2} \frac{\partial \zeta}{\partial t} \right] = \frac{\dot{\phi}^2 a}{H^2} \partial^2 \zeta$$

$$\frac{\partial}{\partial t}\left[a^3 \frac{\partial \gamma_{ij}}{\partial t} \right] = a \partial^2 \gamma_{ij}$$

$$\partial^{-2} \zeta(k) = \frac{-1}{k^2} \zeta(k)$$

Perimeter-A

...UTION FOR ARBITRARY

INVERTIBLE

$\zeta_{\text{UNITARY}} = H(\Delta t)$

"SPATIALLY FLAT GAUGE" $\delta g_{ij} = a^2 \gamma_{ij}$ $\quad \gamma_{ii} = \partial_j \gamma_{ij} = 0$

$\delta\phi \neq 0$

$(\delta\phi)_{\text{SP FLAT}} = -\dot{\bar{\phi}}(\ldots$

$\ldots(k) = \zeta_i(k)$

$\ldots 2A)$ $\quad \dot{\bar{\phi}} = \dot{\phi}$

EINSTEIN'S EQ.S

$$\begin{cases} A = \dfrac{\dot{\zeta}}{H} \\ B_i = \partial_i \left[\dfrac{\zeta}{aH} - \dfrac{\dot{\phi}^2}{2 M_{pl}^2 H^2} \partial^{-2} \dot{\zeta} \right] \end{cases}$$

$$\frac{\partial}{\partial t}\left[\frac{\dot{\phi}^2 a^3}{H^2} \frac{\partial \zeta}{\partial t} \right] = \frac{\dot{\phi}^2 a}{H^2} \partial^2 \zeta$$

$$\frac{\partial}{\partial t}\left[a^3 \frac{\partial \gamma_{ij}}{\partial t} \right] = a \partial^2 \gamma_{ij}$$

$\partial^{-2}\zeta(k) = \dfrac{-1}{k^2}\zeta(k)$

$\ldots(1+2\zeta)\delta_{ij} + \gamma_{ij}]$ $\quad \gamma_{ii} = 0$, $\partial_j \gamma_{ij} = 0$

$G_{\mu\nu} = \dfrac{T_{\mu\nu}}{M_{pl}^2}$

$\nabla^2\phi + V'(\phi) = 0$

$\ldots + \underbrace{3}_{B_i} + \underbrace{1}_{\zeta} + \underbrace{2}_{\gamma_{ij}} = 7$

Perimeter-A

ACTION TO SECOND ORDER

$$S = \int d^4x \sqrt{-g} \left[\frac{M_{pl}^2}{2} R - \frac{1}{2} (\nabla^\mu \phi)(\nabla_\nu \phi) - V(\phi) \right]$$

$$= \frac{1}{2} \int dt\, d^3x\, \frac{\dot{\phi}^2 a^3}{H^2} \left[\dot{\zeta}^2 - a^{-2} (\partial_i \zeta)(\partial_i \zeta) \right]$$

$$+ \frac{M_{pl}^2}{8} \int dt\, d^3x\, a^3 \left[\dot{\gamma}_{ij} \dot{\gamma}_{ij} - a^{-2} (\partial_k \gamma_{ij})(\partial_k \gamma_{ij}) \right]$$

MUKHANOV-SASAKI ACTION

ACTION TO SECOND ORDER

$$\int e^{iS/\hbar}$$

$$S = \int d^4x \sqrt{-g}\left[\frac{M_{pl}^2}{2} R - \frac{1}{2}(\nabla^\mu \phi)(\nabla_\mu \phi) - V(\phi)\right]$$

$$= \frac{1}{2}\int dt\, d^3x\, \frac{\dot\phi^2 a^3}{H^2}\left[\dot\zeta^2 - a^{-2}(\partial_i \zeta)(\partial_i \zeta)\right]$$

$$+ \frac{M_{pl}^2}{8}\int dt\, d^3x\, a^3\left[\dot\gamma_{ij}\dot\gamma_{ij} - a^{-2}(\partial_k \gamma_{ij})(\partial_k \gamma_{ij})\right]$$

MUKHANOV-SASAKI ACTION

Perimeter-A

$e^{iS/\hbar}$

TIME-DEPENDENT HARMONIC OSCILLATOR

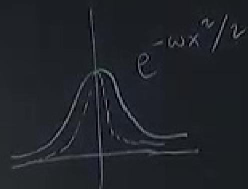
SETUP

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega(t)^2\hat{x}^2$$

$\omega(t) \to \omega_0$ AT EARLY TIMES

STARTS IN $|\psi(t_0)\rangle = |0\rangle$

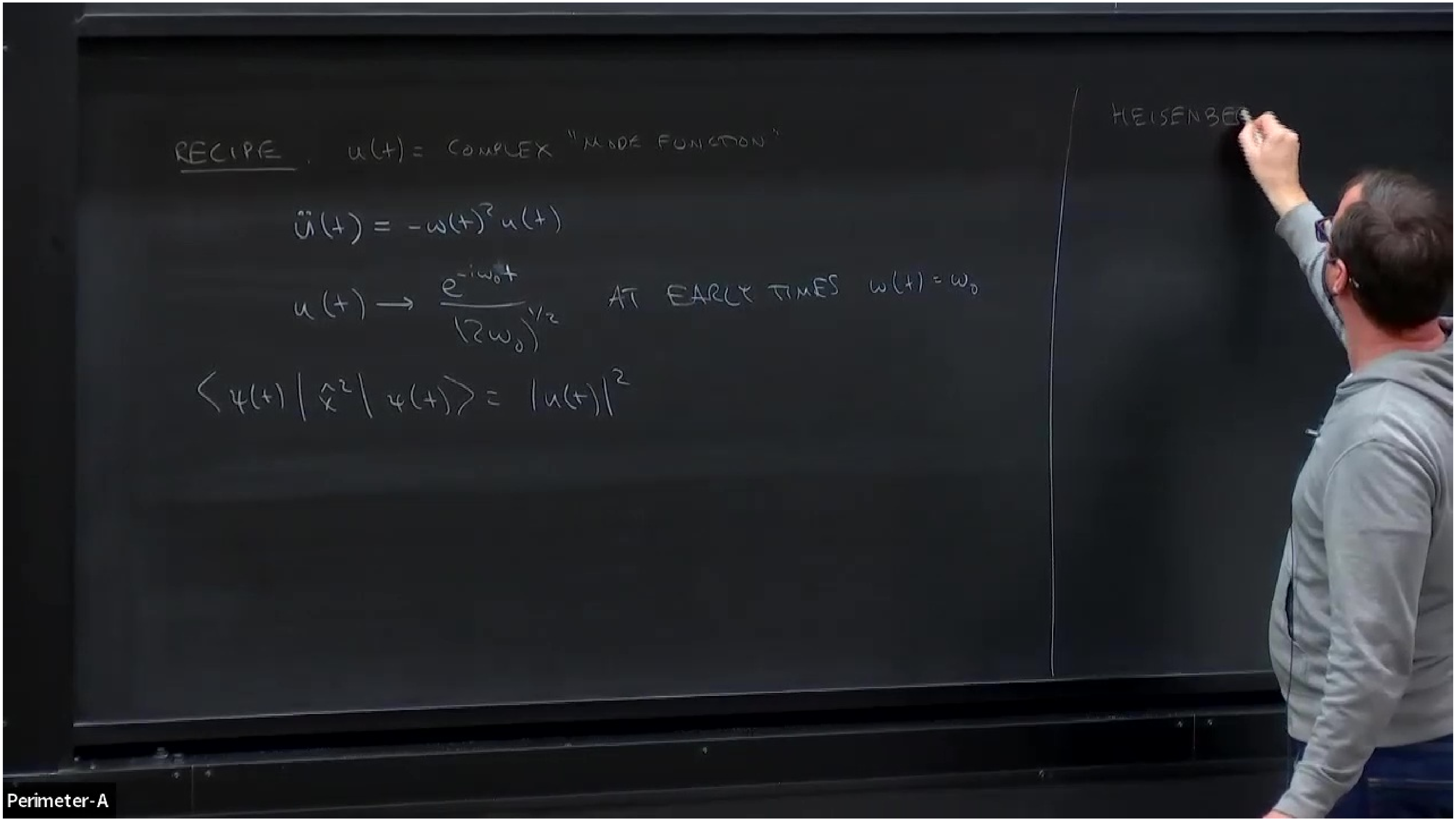
$$\frac{1}{\omega^2}\frac{d\omega}{dt} \ll 1$$

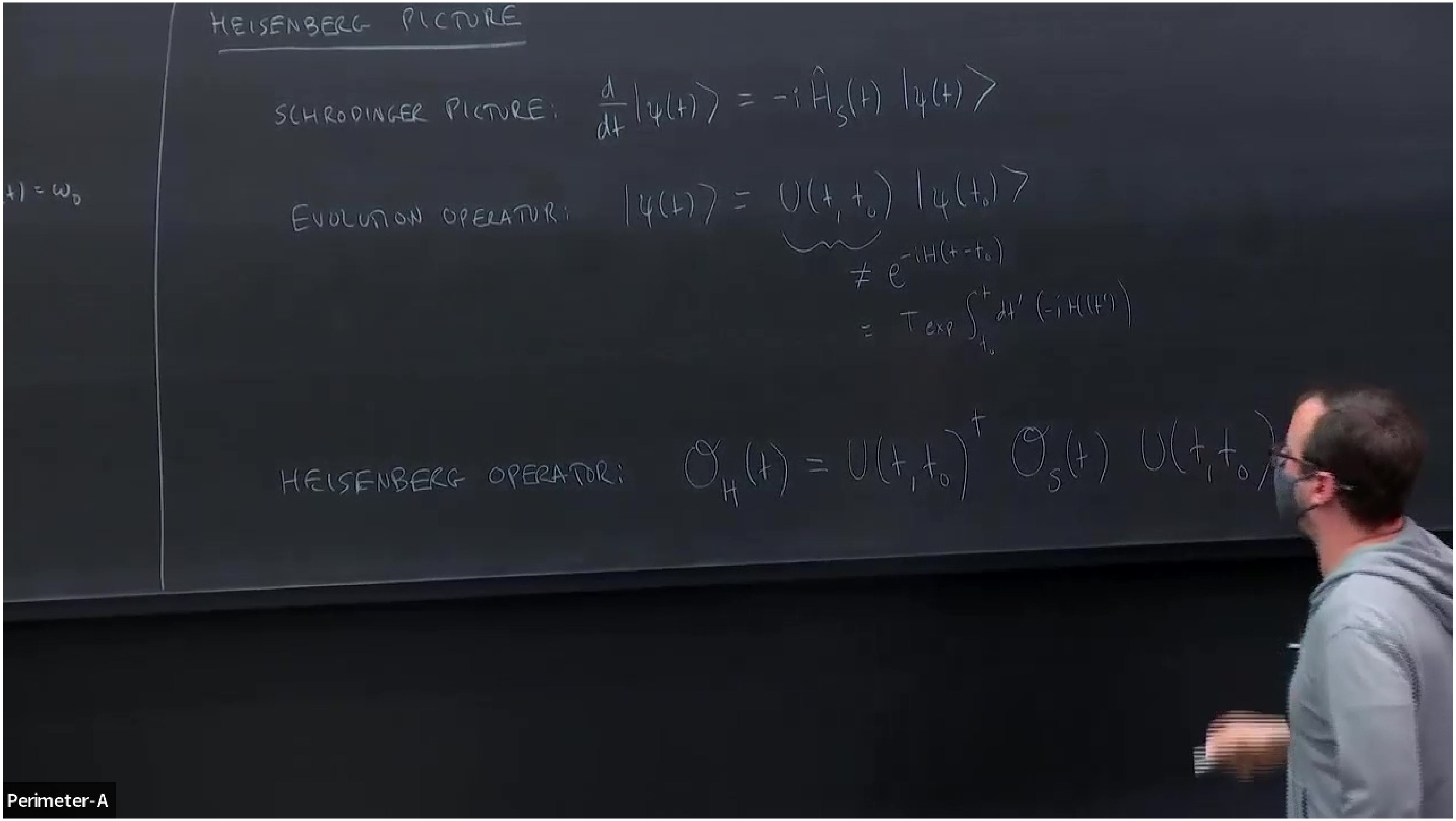


$$\frac{1}{\omega^2}\frac{d\omega}{dt} \gg 1$$

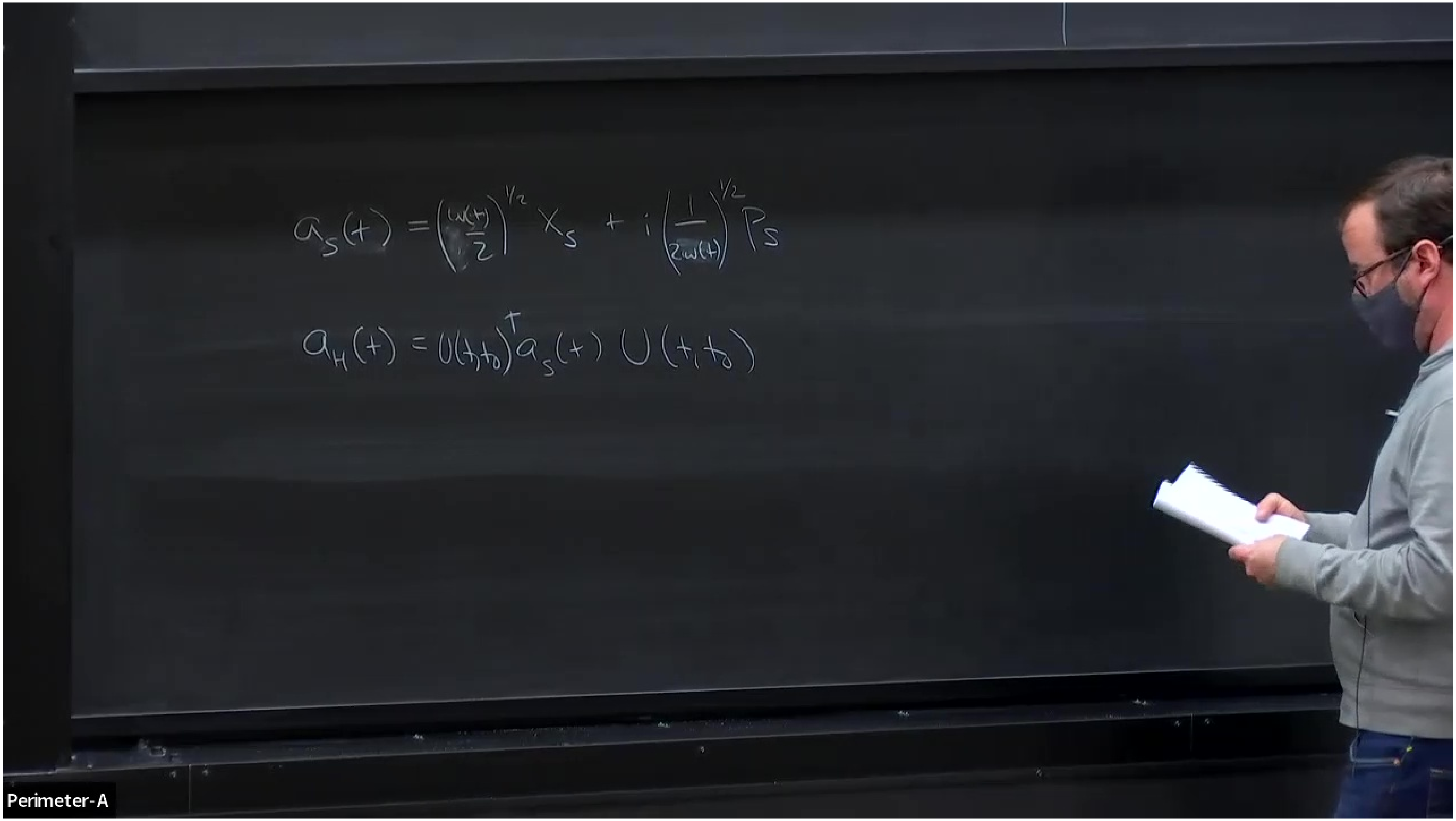
COMPUTE "VARIANCE"

$$\langle\psi(t)|\,\hat{x}^2\,|\psi(t)\rangle$$


RECIPE. $u(t)$ = COMPLEX "MODE FUNCTION"
$\ddot{u}(t) = -\omega(t)^2 u(t)$
$u(t) \rightarrow \frac{e^{-i\omega_0 t}}{(2\omega_0)^{1/2}}$ AT EARLY TIMES $\omega(t) = \omega_0$
$\langle \psi(t) | \hat{x}^2 | \psi(t) \rangle = |u(t)|^2$
HEISENBERG
Perimeter-A



HEISENBERG PICTURE
SCHRODINGER PICTURE: $\frac{d}{dt}|\psi(t)\rangle = -i\hat{H}_S(t)\,|\psi(t)\rangle$
EVOLUTION OPERATOR: $|\psi(t)\rangle = \underbrace{U(t,t_0)}\,|\psi(t_0)\rangle$
$\neq e^{-iH(t-t_0)}$
$= T\exp\int_{t_0}^{t} dt'\,(-iH(t'))$
HEISENBERG OPERATOR: $\mathcal{O}_H(t) = U(t,t_0)^\dagger\,\mathcal{O}_S(t)\,U(t,t_0)$
$t) = \omega_0$
Perimeter-A



$a_S(t) = \left(\frac{\omega(t)}{2}\right)^{1/2} X_S + i\left(\frac{1}{2\omega(t)}\right)^{1/2} P_S$
$a_H(t) = U(t,t_0)^{\dagger} a_S(t)\, U(t,t_0)$
Perimeter-A

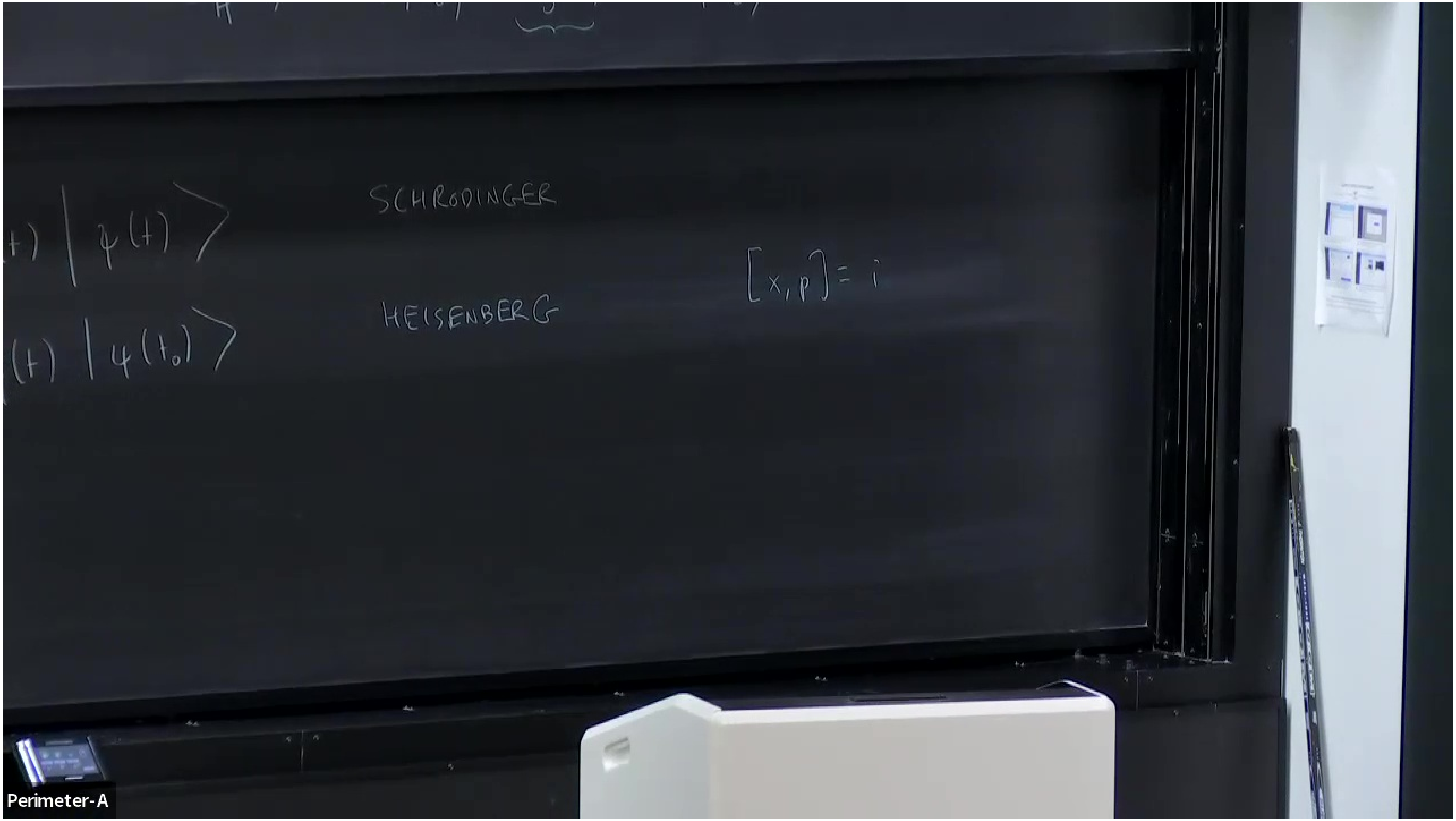
$$\langle\psi|O|\psi\rangle = \langle\psi(t)|\,O_S(t)\,|\psi(t)\rangle$$

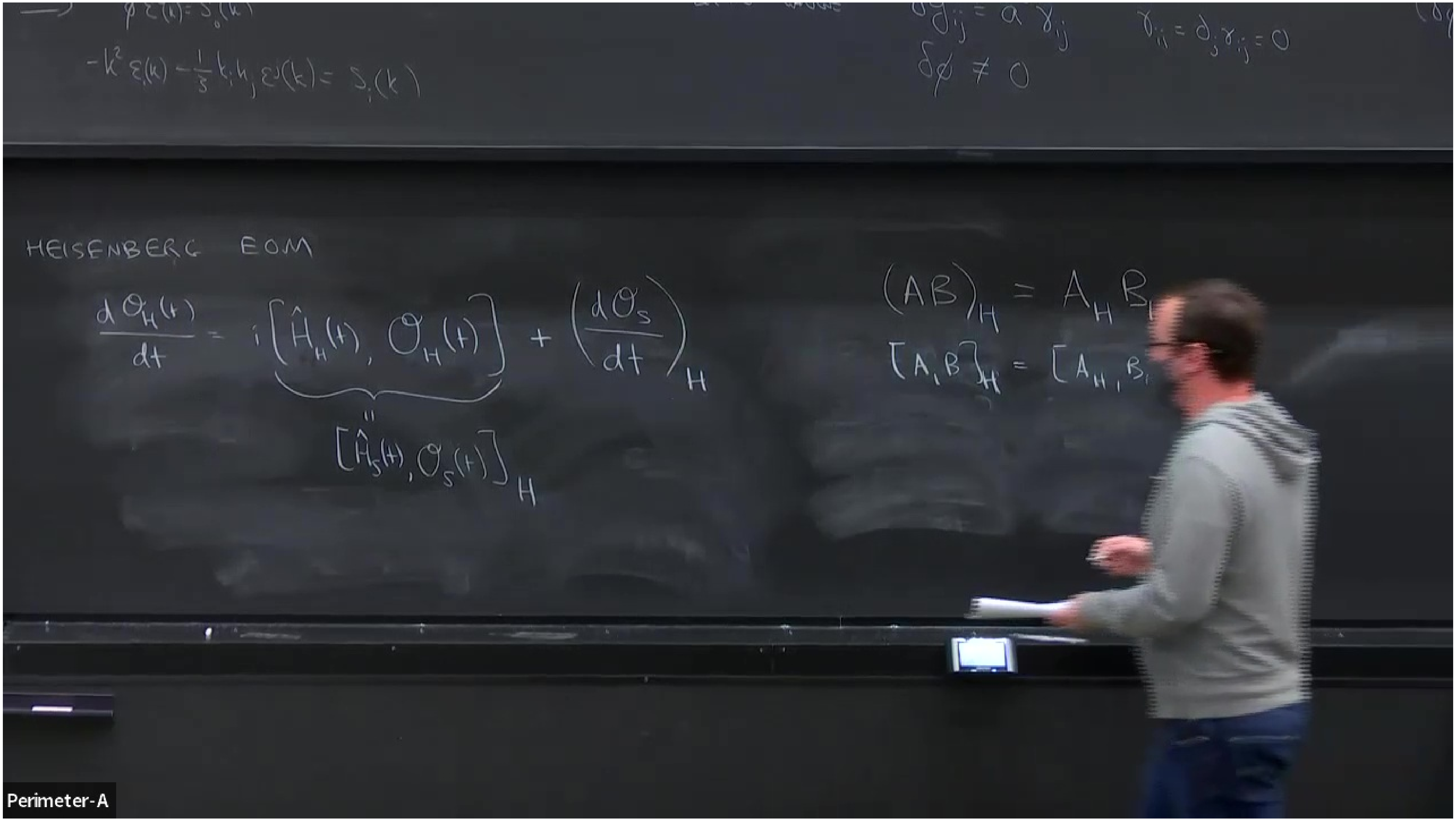
$$= \langle\psi(t_0)|\,O_H(t)\,|\psi(t_0)\rangle$$

SCHRODINGER

HEISENBERG

Perimeter-A


SCHRODINGER
HEISENBERG
[x, p] = i
Perimeter-A


HEISENBERG EOM
$\frac{d\mathcal{O}_H(t)}{dt} = i\underbrace{[\hat{H}_H(t), \mathcal{O}_H(t)]}_{= [\hat{H}_S(t), \mathcal{O}_S(t)]_H} + \left(\frac{d\mathcal{O}_S}{dt}\right)_H$
$(AB)_H = A_H B$
$[A, B]_H = [A_H, B$
$-k^2 \mathcal{E}_i(k) - \frac{1}{3} k_i k_j \mathcal{E}^j(k) = S_i(k)$
$\delta\phi \neq 0$
Perimeter-A

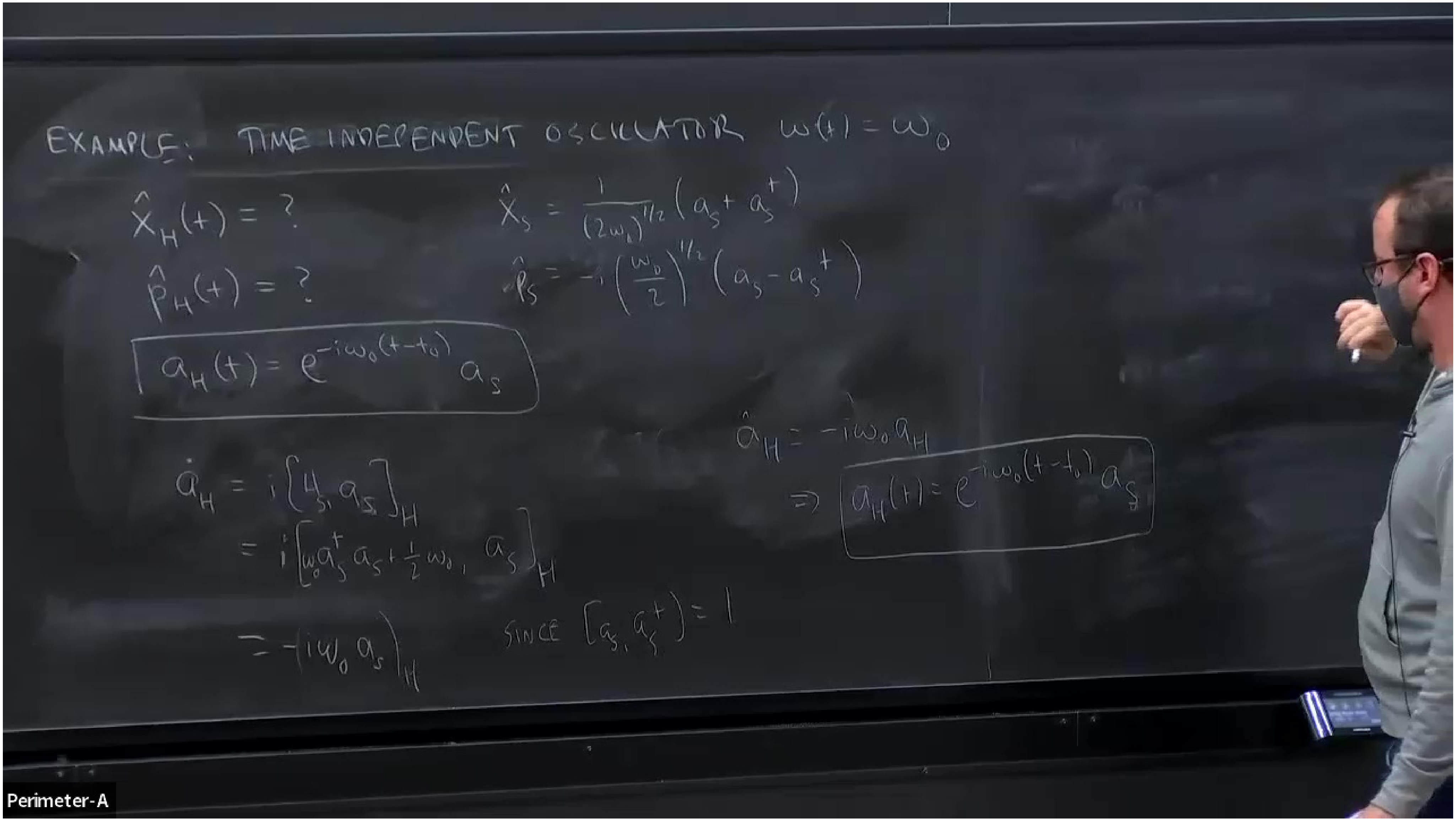
$\partial\phi \neq 0$

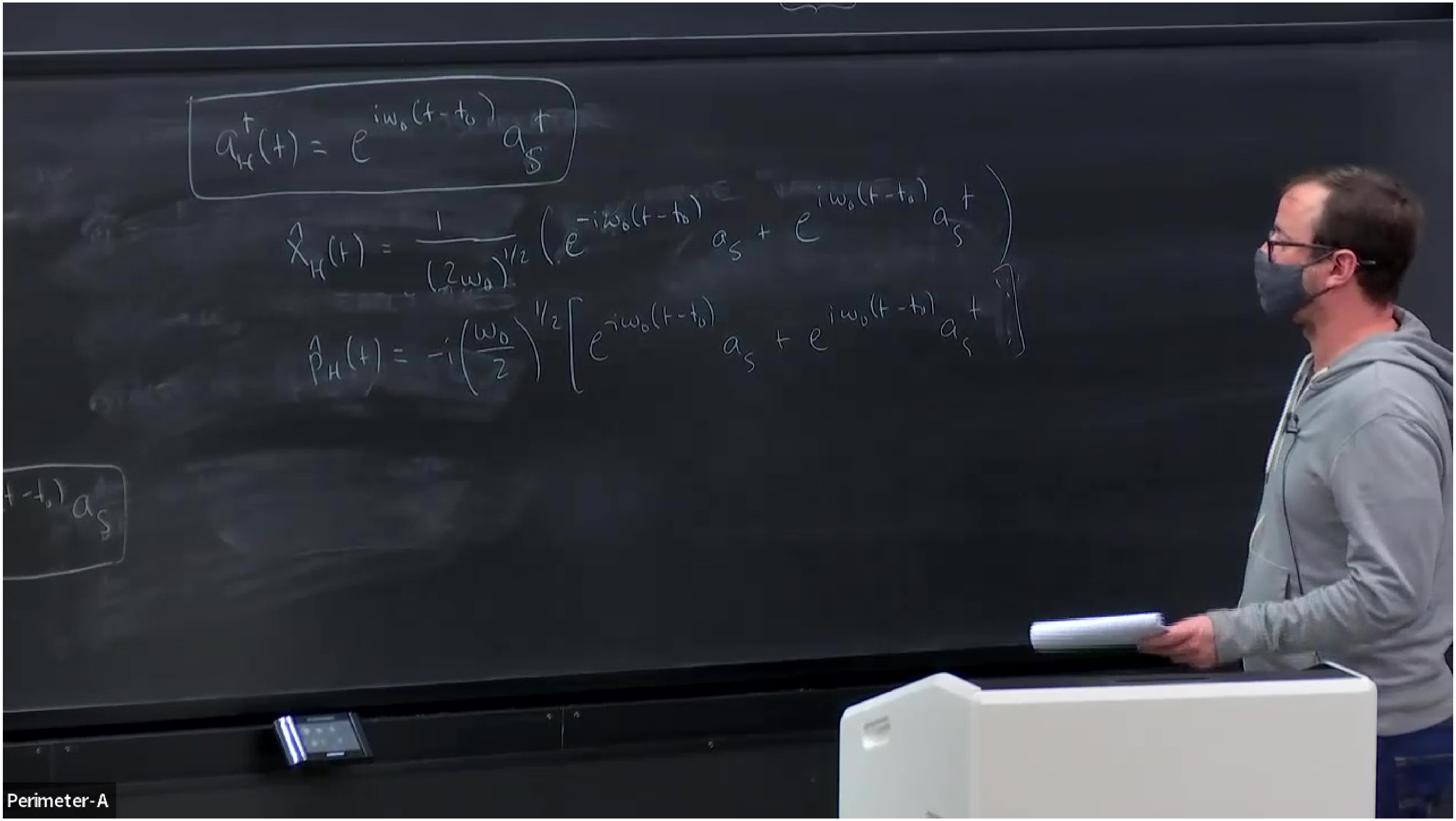
$$\left(\frac{dO_s}{dt}\right)_H$$

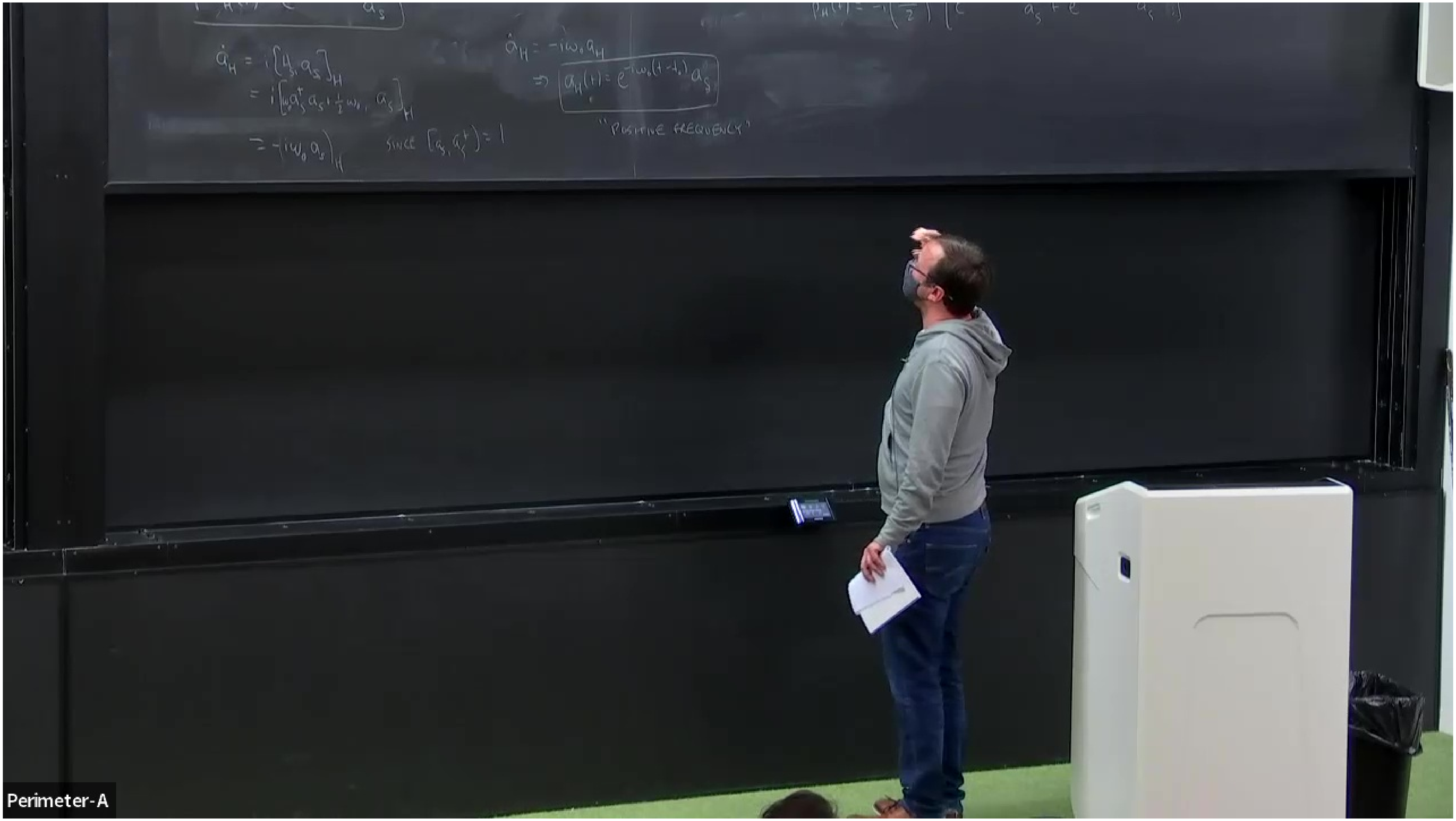
$$(AB)_H = A_H B_H$$

$$[A, B]_H = [A_H, B_H] \Rightarrow [\hat{x}_H(t), \hat{p}_H(t)] = [\hat{x}_S, \hat{p}_S]_H = (i)_H = i$$

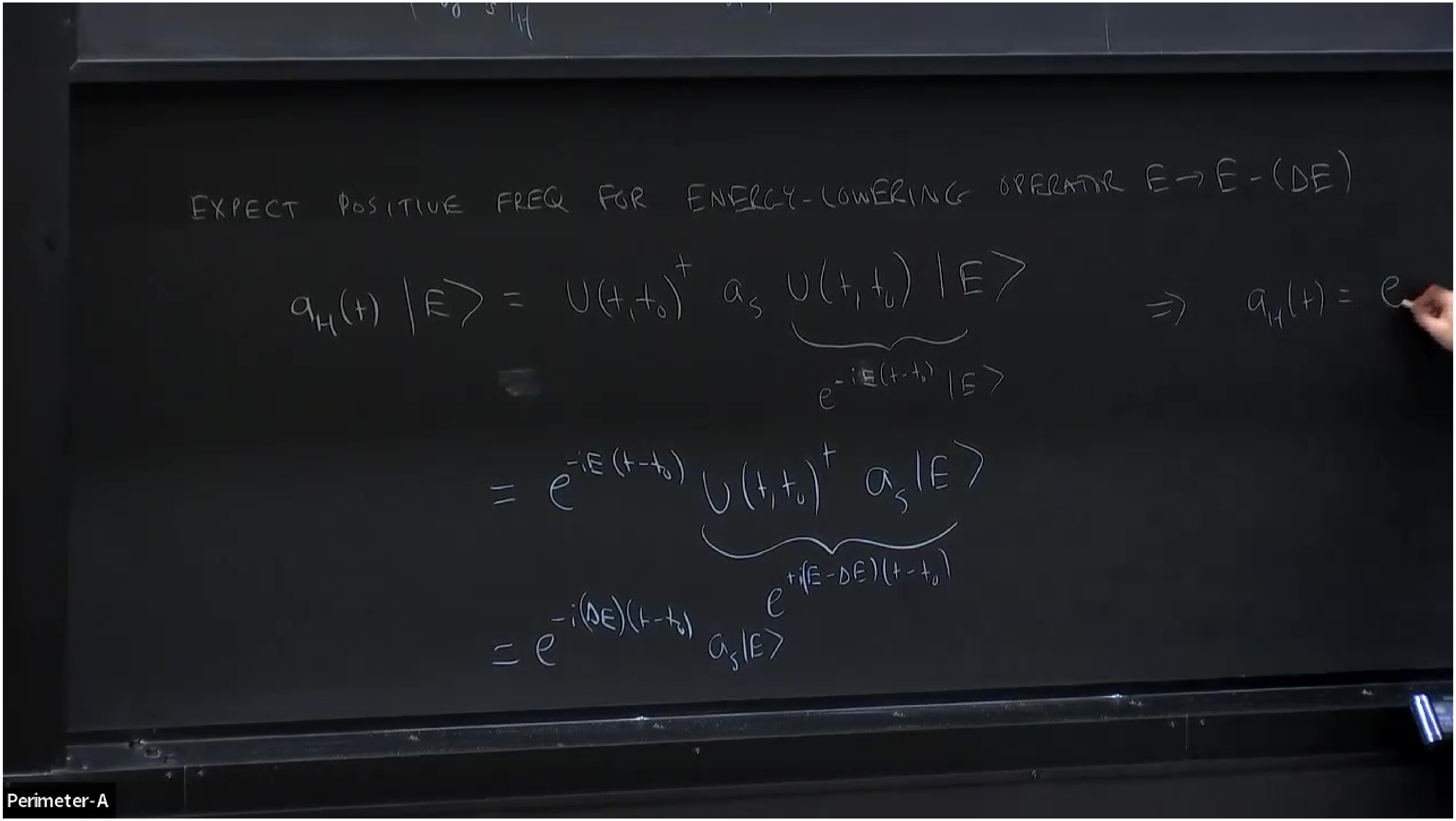
Perimeter-A


EXAMPLE: TIME INDEPENDENT OSCILLATOR $\omega(t) = \omega_0$
$\hat{X}_H(t) = ?$
$\hat{X}_S = \frac{1}{(2\omega_0)^{1/2}}(a_S + a_S^+)$
$\hat{p}_H(t) = ?$
$\hat{p}_S = -i\left(\frac{\omega_0}{2}\right)^{1/2}(a_S - a_S^+)$
$a_H(t) = e^{-i\omega_0(t-t_0)} a_S$
$\dot{a}_H = i[H_S, a_S]_H$
$= i[\omega_0 a_S^+ a_S + \frac{1}{2}\omega_0, a_S]_H$
$= -i\omega_0 a_S|_H$
SINCE $[a_S, a_S^+] = 1$
$\dot{a}_H = -i\omega_0 a_H$
$\Rightarrow a_H(t) = e^{-i\omega_0(t-t_0)} a_S$
Perimeter-A


$\boxed{a_H^\dagger(t) = e^{i\omega_0(t-t_0)} a_S^\dagger}$
$\hat{x}_H(t) = \frac{1}{(2\omega_0)^{1/2}}\left(e^{-i\omega_0(t-t_0)} a_S + e^{i\omega_0(t-t_0)} a_S^\dagger\right)$
$\hat{p}_H(t) = -i\left(\frac{\omega_0}{2}\right)^{1/2}\left[e^{-i\omega_0(t-t_0)} a_S + e^{i\omega_0(t-t_0)} a_S^\dagger\right]$
$t-t_0) a_S$
Perimeter-A



"POSITIVE FREQUENCY"
Perimeter-A


EXPECT POSITIVE FREQ FOR ENERGY-LOWERING OPERATOR $E \to E - (\Delta E)$
$a_H(t)\,|E\rangle = U(t,t_0)^\dagger\, a_S \underbrace{U(t,t_0)\,|E\rangle}_{e^{-iE(t-t_0)}|E\rangle}$
$\Rightarrow \quad a_H(t) = e$
$= e^{-iE(t-t_0)} \underbrace{U(t,t_0)^\dagger\, a_S}_{e^{+i(E-\Delta E)(t-t_0)}} |E\rangle$
$= e^{-i(\Delta E)(t-t_0)}\, a_S|E\rangle$
Perimeter-A

E FREQ FOR ENERGY-LOWERING OPERATOR $E \to E - (\Delta E)$

$$E\rangle = U(t,t_0)^{\dagger}\, a_s \underbrace{U(t,t_0)\,|E\rangle}_{e^{-iE(t-t_0)}|E\rangle} \quad \Rightarrow \quad \boxed{a_H(t) = e^{-i(\Delta E)(t-t_0)}\, a_s}$$

$$= e^{-iE(t-t_0)} \underbrace{U(t,t_0)^{\dagger}\, a_s|E\rangle}_{e^{+i(E-\Delta E)(t-t_0)}}$$

$$= e^{-i(\Delta E)(t-t_0)}\, a_s|E\rangle$$

Perimeter-A