Title: Quantum many-body physics beyond the low complexity regime

Speakers: Philippe Faist

Series: Colloquium

Date: April 20, 2022 - 2:00 PM

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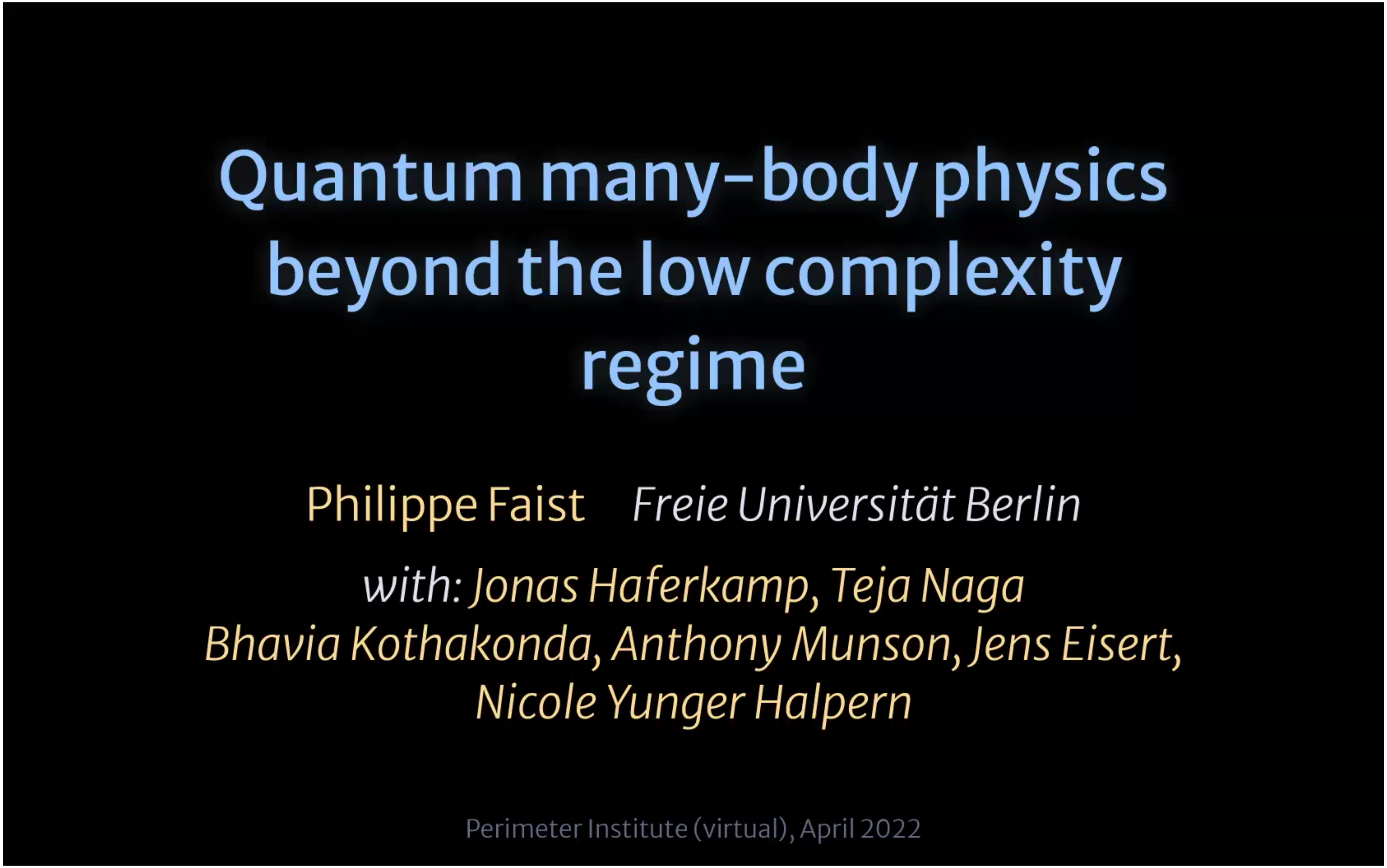
Abstract: Quantifying quantum states' complexity is a key problem in various subfields of science, from quantum computing to black-hole physics. I'll explain two approaches to understanding the behavior and the operational significance of quantum complexity in many-body systems. First, I'll consider a simple model on n qubits: We create a random quantum circuit by randomly sampling the gates that compose it. In this model, quantum complexity can be shown to grow linearly in the number of gates until saturating at a value that is exponential in n. This result proves a version of a conjecture by Brown and Susskind in the context of quantum gravity, thereby reinforcing our understanding of the evolution of wormholes in holography. Second, I'll discuss how quantum complexity manifests itself in the operational processes that we can carry out on an n-qubit system. For instance, what resources are necessary to reset an n-qubit memory register to the pure all-zero computational basis state? This approach reveals a connection between thermodynamics and complexity, as we exhibit a trade-off between the thermodynamic work cost that is necessary for the reset procedure and the complexity cost of the procedure. The general trade-off is quantified by a new measure of entropy which directly connects complexity with entropy. I'll discuss the implications of our results and new prospects for many-body physics in the regime where quantum states are of ever increasing complexity.

Joint work with: Jonas Haferkamp, Teja Naga Bhavia Kothakonda, Anthony Munson, Jens Eisert, Nicole Yunger Halpern



Quantum many-body physics beyond the low complexity regime

Philippe Faist   *Freie Universität Berlin*

with: Jonas Haferkamp, Teja Naga
Bhavia Kothakonda, Anthony Munson, Jens Eisert,
Nicole Yunger Halpern

Perimeter Institute (virtual), April 2022

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Freie Universität Berlin

Master’s project: Complexity/work trade-offs

Jonas Haferkamp

Freie U

Linear growth of quantum complexity

Jens Eisert

reie Universität Berlin

Anthony Munson

University of Maryland

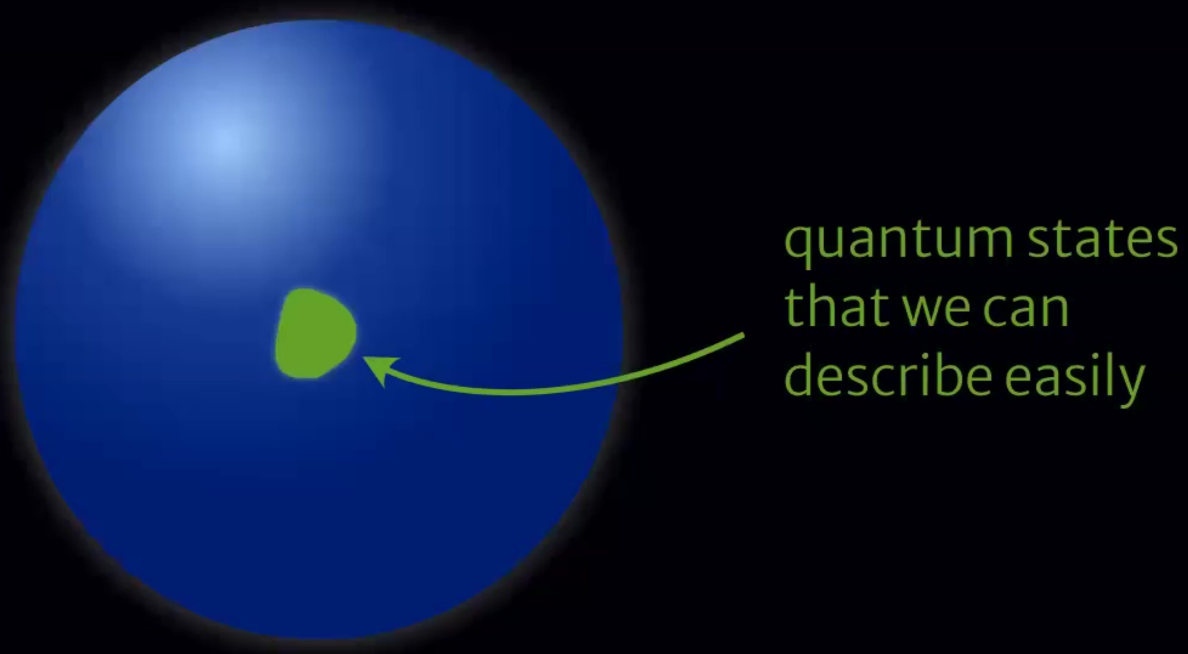
Nicole Yunger Halpern

University of Maryland

Physics at the complexity frontier

Preskill
e.g. Quantum 2018

How can we make sense of the immensity of the Hilbert space of *n* qubits?



quantum states that we can describe easily

Physics at the complexity frontier

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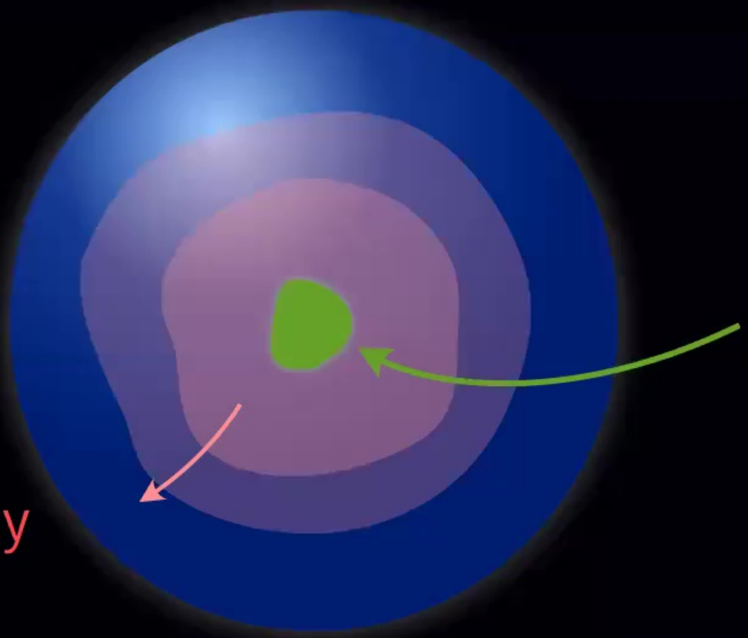
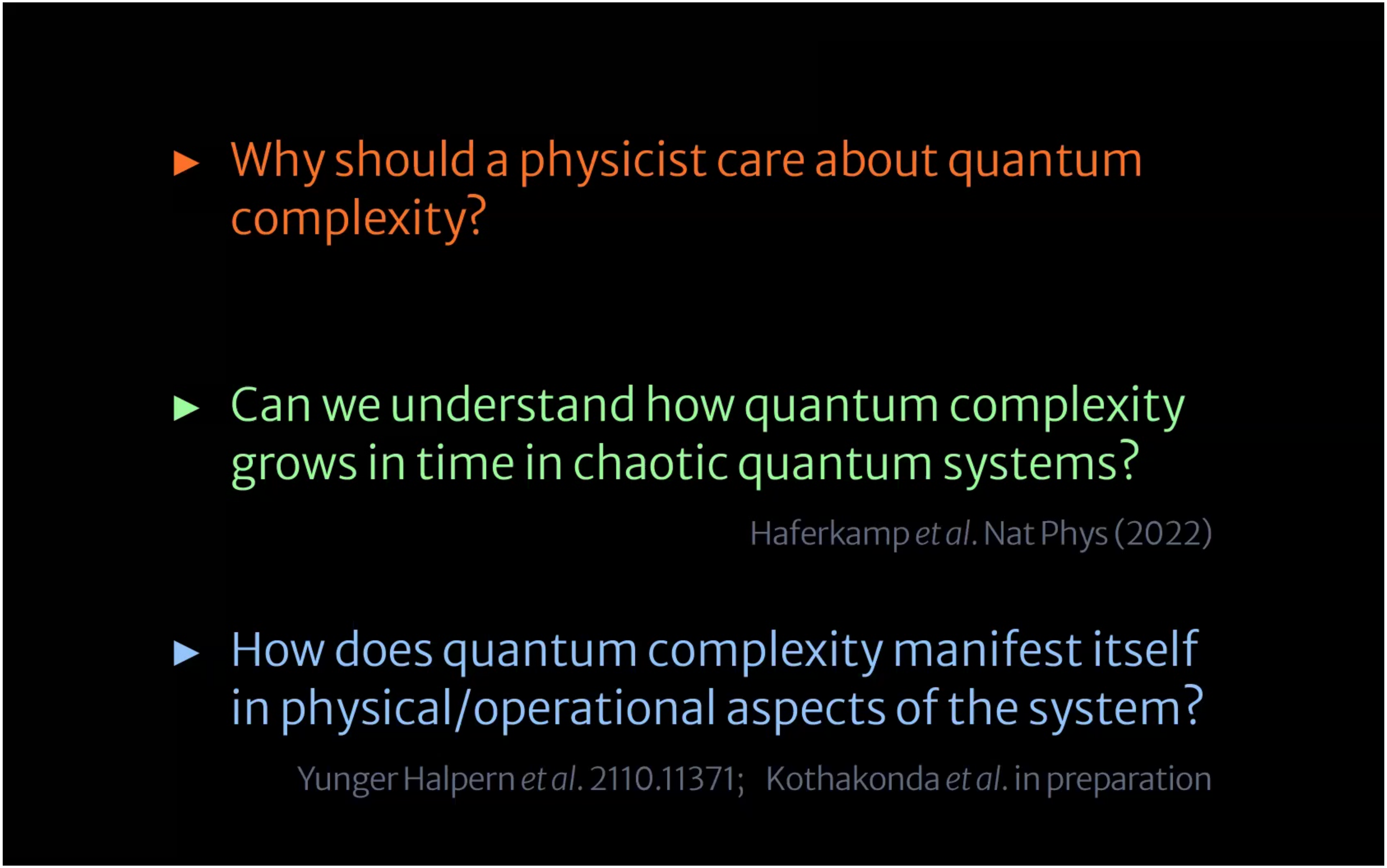


quantum states that we can describe easily

high complexity states



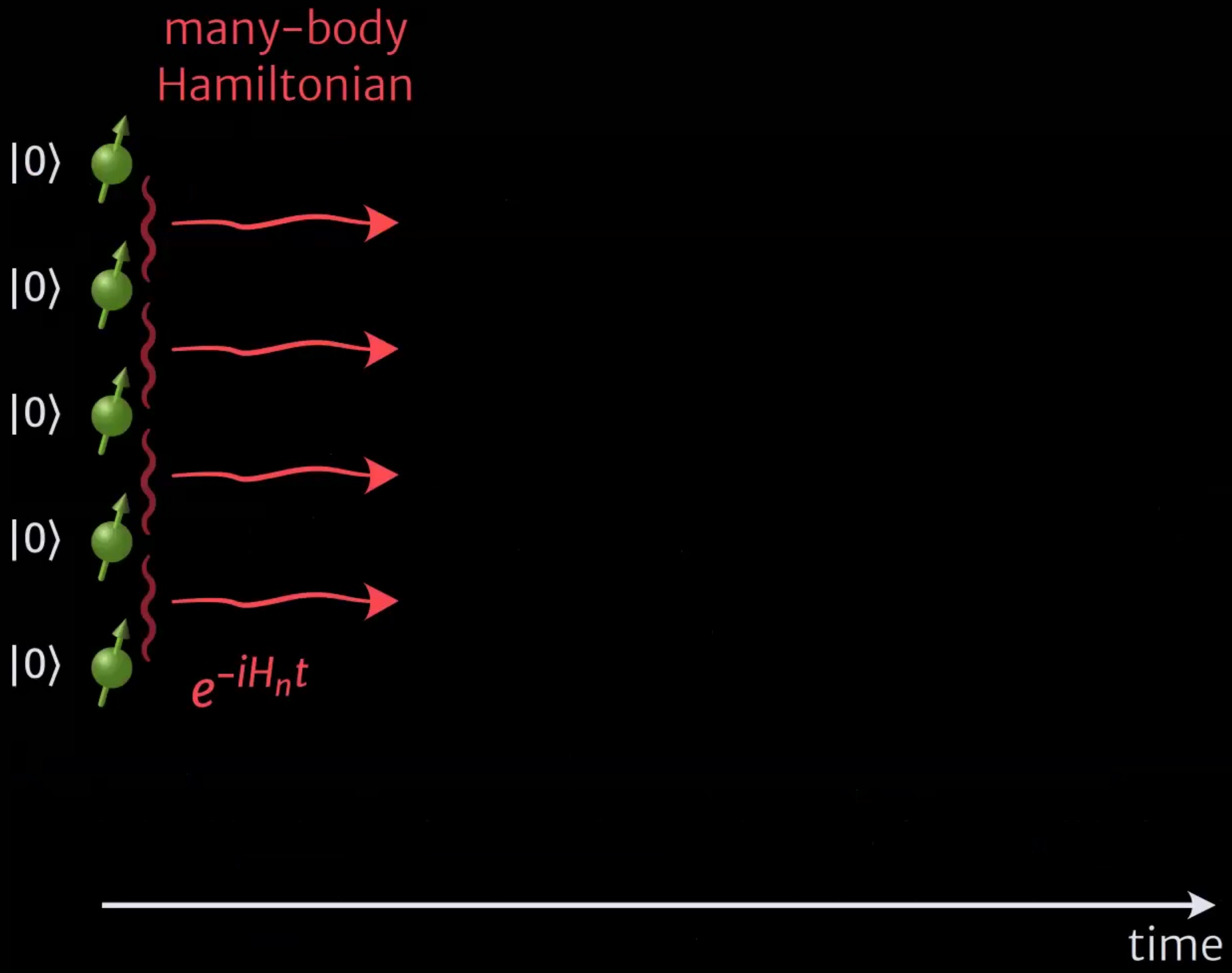
- Why should a physicist care about quantum complexity?

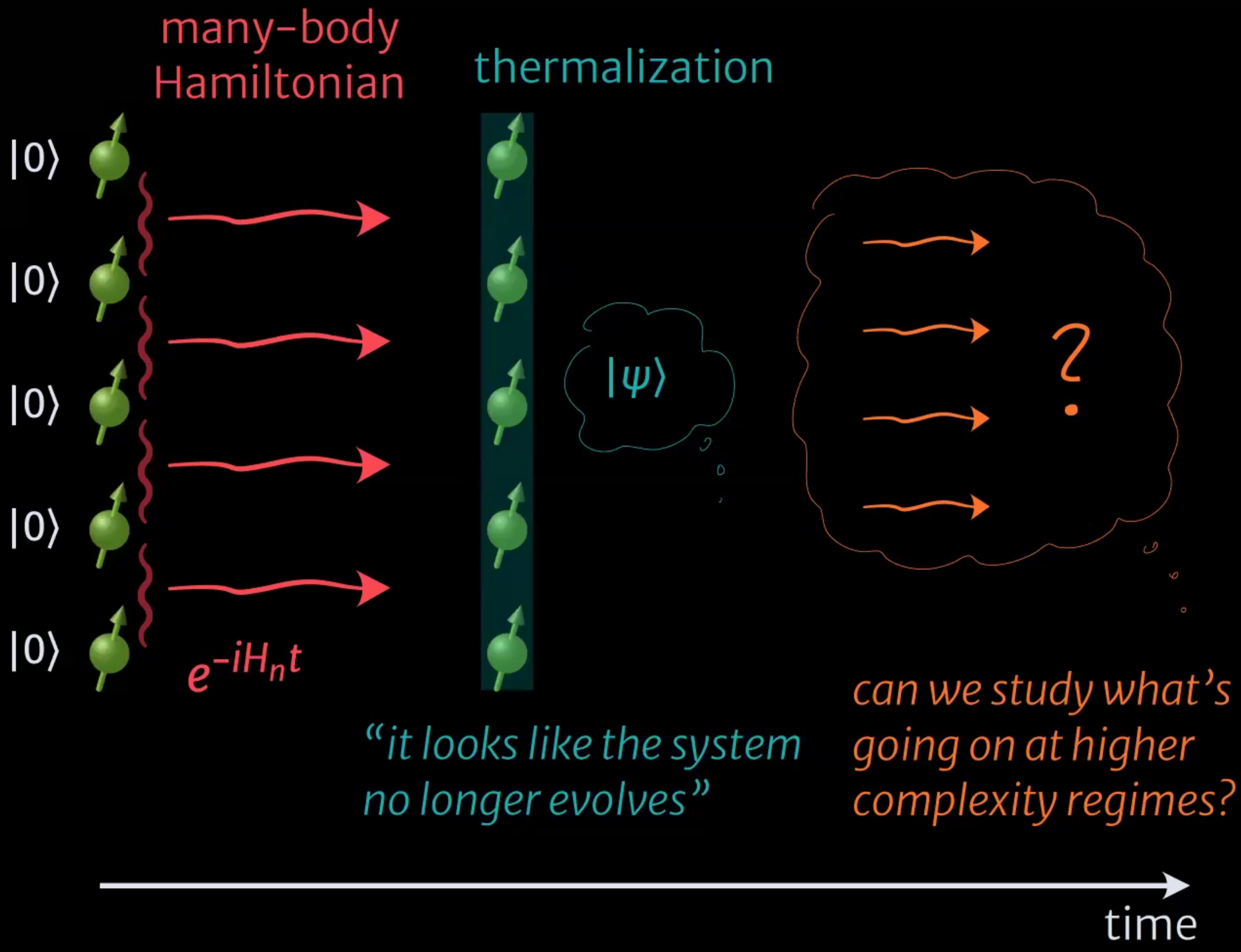
- Can we understand how quantum complexity grows in time in chaotic quantum systems?

Haferkamp *et al*. Nat Phys (2022)

- How does quantum complexity manifest itself in physical/operational aspects of the system?

Yunger Halpern *et al*. 2110.11371; Kothakonda *et al*. in preparation


many-body
Hamiltonian
|0⟩
|0⟩
|0⟩
|0⟩
|0⟩
$e^{-iH_n t}$
time



many-body Hamiltonian

thermalization

$|0\rangle$

$|0\rangle$

$|0\rangle$

$|0\rangle$

$|0\rangle$

$e^{-iH_n t}$

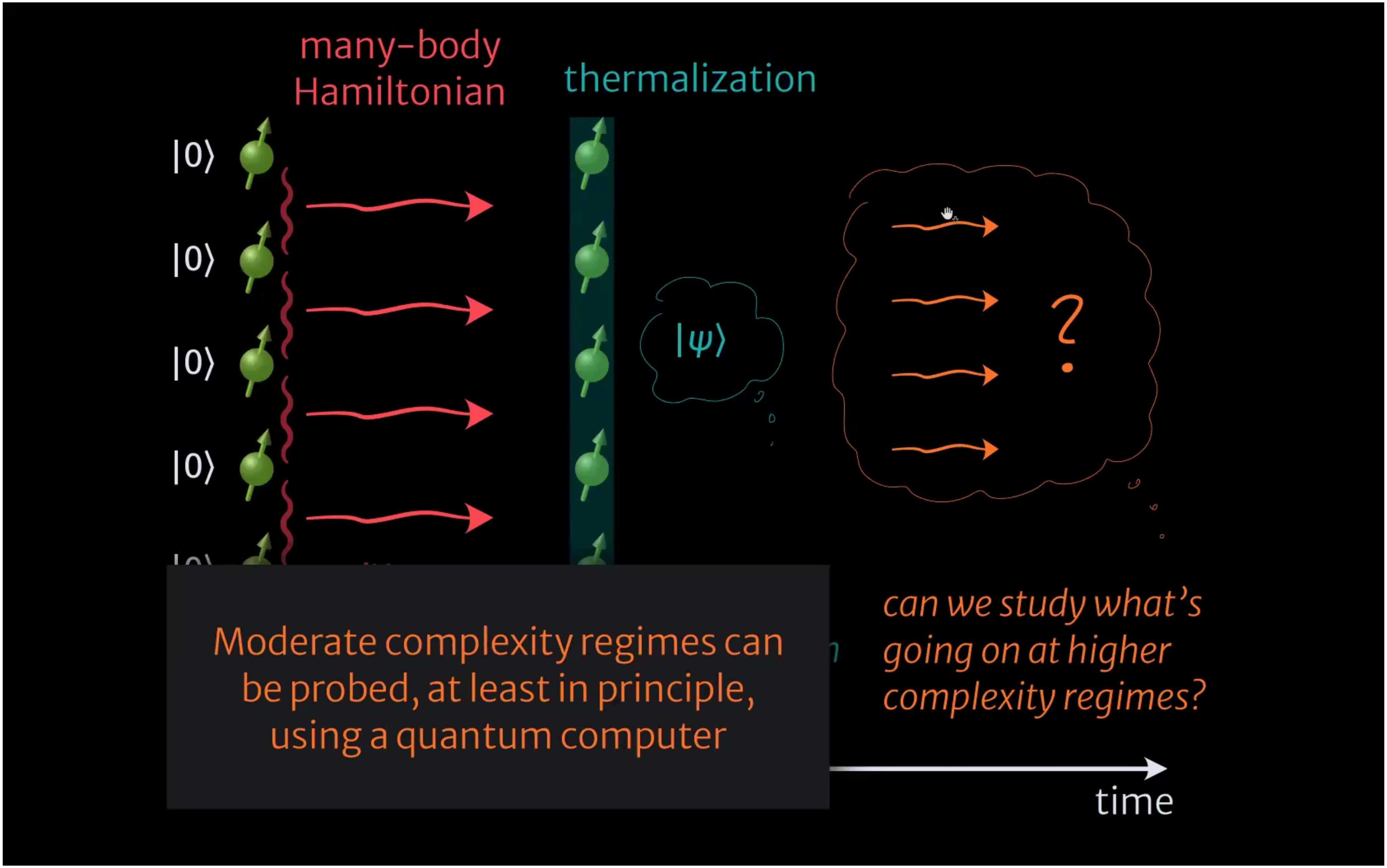
$|\psi\rangle$

?

"it looks like the system no longer evolves"

can we study what's going on at higher complexity regimes?

time

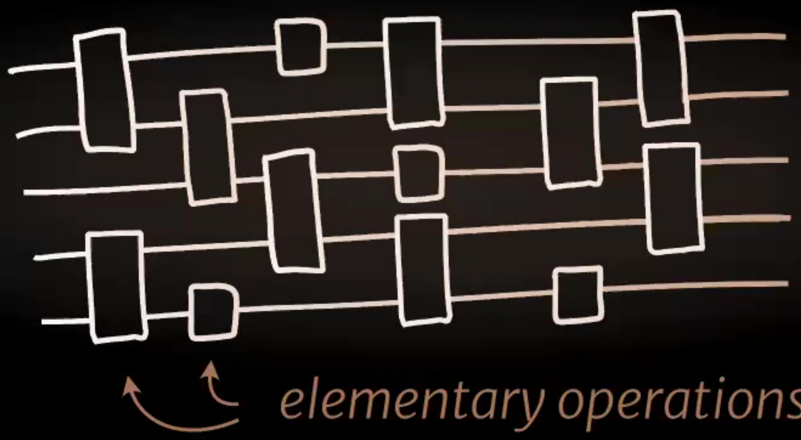


Moderate complexity regimes can be probed, at least in principle, using a quantum computer

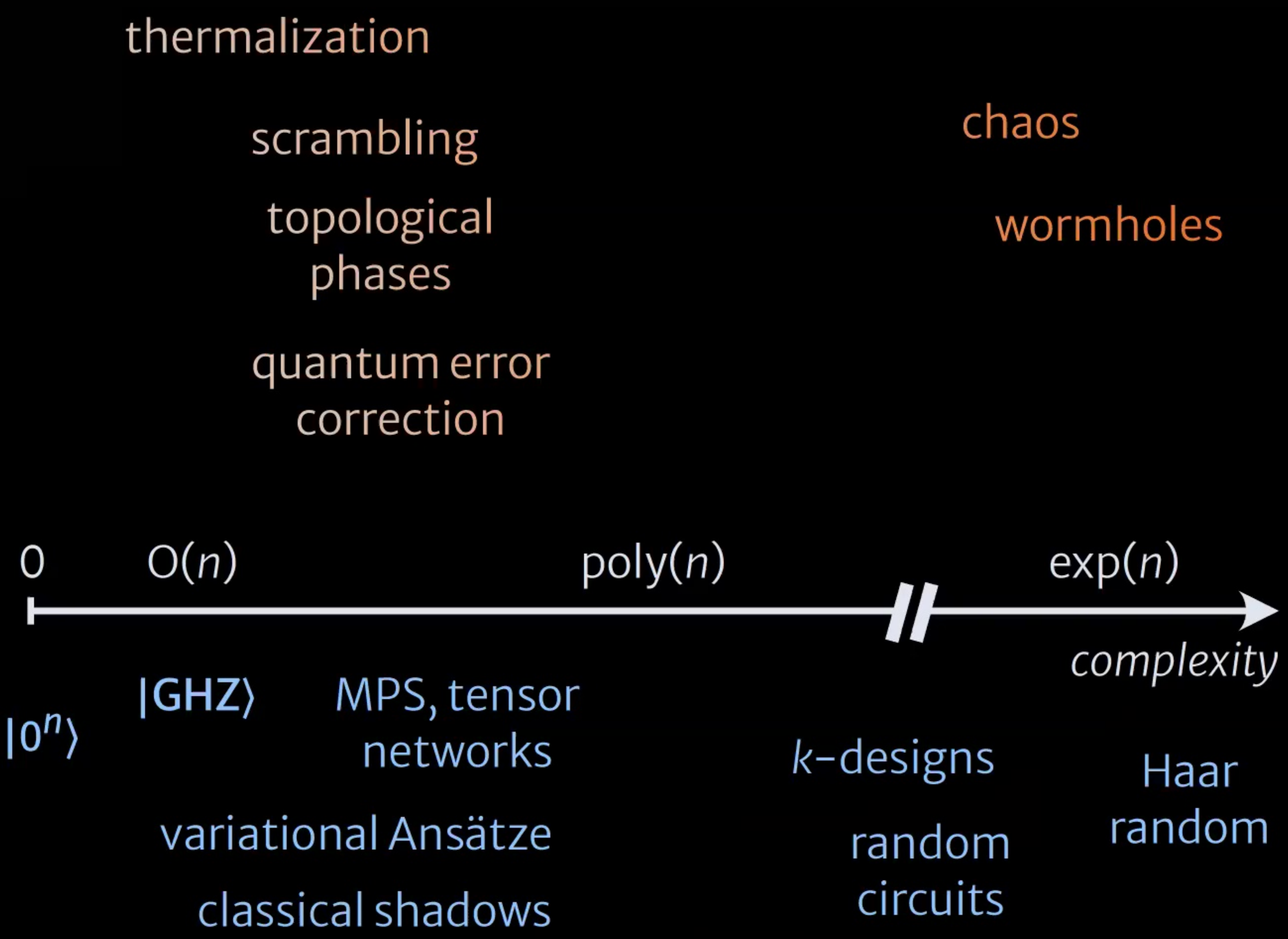
can we study what's going on at higher complexity regimes?

Quantum circuit complexity of an n-qubit unitary U = Minimal number of gates required to implement U on a quantum computer

$U =$



elementary operations
(any 2-qubit unitary)



thermalization

scrambling

topological phases

quantum error correction

chaos

wormholes

0   O(n)   poly(n)   exp(n)

complexity

$|0^n\rangle$

$|\mathrm{GHZ}\rangle$

MPS, tensor networks

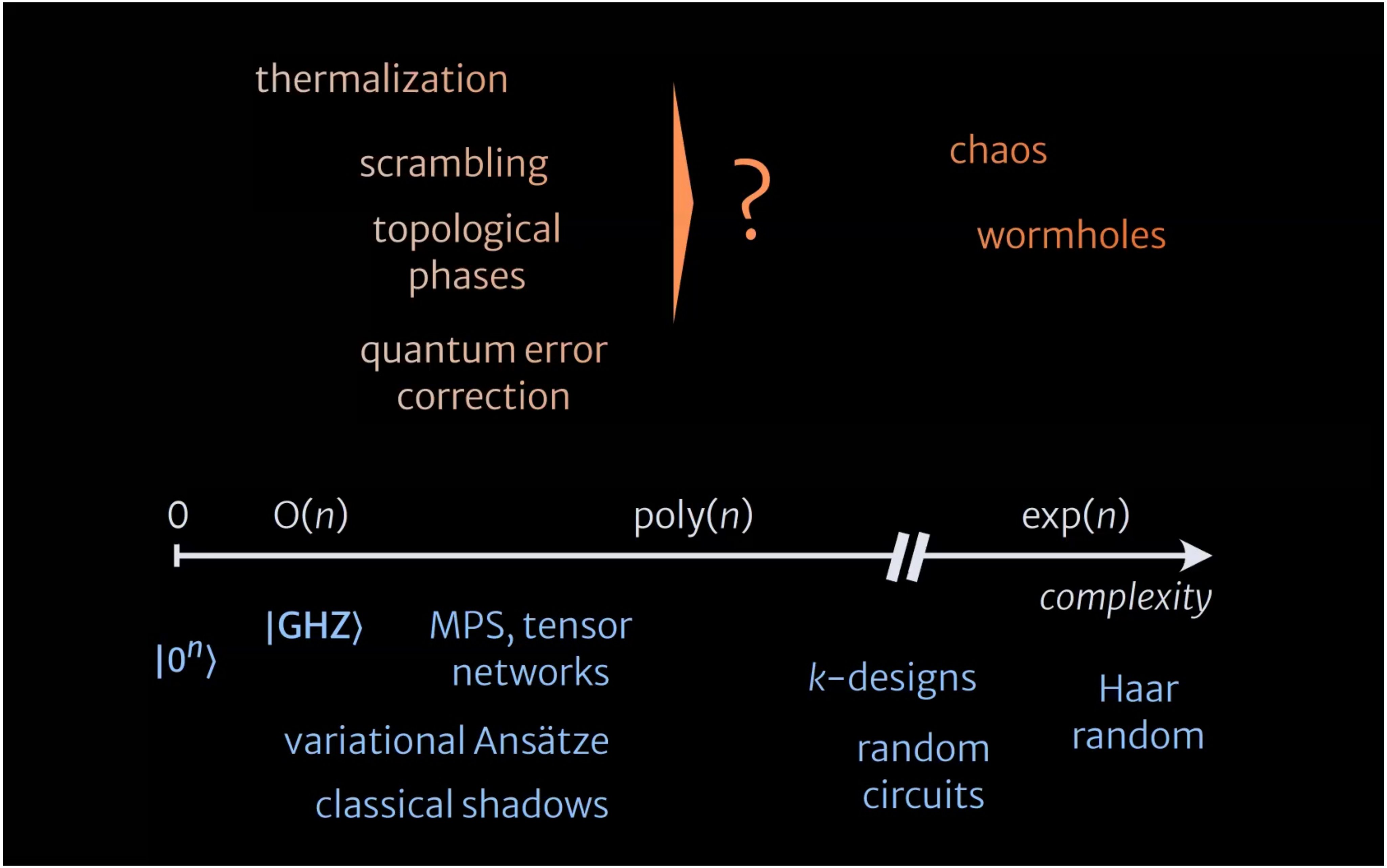
variational Ansätze

classical shadows

k-designs

random circuits

Haar random



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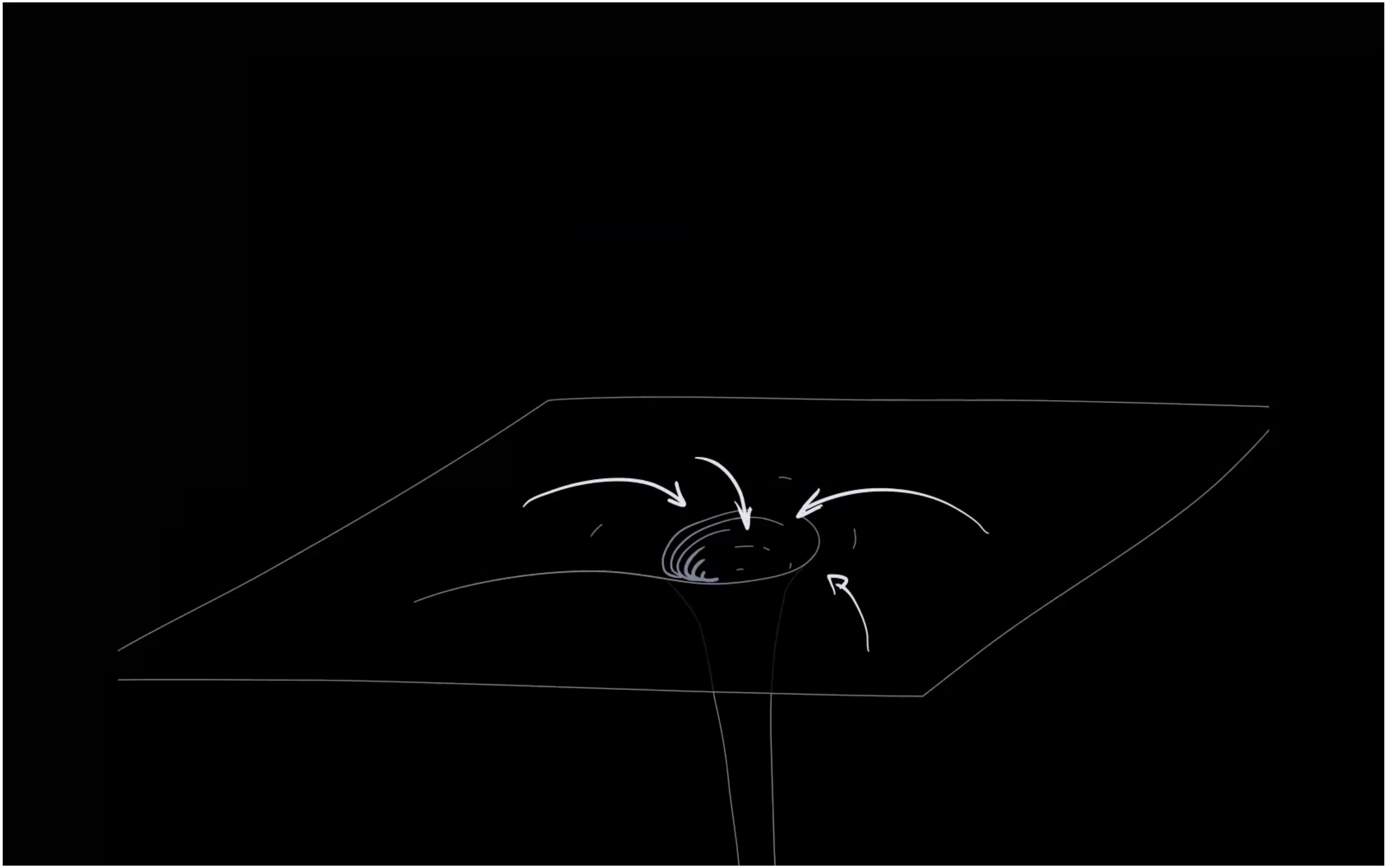
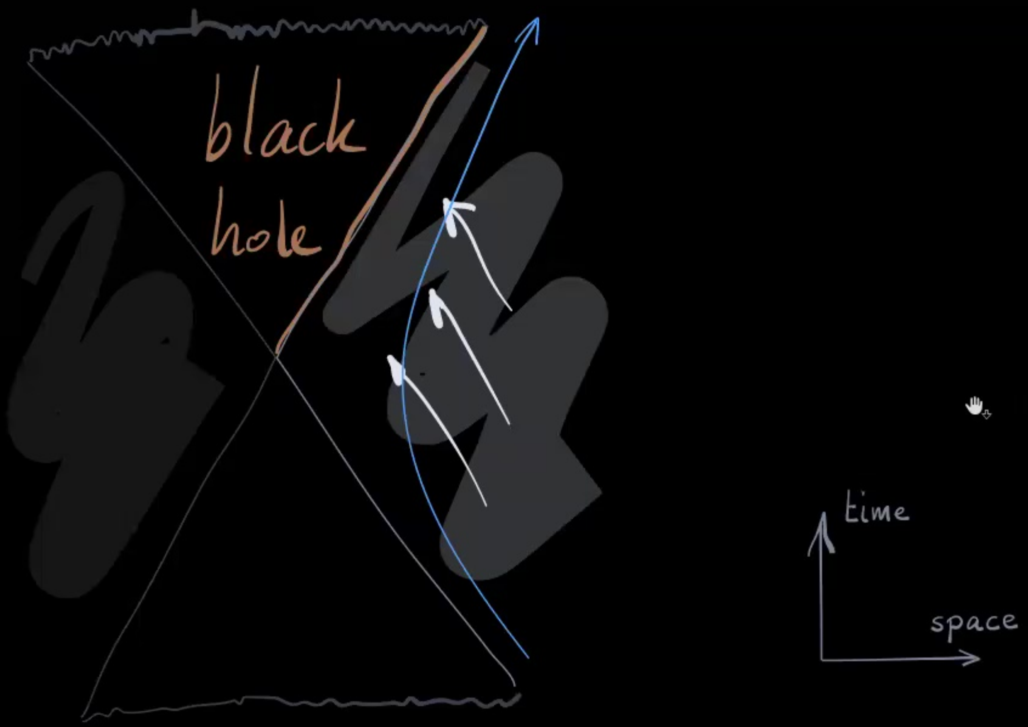
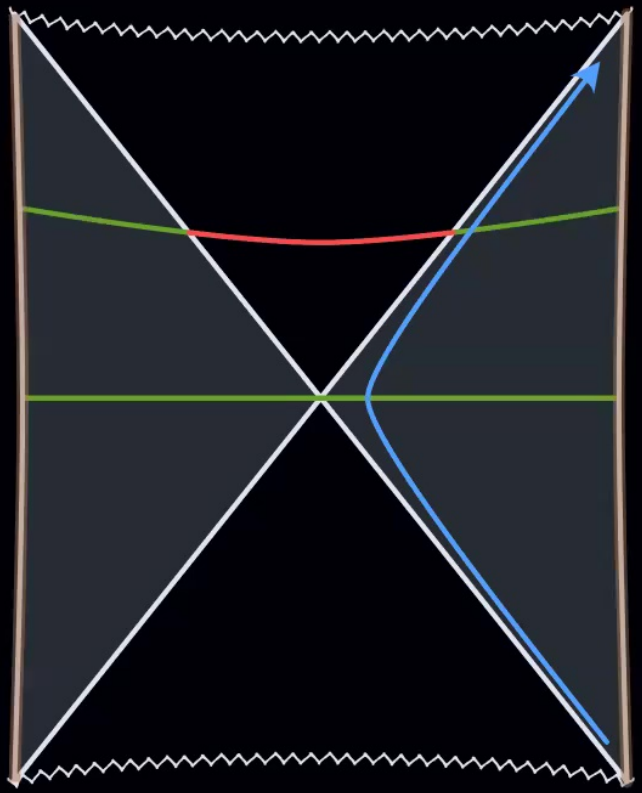



black hole
time
space

In AdS/CFT, physics in the bulk (AdS) has a dual description on the boundary (CFT).



Two entangled sides of a black hole connected by a "wormhole"

$$|\psi_0\rangle \propto \sum_k e^{-\beta E_k/2} |E_k\rangle_L |E_k\rangle_R$$

thermofield double

Susskind, PiTP (2018) 1810.11563; …

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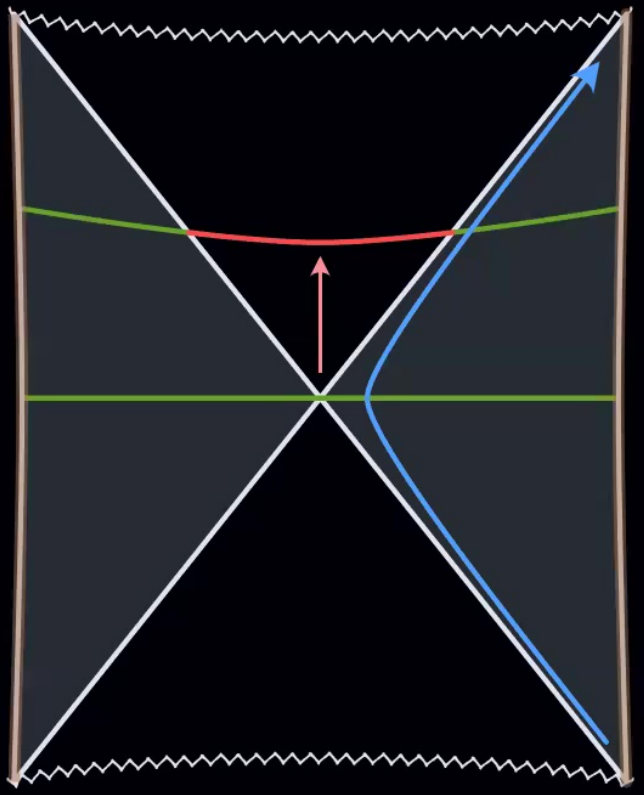


Two entangled sides of a black hole connected by a "wormhole"

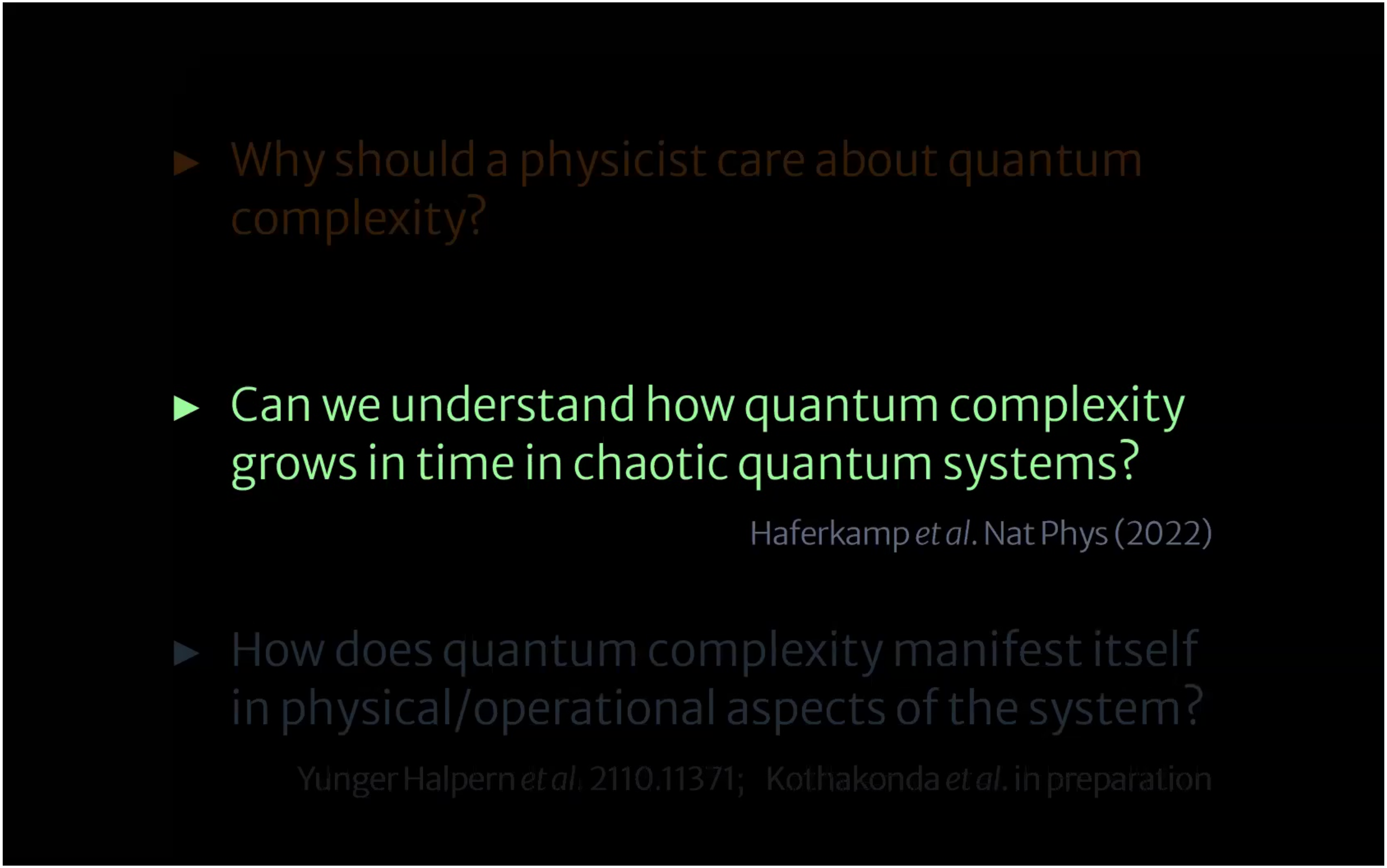
$|\psi_t\rangle = U_L(t)\, U_R(t)\, |\psi_0\rangle$

$|\psi_0\rangle \propto \sum_k e^{-\beta E_k/2} |E_k\rangle_L |E_k\rangle_R$

thermofield double

Complexity expected to be the CFT dual quantity to the wormhole length & to grow linearly in time

Susskind, PiTP (2018) 1810.11563; …



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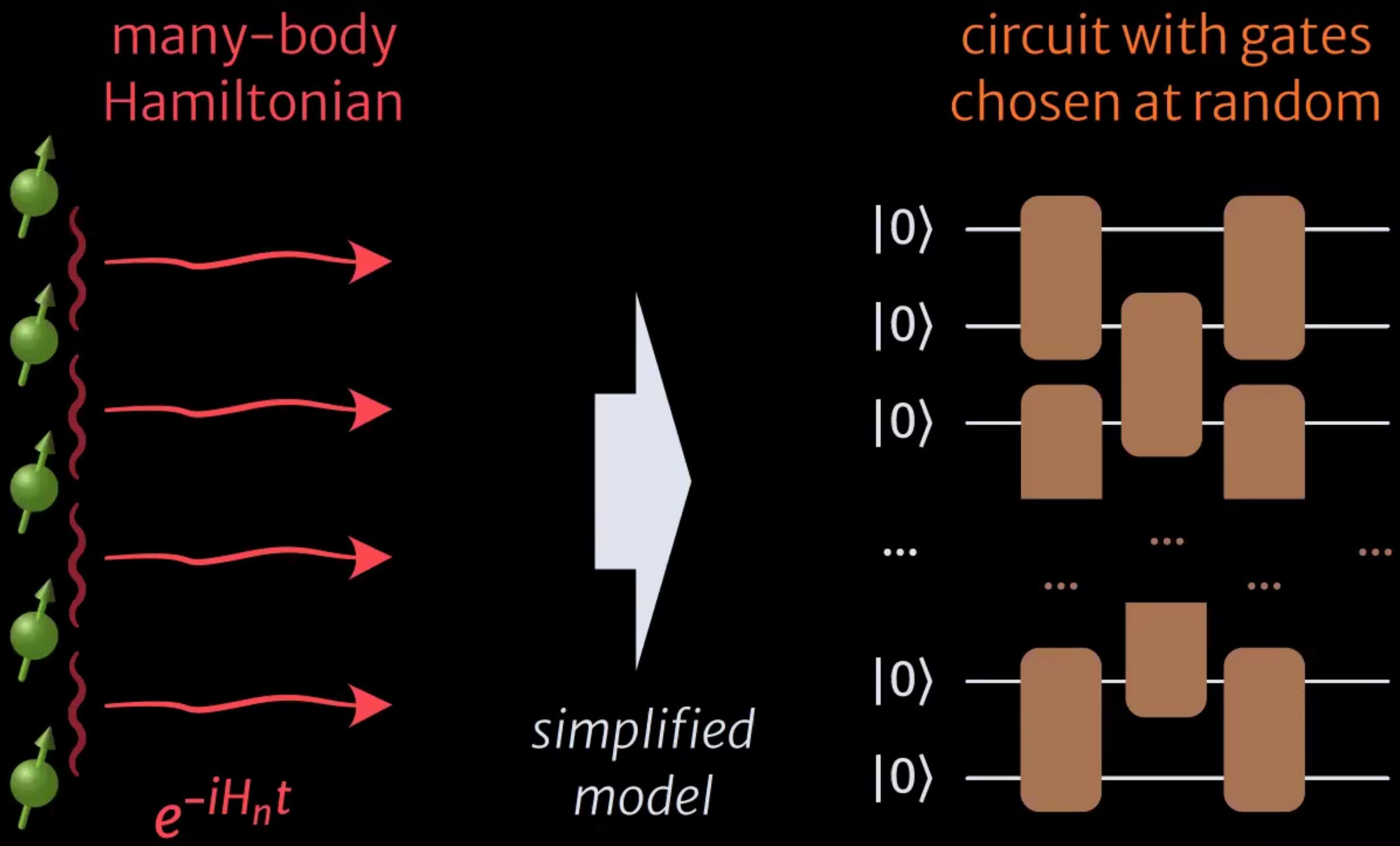


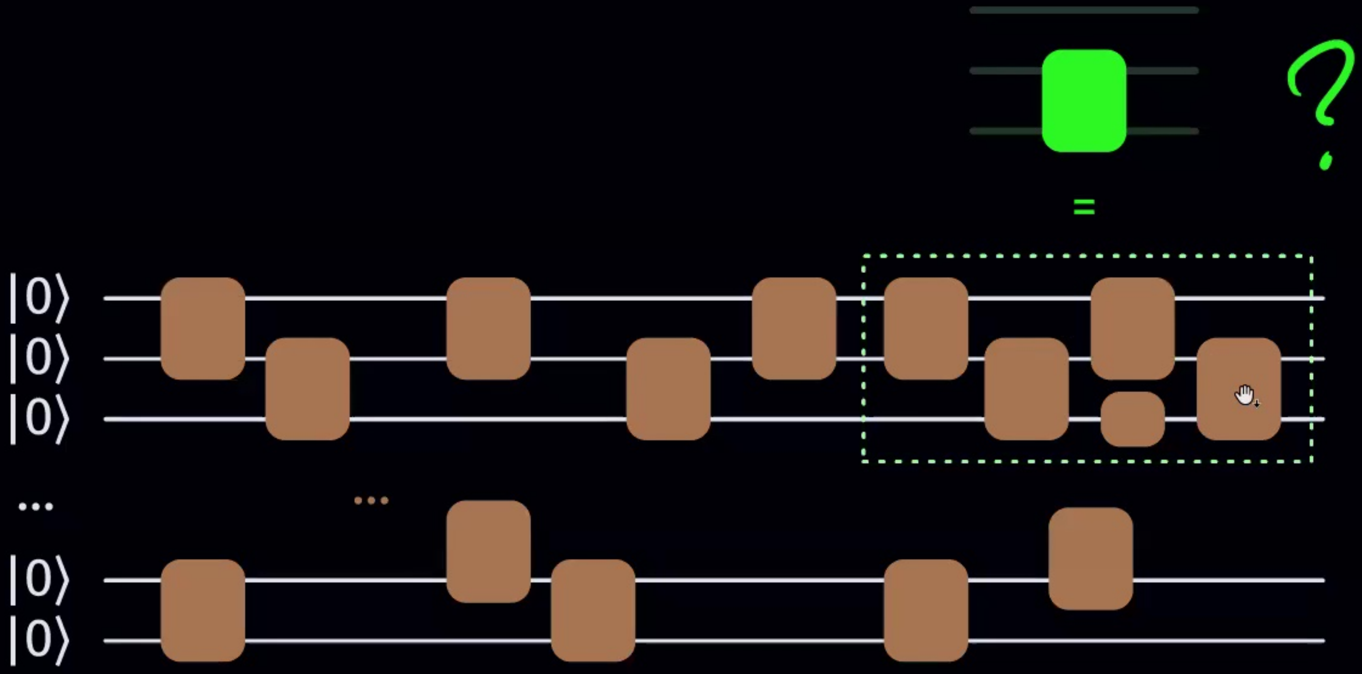
many-body Hamiltonian

$e^{-iH_n t}$

simplified model

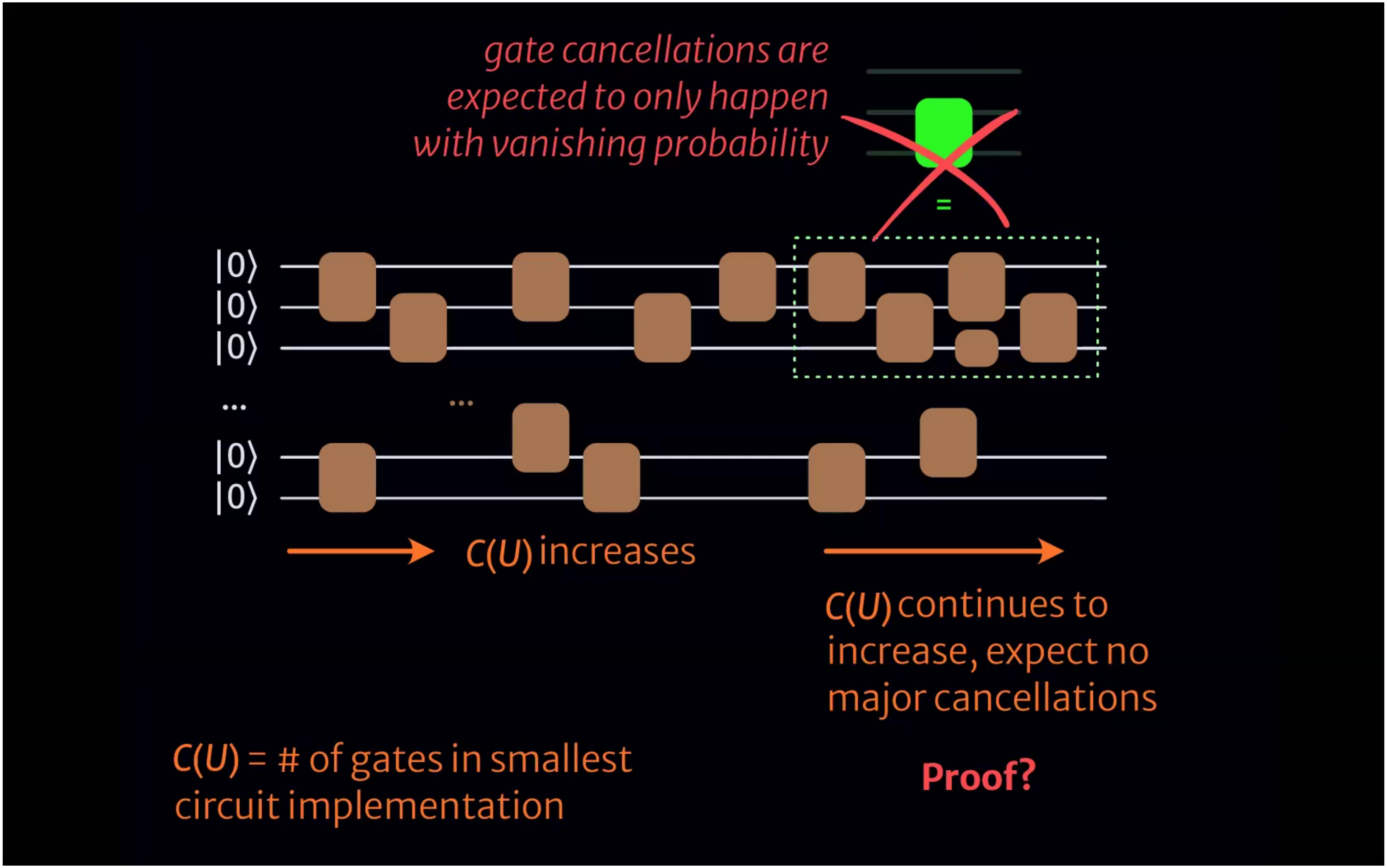
circuit with gates chosen at random

Hayden & Preskill JHEP 2007; Nahum *et al* PRX 2018;
Susskind PiTP (2018) 1810.11563; …



→ $C(U)$ increases

$C(U)$ = # of gates in smallest circuit implementation



gate cancellations are expected to only happen with vanishing probability

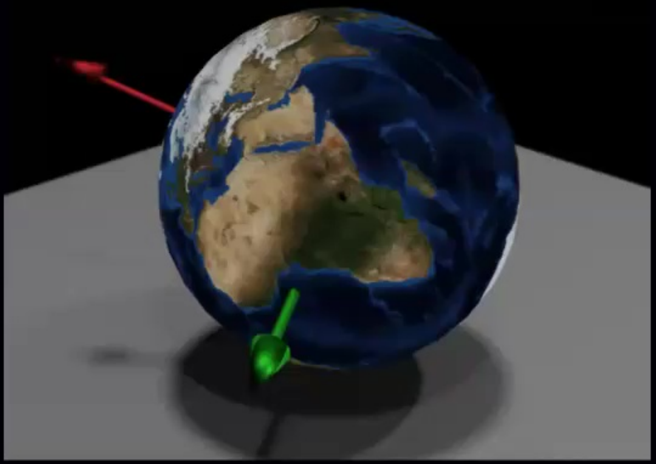
$C(U)$ increases

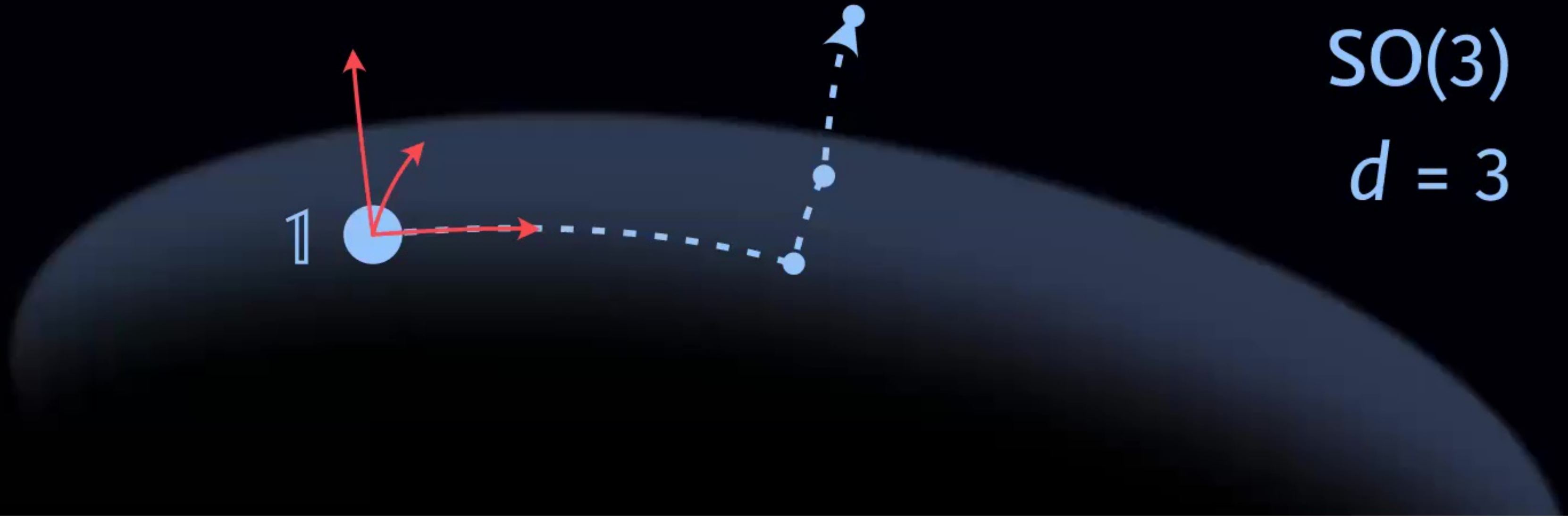
$C(U)$ continues to increase, expect no major cancellations

$C(U)$ = # of gates in smallest circuit implementation

Proof?

- the first few rotations explore "new directions in rotation space"
- Any rotation of the 2-sphere can be decomposed into at most 3 rotations around X, Z



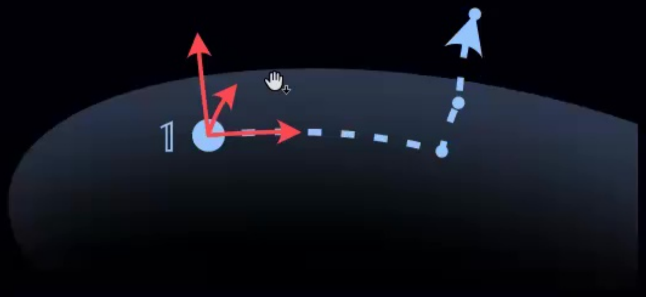


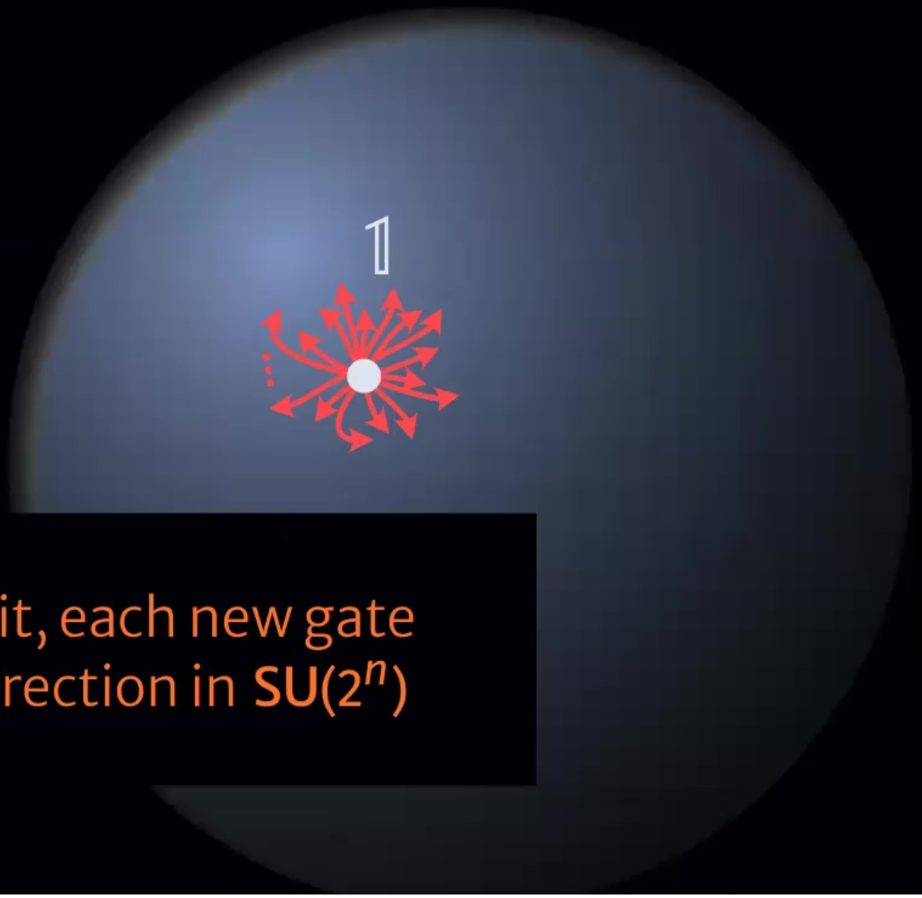
SO(3)

$d = 3$

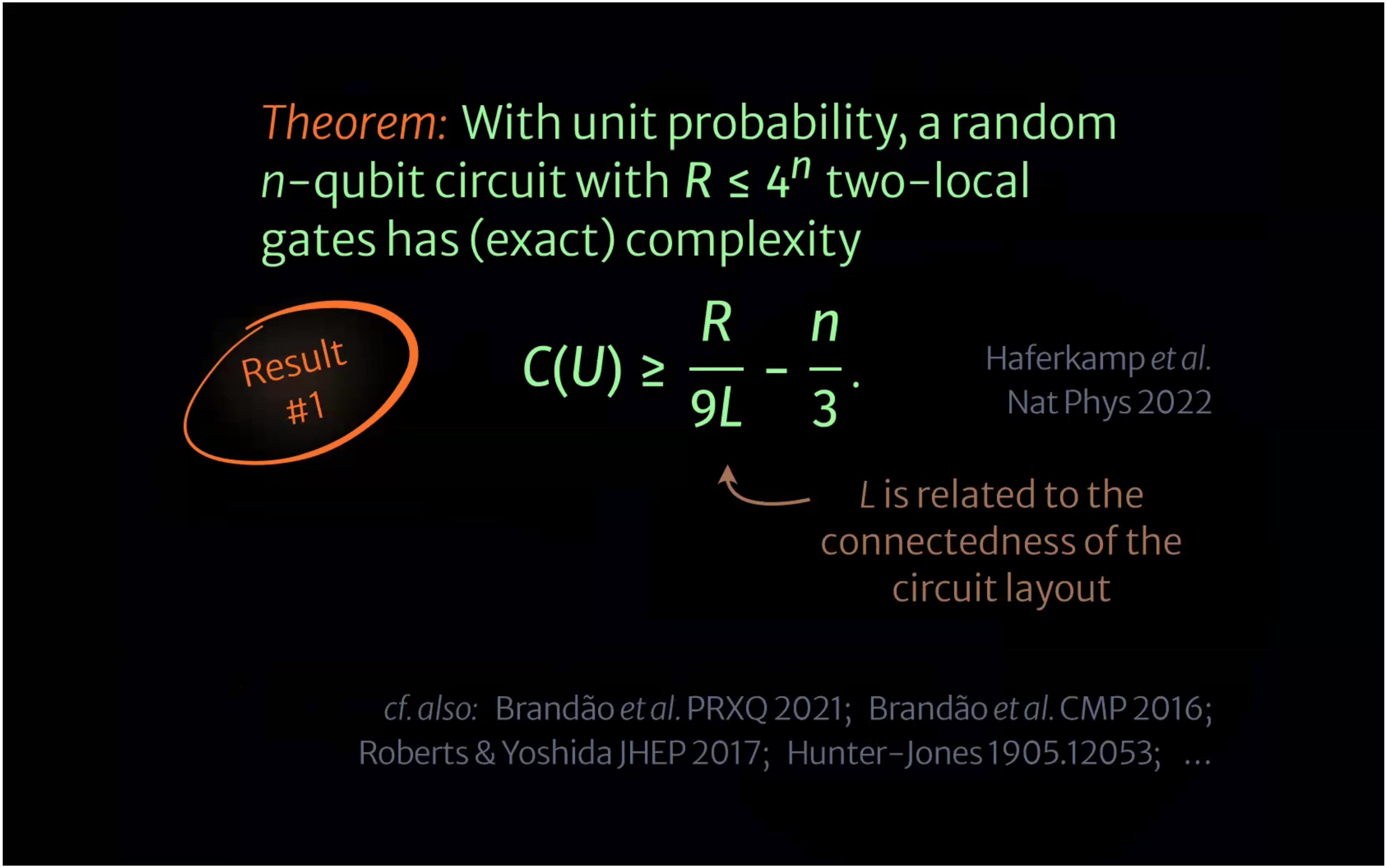
SO(3) → SU(2^n)

$d = 3$ $\qquad$ $d \sim 4^n$





- Idea: in a random circuit, each new gate likely explores a new direction in **SU(2^n)**



Theorem: With unit probability, a random n-qubit circuit with $R \leq 4^n$ two-local gates has (exact) complexity

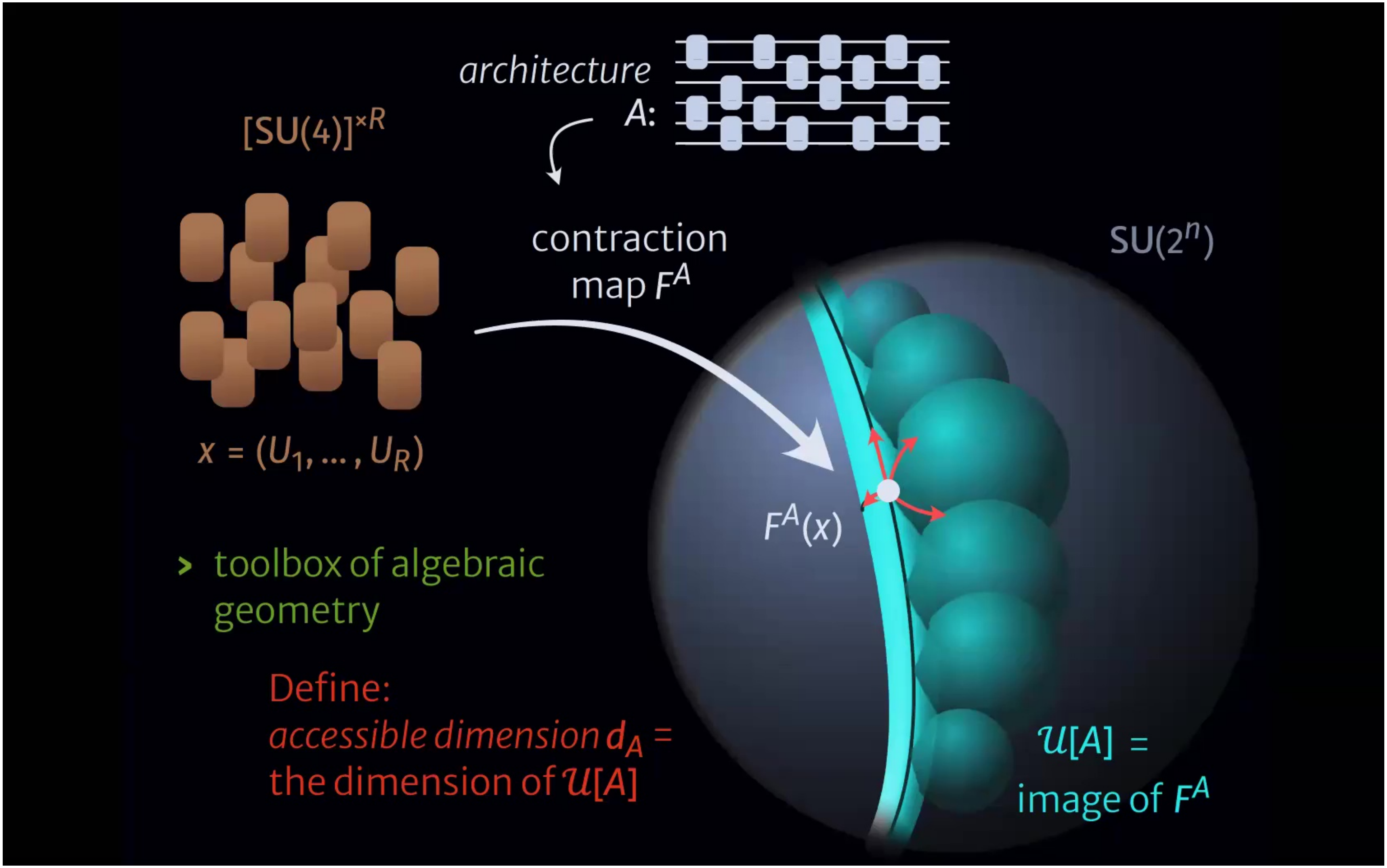
Result #1

$$C(U) \geq \frac{R}{9L} - \frac{n}{3}.$$

Haferkamp *et al.*
Nat Phys 2022

L is related to the connectedness of the circuit layout

cf. also: Brandão *et al.* PRXQ 2021; Brandão *et al.* CMP 2016; Roberts & Yoshida JHEP 2017; Hunter-Jones 1905.12053; …



architecture A:

$[SU(4)]^{\times R}$

contraction map F^A

$SU(2^n)$

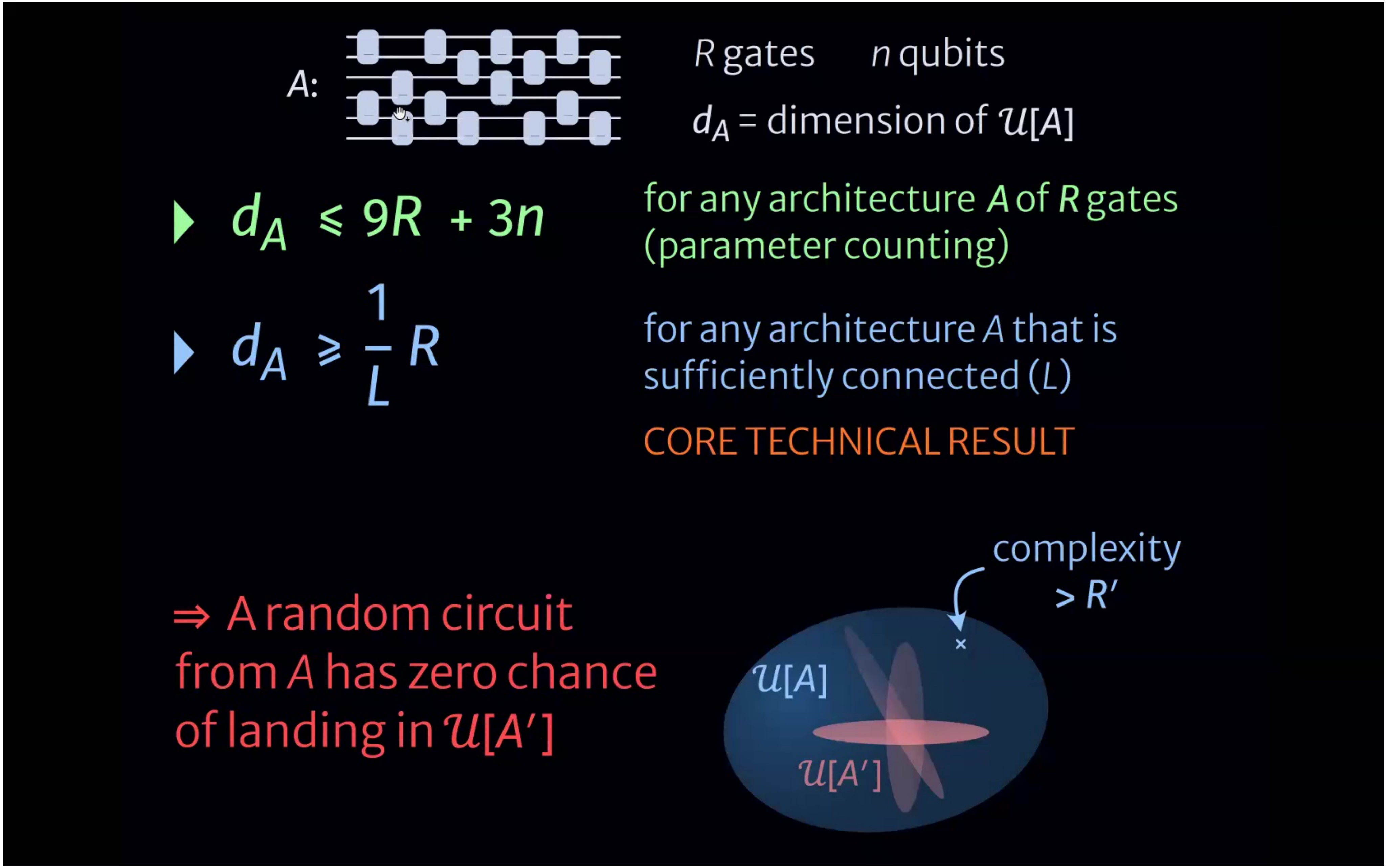
$x = (U_1, \ldots, U_R)$

$F^A(x)$

- toolbox of algebraic geometry

Define:
accessible dimension $\boldsymbol{d_A}$ =
the dimension of $\mathcal{U}[A]$

$\mathcal{U}[A]$ =
image of F^A



A:

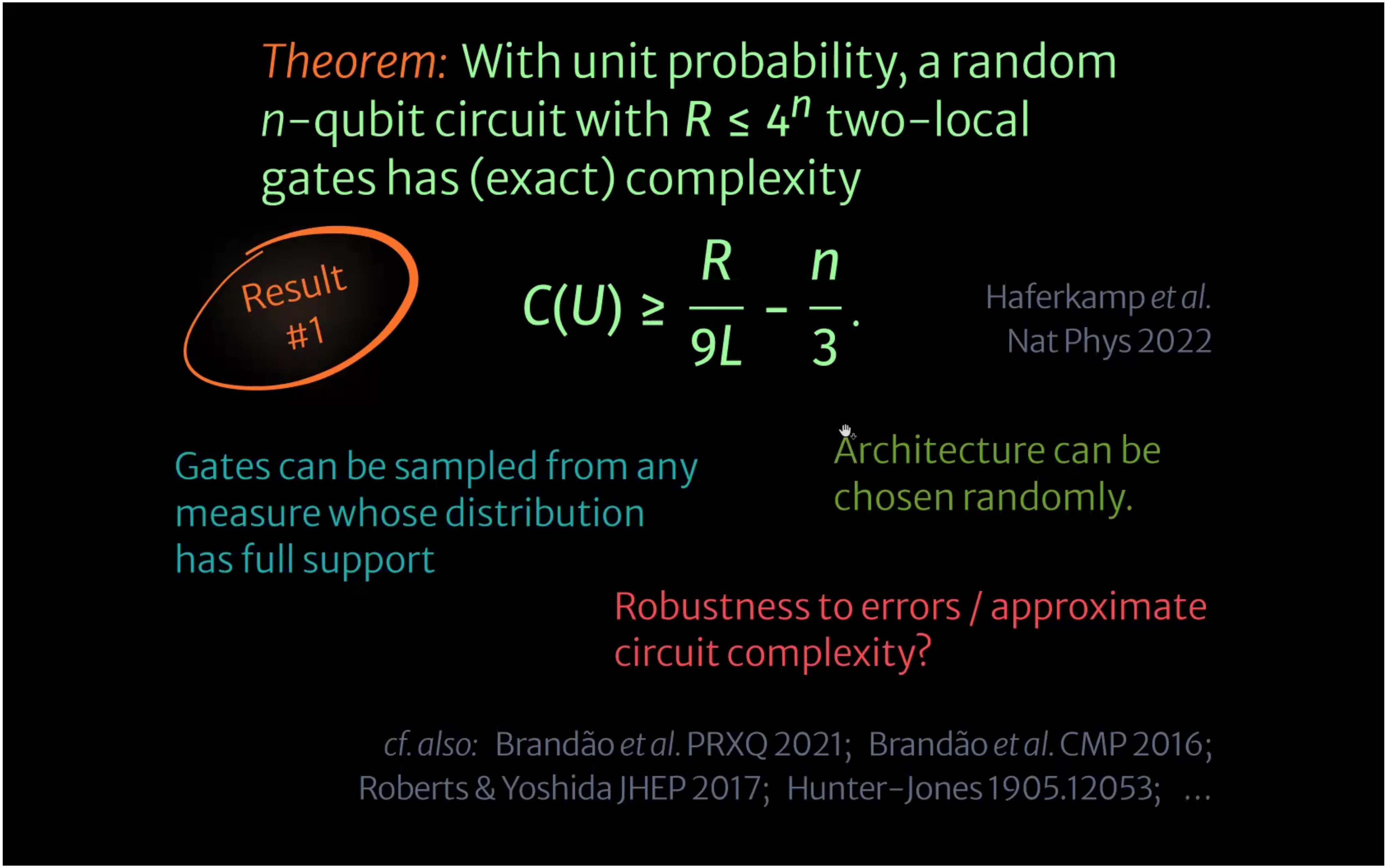
R gates *n* qubits

d_A = dimension of $\mathcal{U}[A]$

- $d_A \leqslant 9R + 3n$ for any architecture *A* of *R* gates (parameter counting)
- $d_A \geqslant \frac{1}{L} R$ for any architecture *A* that is sufficiently connected (L)

CORE TECHNICAL RESULT

⇒ A random circuit from *A* has zero chance of landing in $\mathcal{U}[A']$



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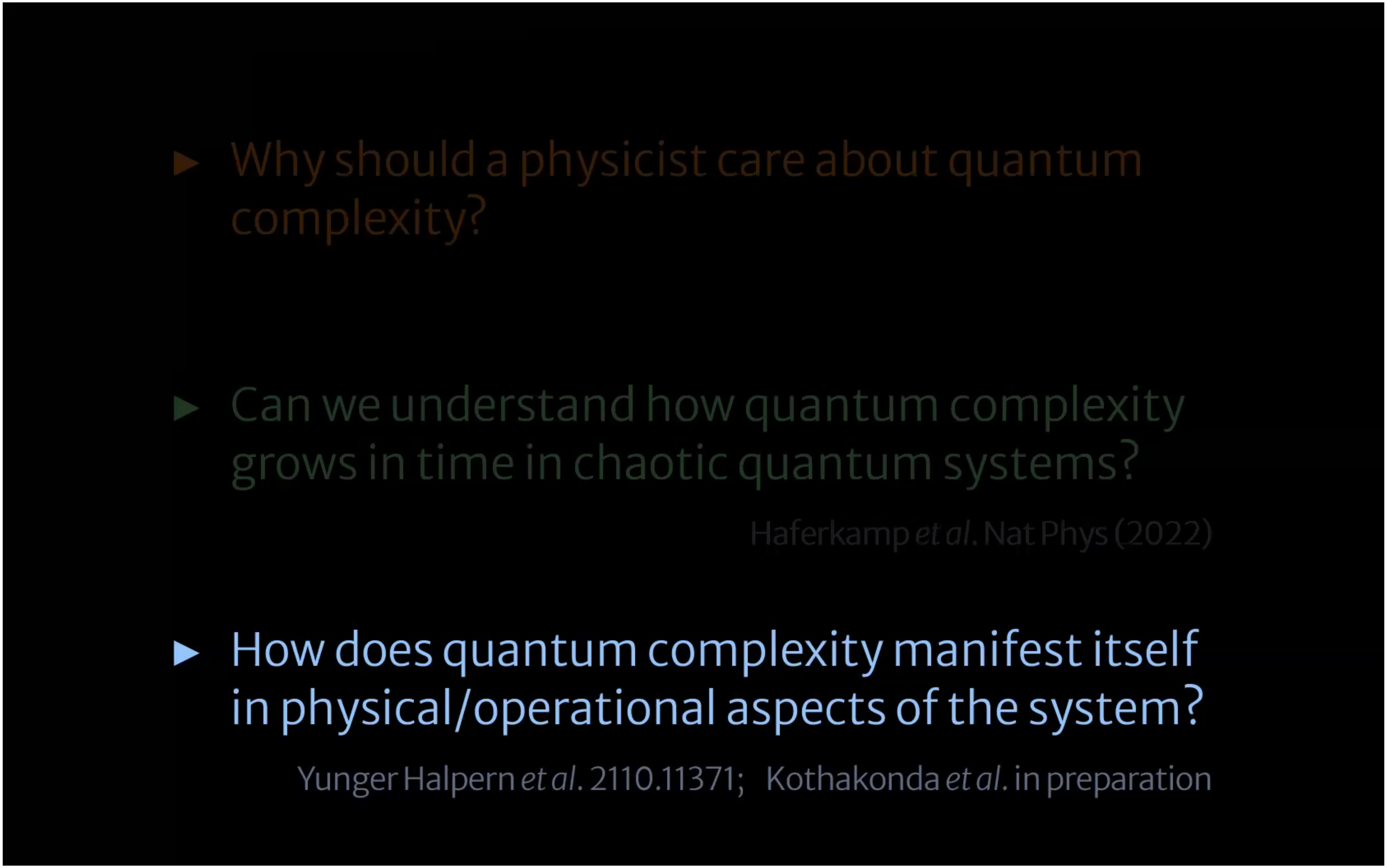
Haferkamp *et al.*
Nat Phys 2022

Gates can be sampled from any measure whose distribution has full support

Architecture can be chosen randomly.

Robustness to errors / approximate circuit complexity?

cf. also: Brandão *et al.* PRXQ 2021; Brandão *et al.* CMP 2016; Roberts & Yoshida JHEP 2017; Hunter-Jones 1905.12053; …



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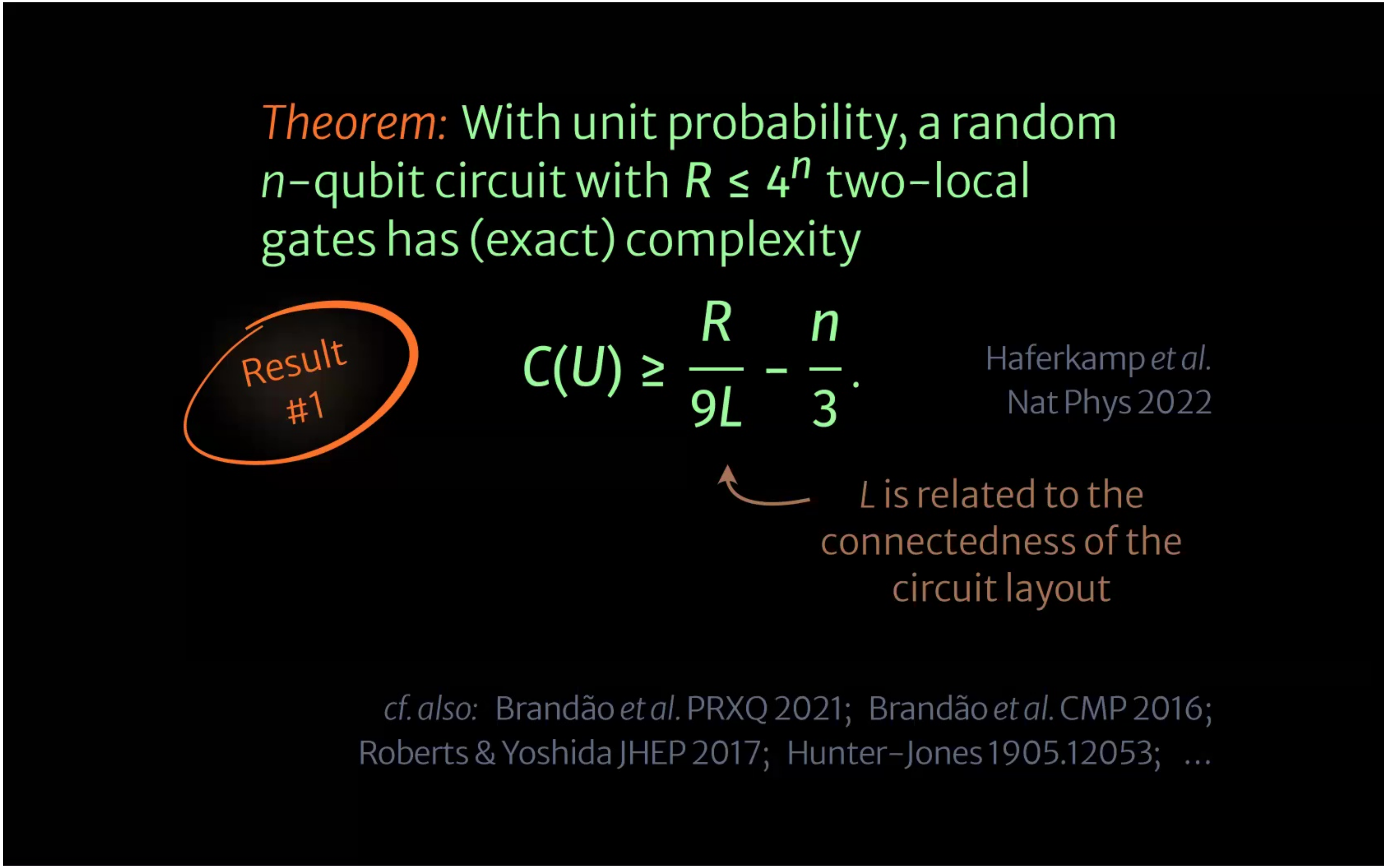

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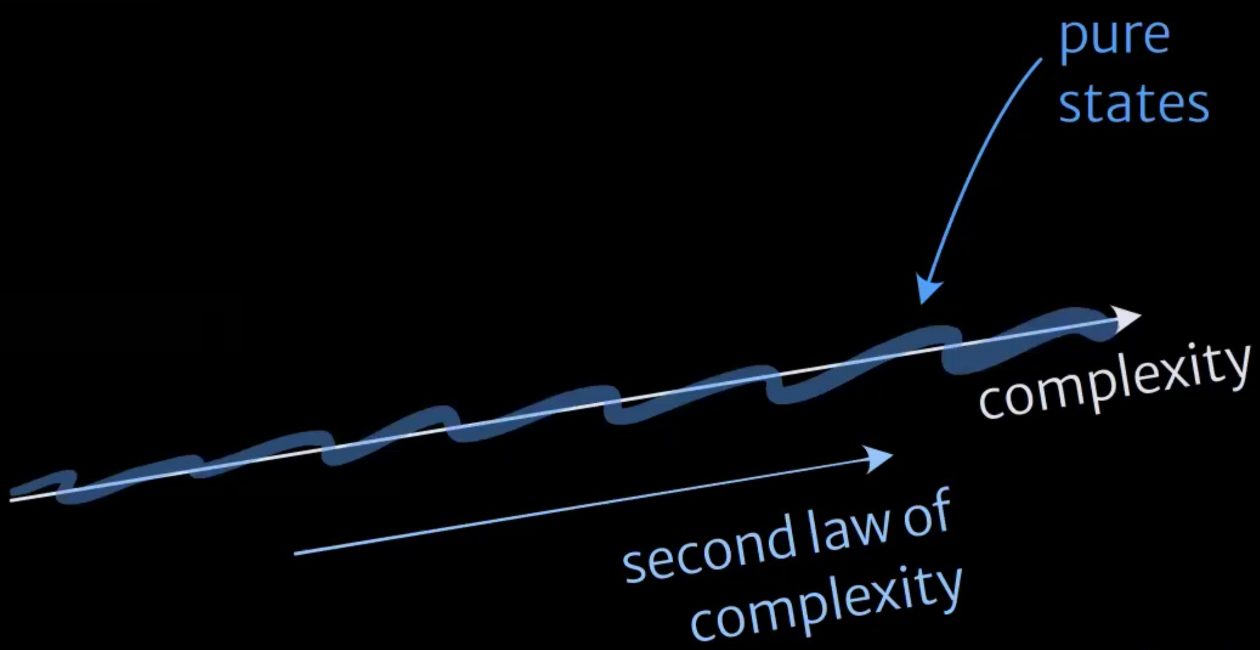
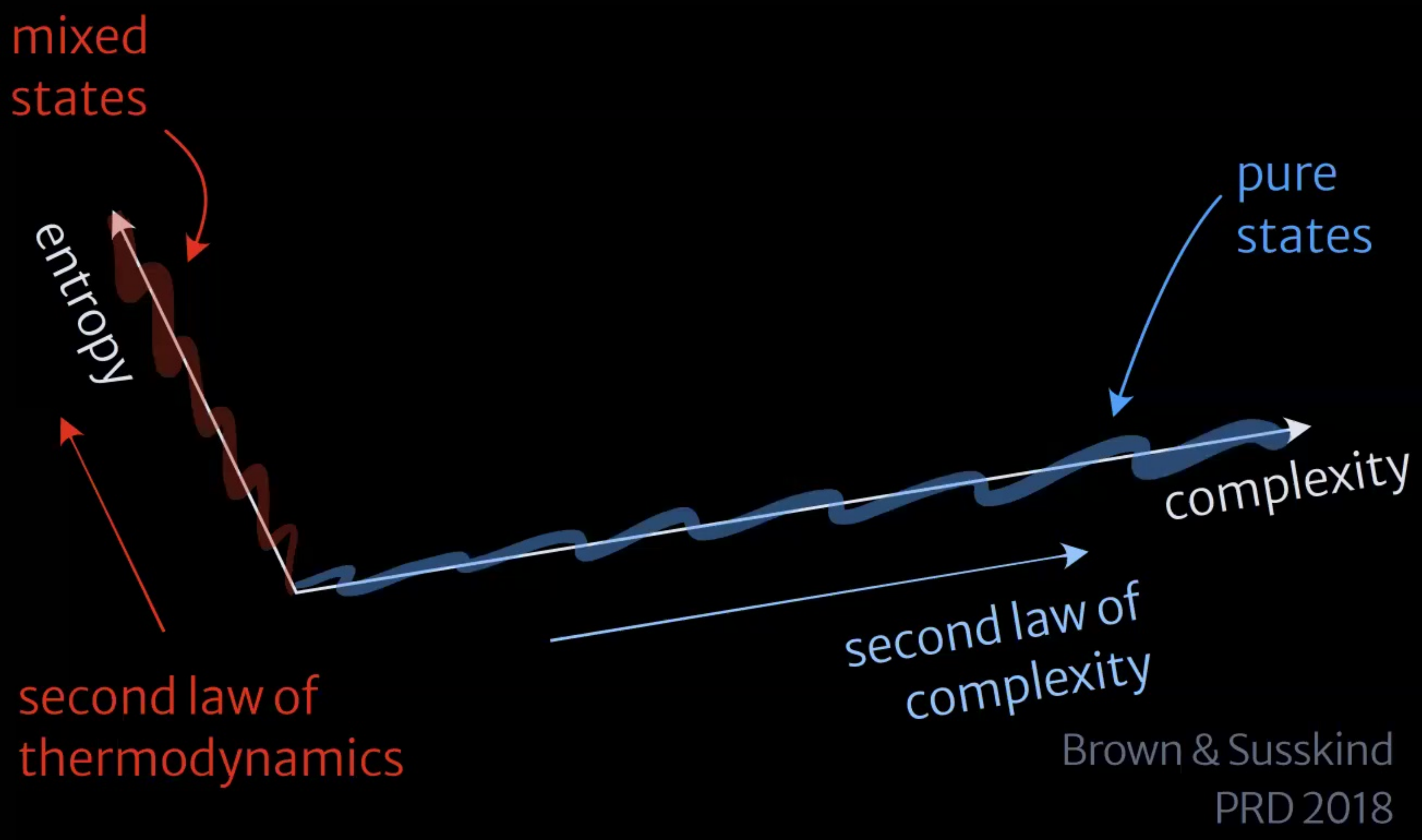


Brown & Susskind
PRD 2018



mixed
states

entropy

second law of
thermodynamics

pure
states

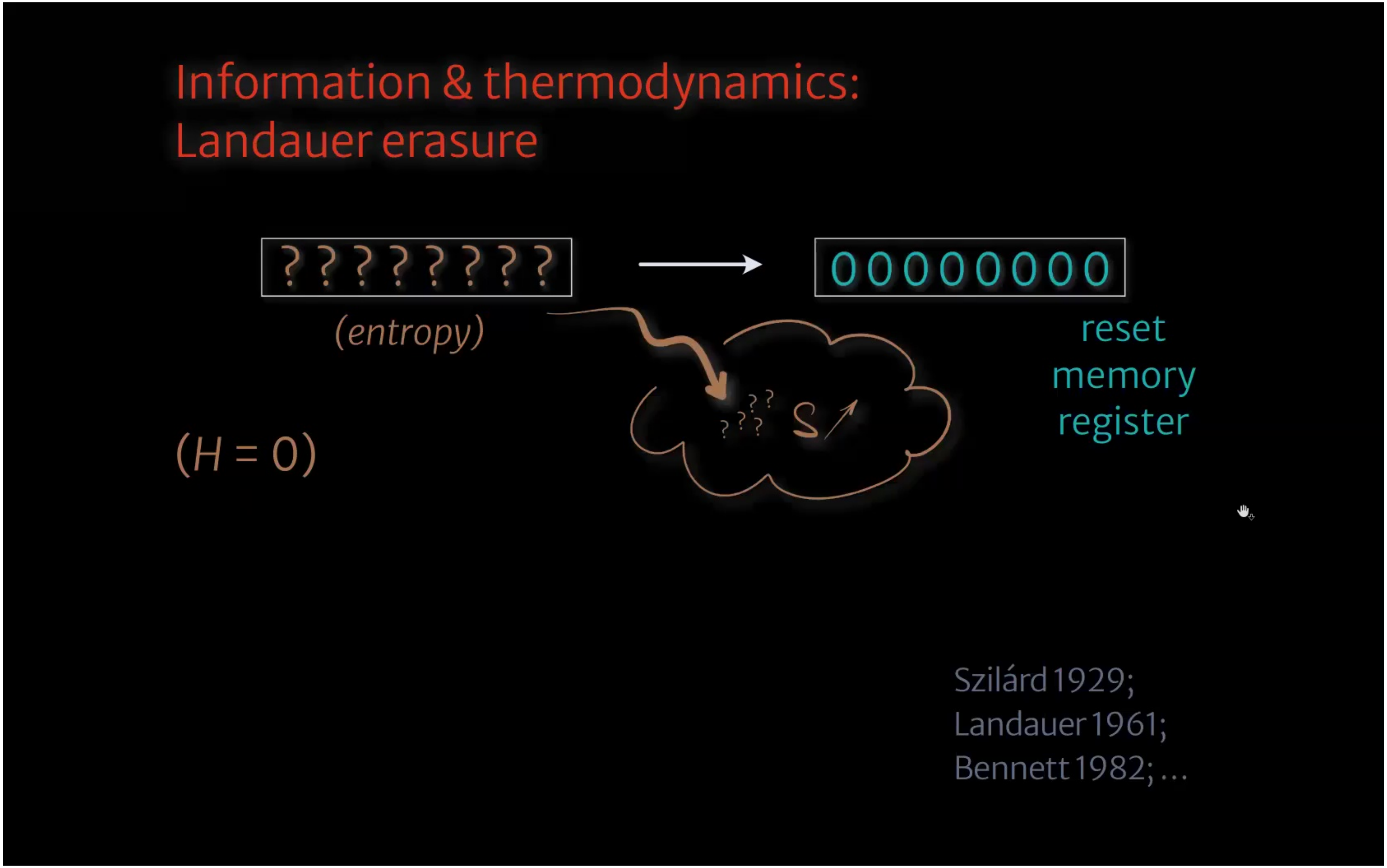
complexity

second law of
complexity

Brown & Susskind
PRD 2018

- Thermodynamics describes what processes can be performed on a system by a macroscopic observer.

- Thermodynamics describes what processes can be performed on a system by a macroscopic observer.
- How does quantum complexity manifest itself in the resources required to carry out a process on a system?



Information & thermodynamics: Landauer erasure

? ? ? ? ? ? ? ? → 0 0 0 0 0 0 0 0

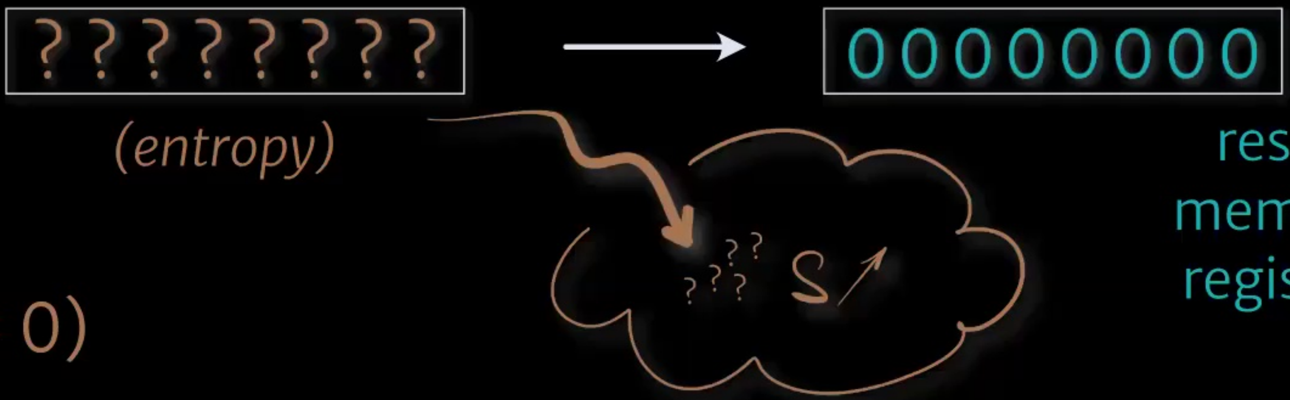
(entropy)

reset
memory
register

$(H = 0)$

Szilárd 1929;
Landauer 1961;
Bennett 1982; …

Information & thermodynamics: Landauer erasure



(entropy)

reset memory register

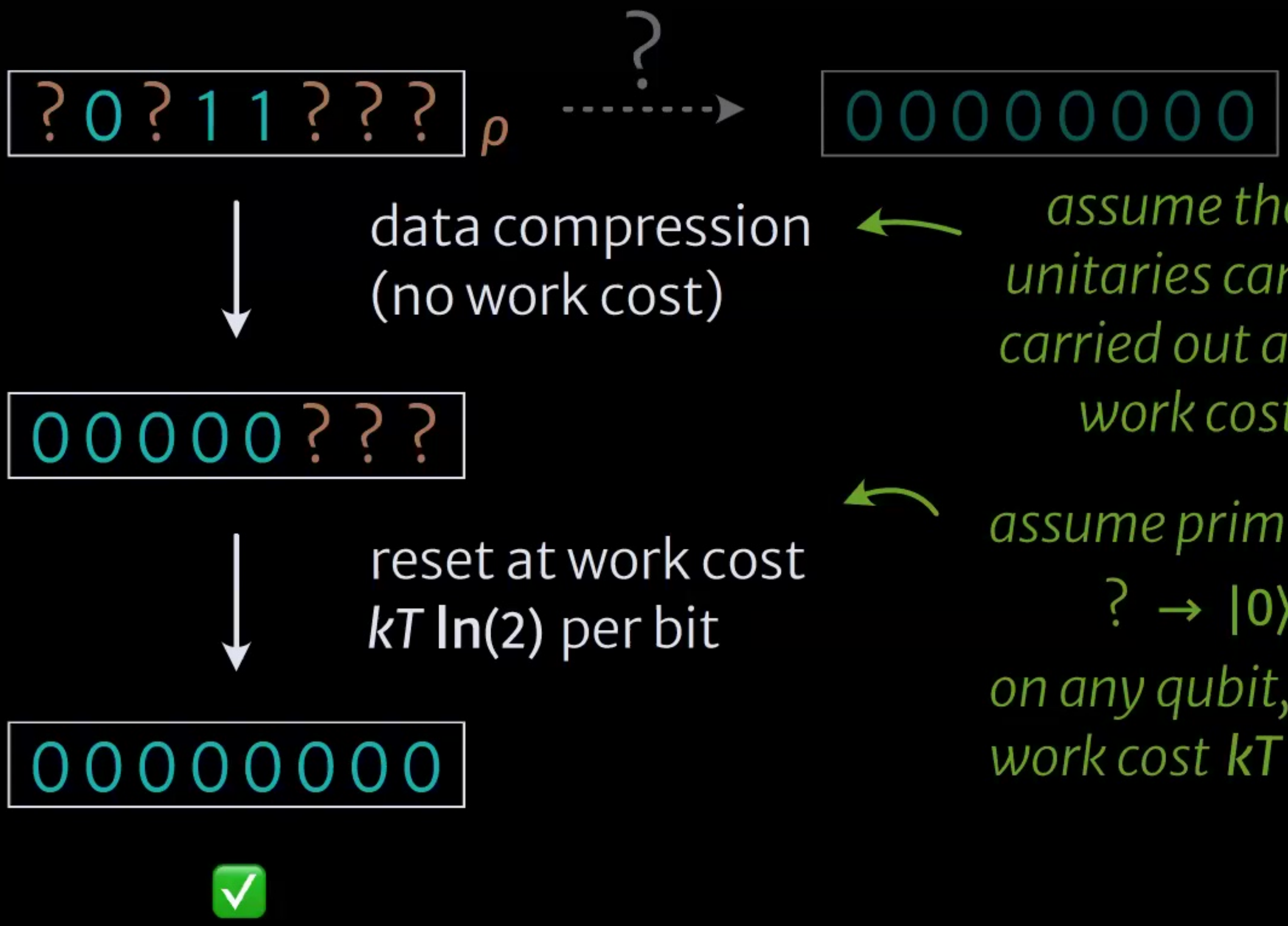
$(H = 0)$

environment at temperature T

Information erasure requires thermodynamic work:

Work ≥ $kT \ln(2)$ per bit discarded in the environment

Szilárd 1929;
Landauer 1961;
Bennett 1982; …



? 0 ? 1 1 ? ? ? ρ $\xrightarrow{?}$ 0 0 0 0 0 0 0 0

data compression
(no work cost)

assume that unitaries can be carried out at no work cost

0 0 0 0 0 ? ? ?

reset at work cost
$kT \ln(2)$ per bit

assume primitive:

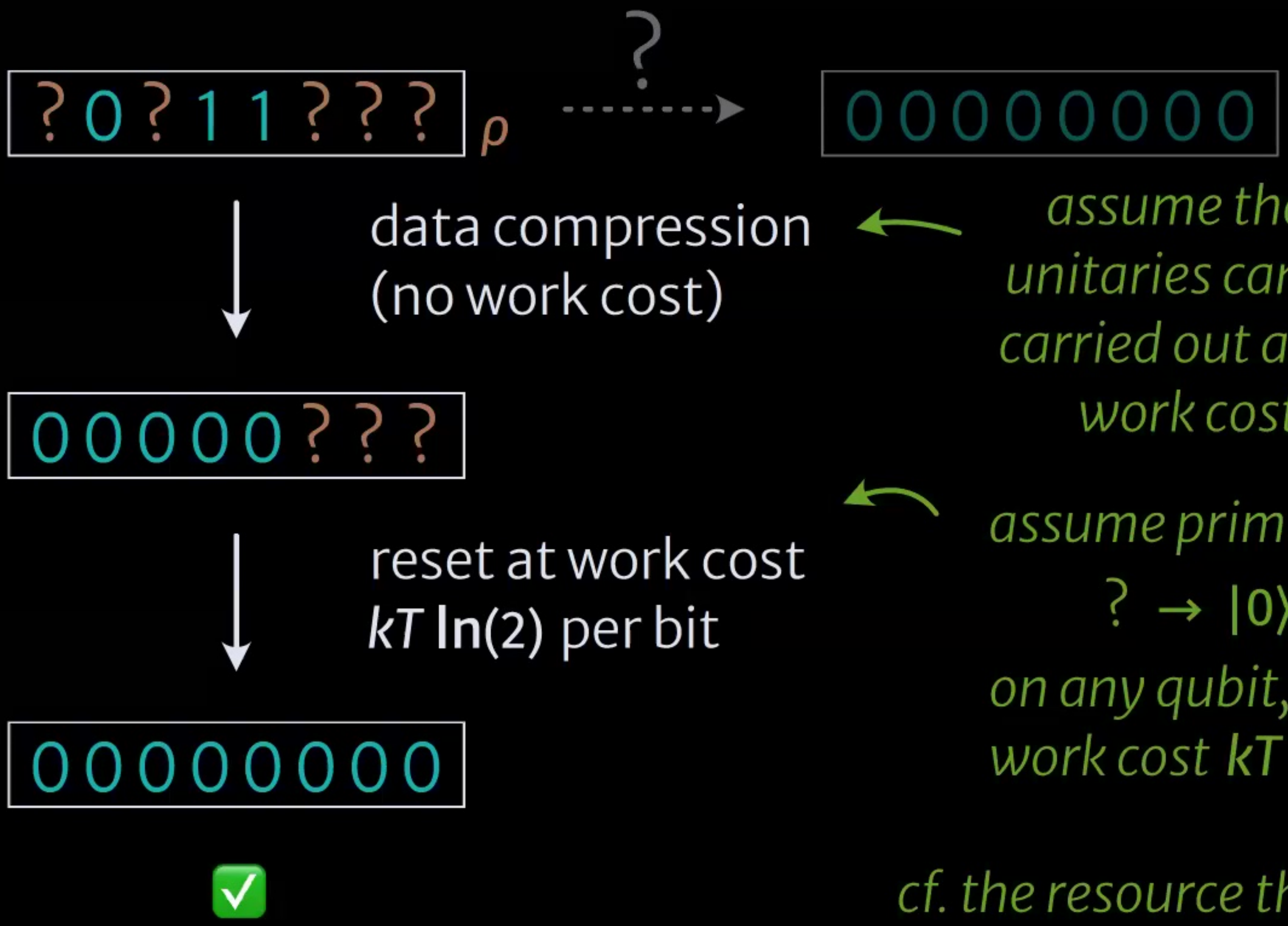
$? \rightarrow |0\rangle$

on any qubit, at work cost $kT \ln(2)$

0 0 0 0 0 0 0 0

✅

$$\text{Total work cost} = kT\ln(2)\,\underbrace{H^{1-\epsilon}(\rho)}_{\#\text{ of qubits needed to compress }\rho}$$



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reset at work cost $kT\ln(2)$ per bit

assume primitive:

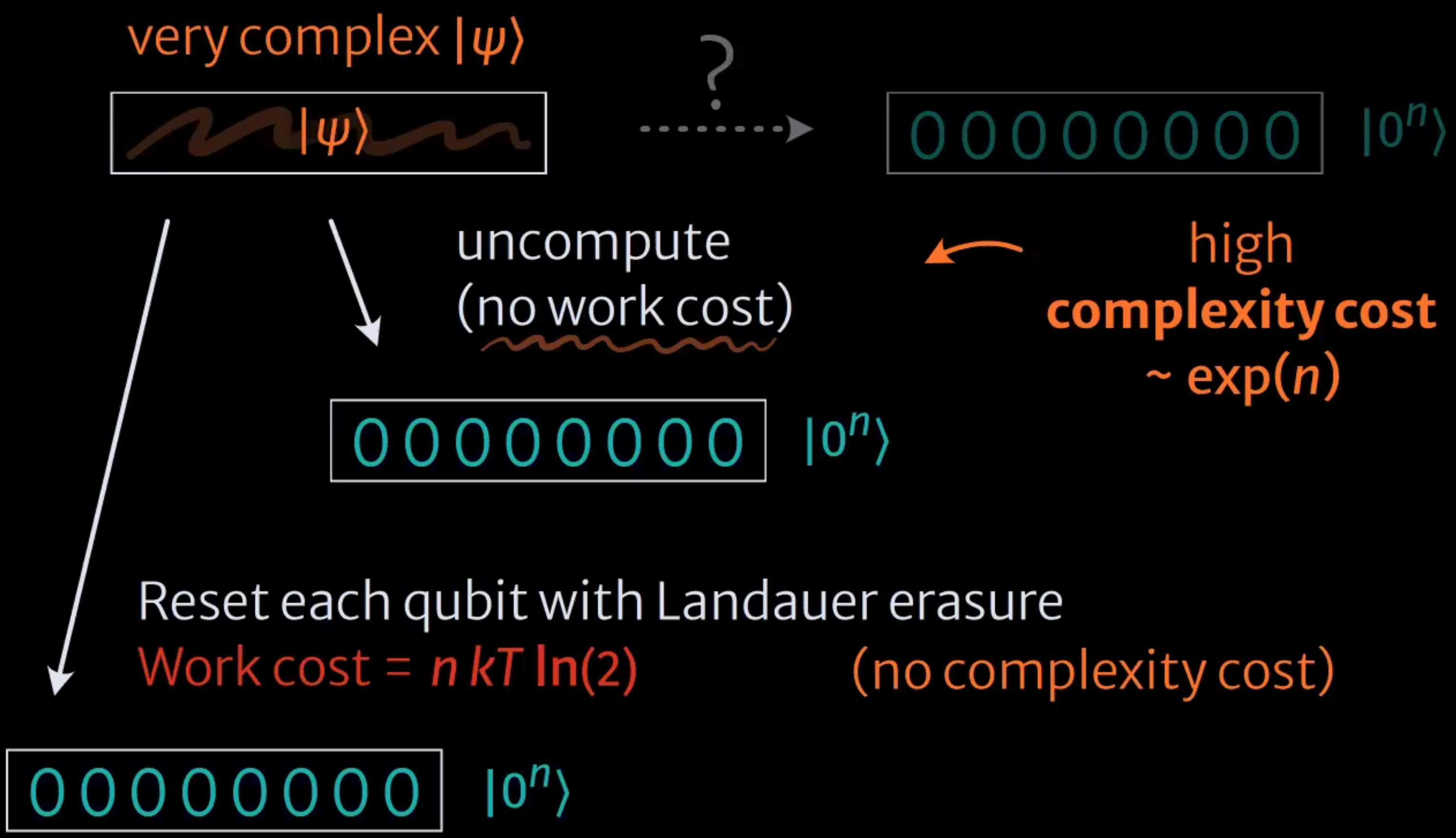
$? \rightarrow |0\rangle$

on any qubit, at work cost $kT\ln(2)$

Total work cost = $kT\ln(2)\ \underbrace{H^{1-\epsilon}(\rho)}_{\text{\# of qubits needed to compress } \rho}$

cf. the resource theory of thermodynamics

Brandão+ PRL 2013;
PhF+ Nat Comm 2015;
Chitambar+ RMP 2019; …



very complex $|\psi\rangle$

$|\psi\rangle$

?

0 0 0 0 0 0 0 0 0 $|0^n\rangle$

uncompute
(no work cost)

high
complexity cost
~ $\exp(n)$

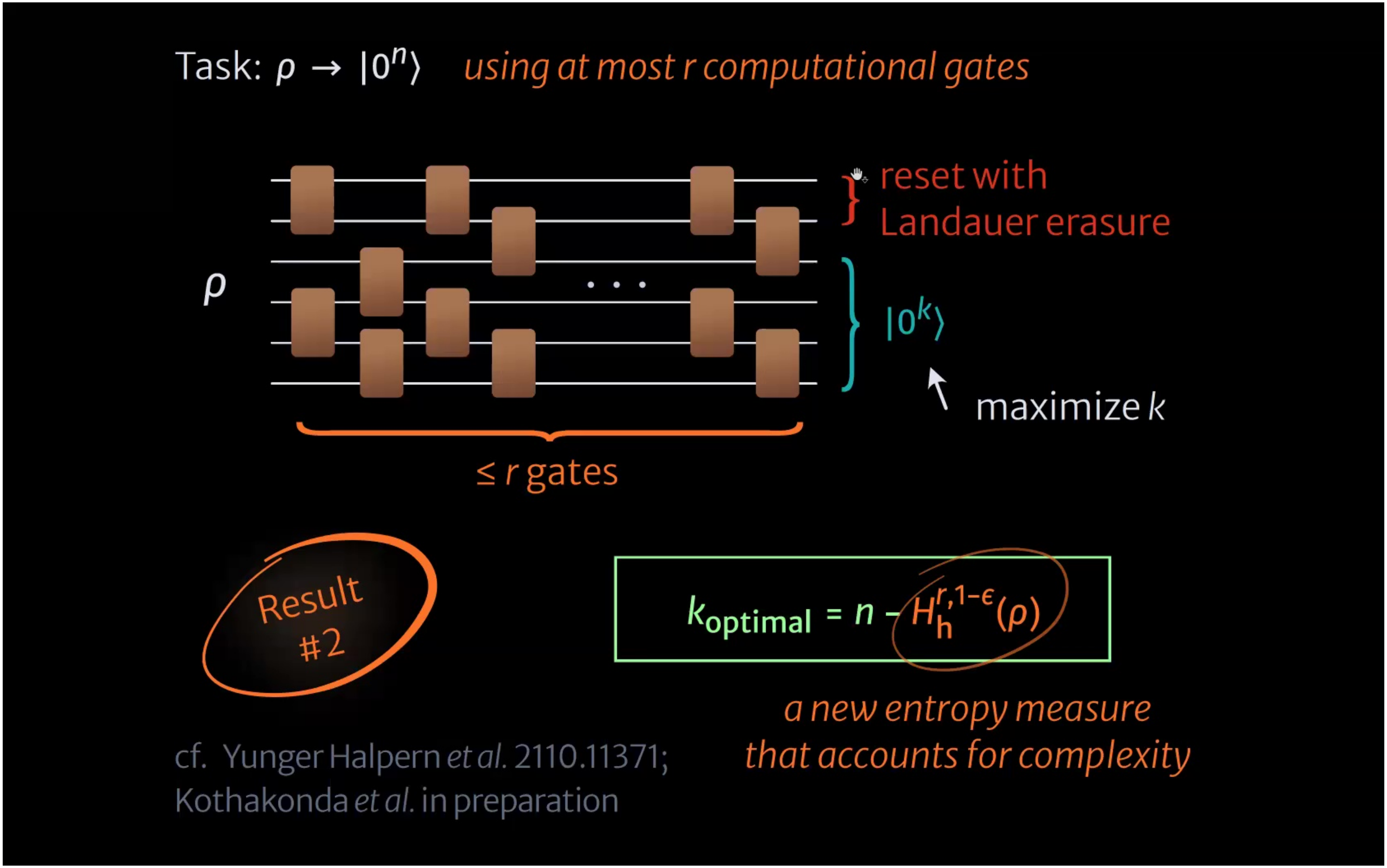
0 0 0 0 0 0 0 0 0 $|0^n\rangle$

Reset each qubit with Landauer erasure

Work cost = $n\,kT\ln(2)$ (no complexity cost)

0 0 0 0 0 0 0 0 0 $|0^n\rangle$

- ▶ General trade-off between work cost and complexity cost to reset a state to zero



Task: $\rho \rightarrow |0^n\rangle$ *using at most r computational gates*

ρ

reset with Landauer erasure

$|0^k\rangle$

maximize k

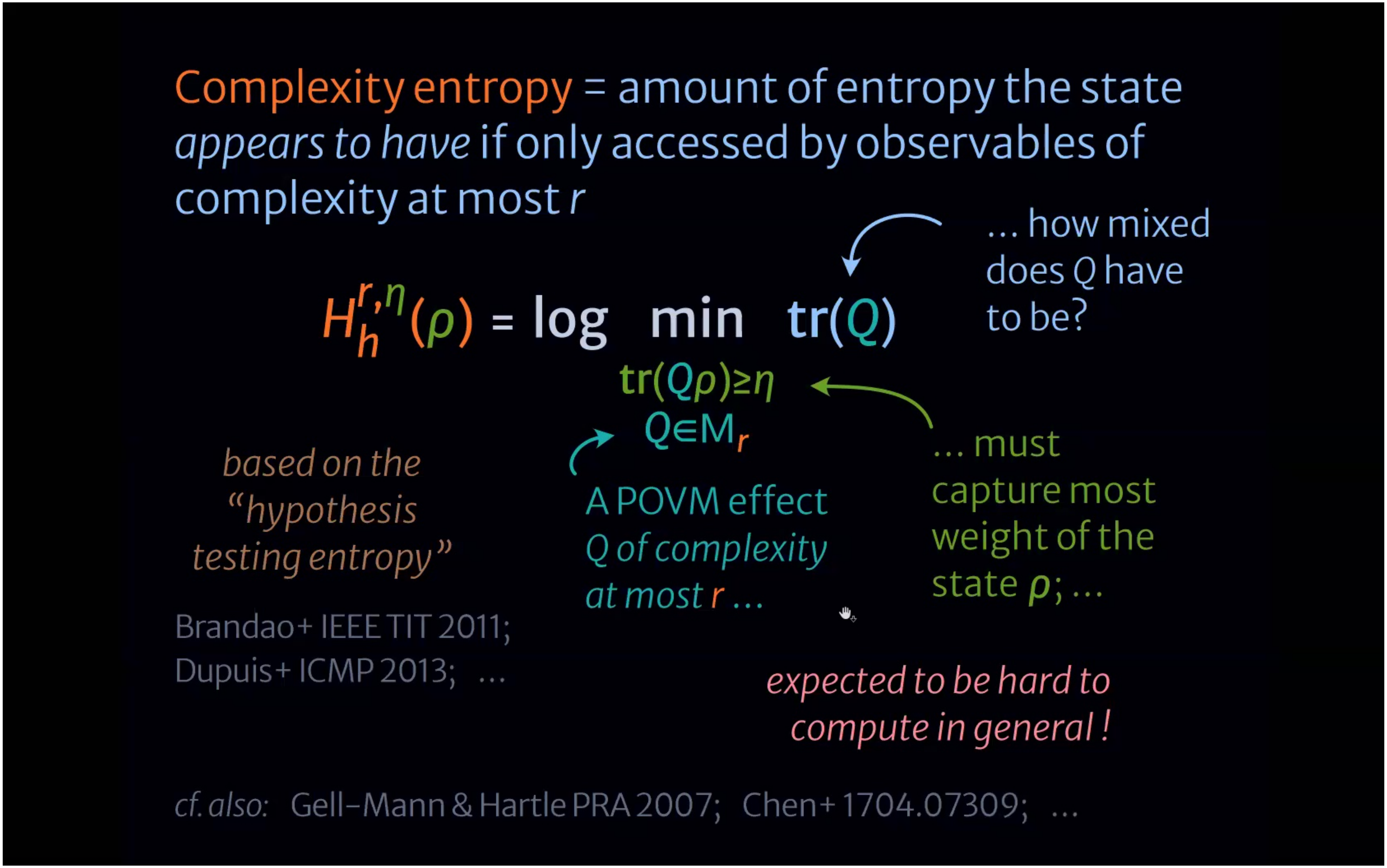
$\leq r$ gates

Result #2

$$\boxed{k_{\text{optimal}} = n - H_{\text{h}}^{r,1-\epsilon}(\rho)}$$

a new entropy measure that accounts for complexity

cf. Yunger Halpern *et al.* 2110.11371;
Kothakonda *et al.* in preparation



Complexity entropy = amount of entropy the state *appears to have* if only accessed by observables of complexity at most r

$$H_h^{r,\eta}(\rho) = \log \min_{\substack{\mathrm{tr}(Q\rho)\geq\eta \\ Q\in \mathrm{M}_r}} \mathrm{tr}(Q)$$

… how mixed does Q have to be?

… must capture most weight of the state ρ; …

A POVM effect *Q of complexity at most r …*

based on the "hypothesis testing entropy"

Brandao+ IEEE TIT 2011;
Dupuis+ ICMP 2013; …

expected to be hard to compute in general !

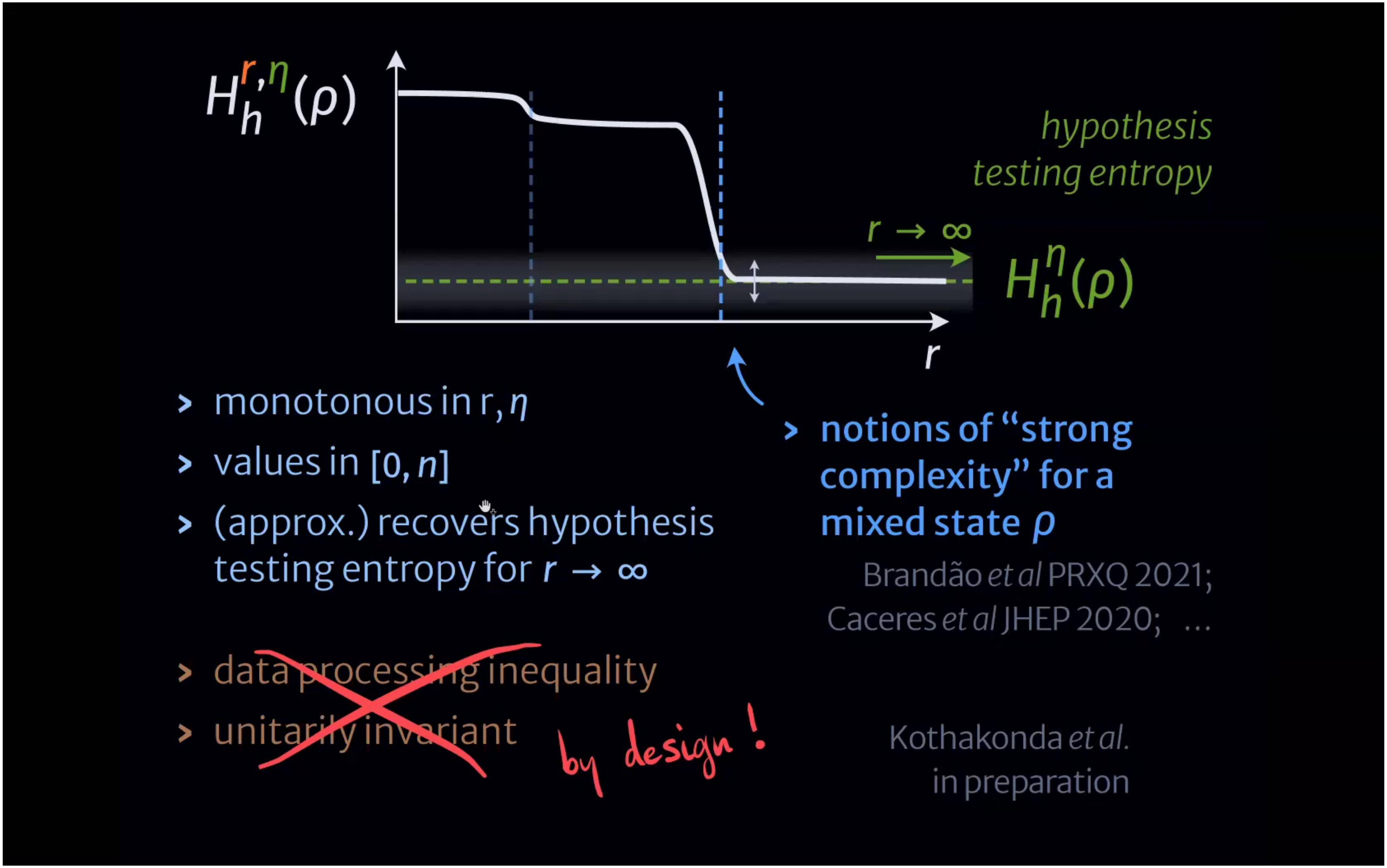
cf. also: Gell-Mann & Hartle PRA 2007; Chen+ 1704.07309; …

ρ looks very mixed at this complexity scale: high work cost for erasure

ρ has low entropy in the eyes of a computationally powerful agent: low work cost for erasure

- Operational meaning of complexity entropy = amount of work required to reset a state *in the eyes of an agent who can perform at most r gates*

Kothakonda *et al.* in preparation



- monotonous in r, η
- values in $[0, n]$
- (approx.) recovers hypothesis testing entropy for $r \to \infty$

- **notions of "strong complexity" for a mixed state ρ**

Brandão *et al* PRXQ 2021;
Caceres *et al* JHEP 2020; …

- ~~data processing inequality~~
- ~~unitarily invariant~~

by design!

Kothakonda *et al.*
in preparation

- Towards a unification of the second laws of thermodynamics and of complexity:
  $H_h^{r,\eta}(\rho)$ tends to only ever increase ?

- Resource theory of complexity: monotone?

Yunger Halpern *et al* 2110.11371

- Complexity-aware replacement of "entropy" for physics in high complexity regimes?
  (e.g. spin-glass, phase transitions, exotic phases of matter?)

Quantifier for **data compression** with computational limitations & quantum **pseudorandomness**

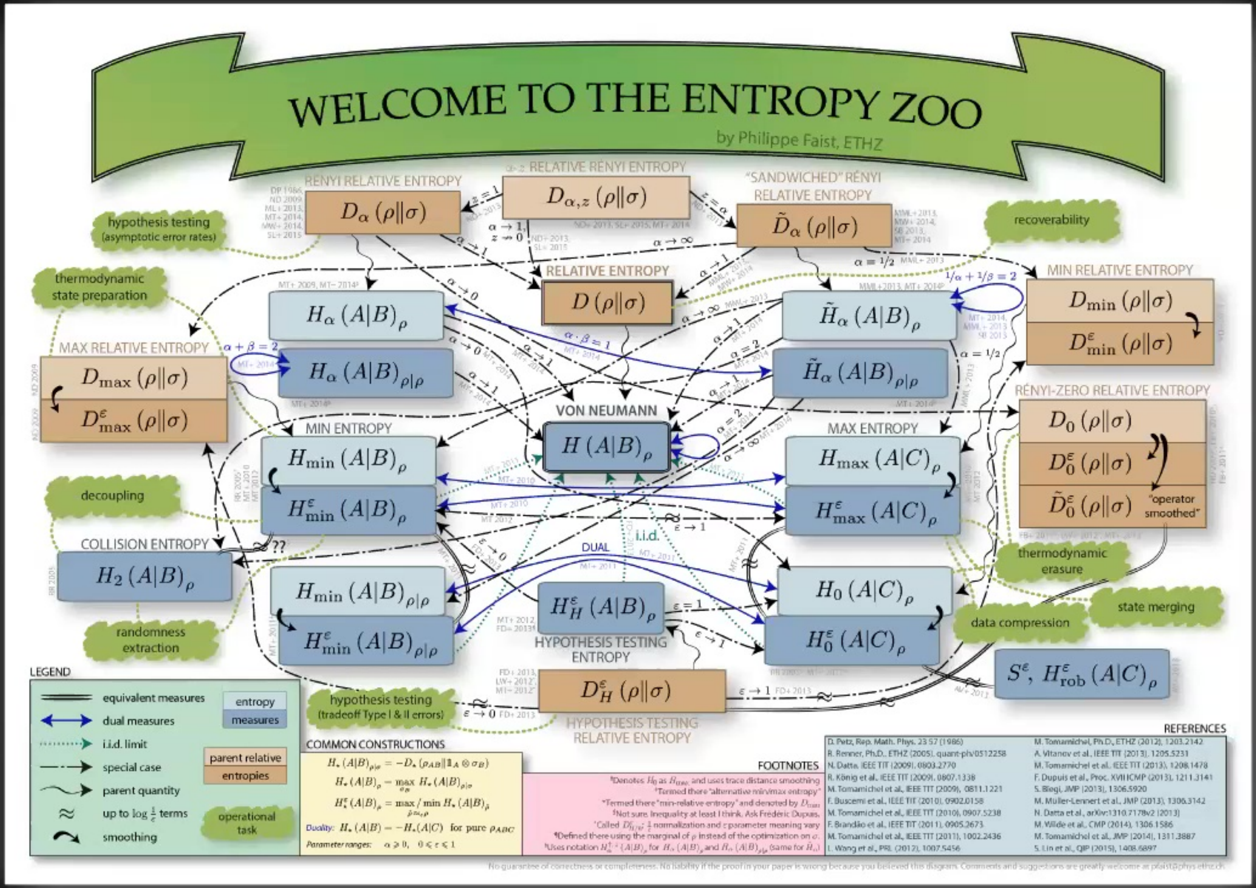
Ji, Liu, Song CRYPTO '18

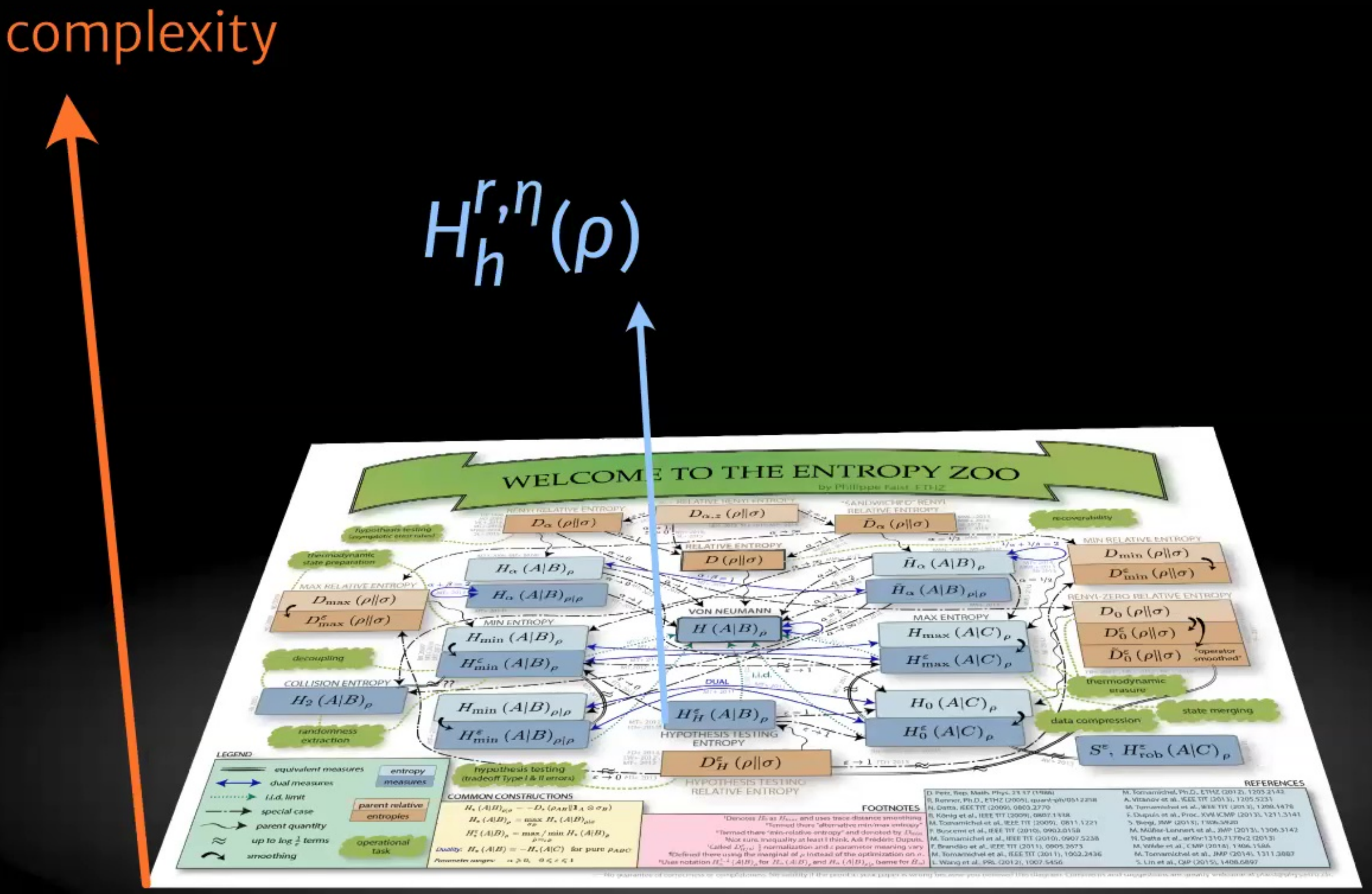
Thermodynamic erasure with a **quantum memory**? Work cost of general **quantum processes**? With nontrivial **Hamiltonians**?

del Rio *et al.* Nat 2011
PhF *et al.* Nat Comm 2015
PhF *et al.* PRX 2018

Complexity in the resource theory of thermodynamics?

cf. Taranto *et al* 2106.05151


WELCOME TO THE ENTROPY ZOO
by Philippe Faist, ETHZ
RÉNYI RELATIVE ENTROPY
$D_\alpha(\rho\|\sigma)$
α-z RELATIVE RÉNYI ENTROPY
$D_{\alpha,z}(\rho\|\sigma)$
"SANDWICHED" RÉNYI RELATIVE ENTROPY
$\tilde{D}_\alpha(\rho\|\sigma)$
RELATIVE ENTROPY
$D(\rho\|\sigma)$
MIN RELATIVE ENTROPY
$D_{\min}(\rho\|\sigma)$
$D^\varepsilon_{\min}(\rho\|\sigma)$
MAX RELATIVE ENTROPY
$D_{\max}(\rho\|\sigma)$
$D^\varepsilon_{\max}(\rho\|\sigma)$
RÉNYI-ZERO RELATIVE ENTROPY
$D_0(\rho\|\sigma)$
$D^\varepsilon_0(\rho\|\sigma)$
$\tilde{D}^\varepsilon_0(\rho\|\sigma)$
"operator smoothed"
$H_\alpha(A|B)_\rho$
$H_\alpha(A|B)_{\rho|\rho}$
$\tilde{H}_\alpha(A|B)_\rho$
$\tilde{H}_\alpha(A|B)_{\rho|\rho}$
VON NEUMANN
$H(A|B)_\rho$
MIN ENTROPY
$H_{\min}(A|B)_\rho$
$H^\varepsilon_{\min}(A|B)_\rho$
MAX ENTROPY
$H_{\max}(A|C)_\rho$
$H^\varepsilon_{\max}(A|C)_\rho$
COLLISION ENTROPY
$H_2(A|B)_\rho$
$H_{\min}(A|B)_{\rho|\rho}$
$H^\varepsilon_{\min}(A|B)_{\rho|\rho}$
$H^\varepsilon_H(A|B)_\rho$
HYPOTHESIS TESTING ENTROPY
$H_0(A|C)_\rho$
$H^\varepsilon_0(A|C)_\rho$
$S^\varepsilon,\ H^\varepsilon_{\mathrm{rob}}(A|C)_\rho$
$D^\varepsilon_H(\rho\|\sigma)$
HYPOTHESIS TESTING RELATIVE ENTROPY
hypothesis testing (asymptotic error rates)
thermodynamic state preparation
recoverability
decoupling
randomness extraction
hypothesis testing (tradeoff Type I & II errors)
thermodynamic erasure
state merging
data compression
$\alpha \to 1$
$\alpha \to \infty$
$\alpha \to 0$
$\alpha = 1/2$
$\alpha = 2$
$z = \alpha$
$z = 1$
$\alpha + \beta = 2$
$1/\alpha + 1/\beta = 2$
$\alpha \cdot \beta = 1$
$\varepsilon \to 0$
$\varepsilon \to 1$
$\varepsilon = 1$
i.i.d.
DUAL
??
LEGEND
equivalent measures
dual measures
i.i.d. limit
special case
parent quantity
$\approx$ up to $\log\frac{1}{\varepsilon}$ terms
smoothing
entropy measures
parent relative entropies
operational task
COMMON CONSTRUCTIONS
$H_*(A|B)_{\rho|\sigma} = -D_*(\rho_{AB}\|\mathbb{1}_A\otimes\sigma_B)$
$H_*(A|B)_\rho = \max_{\sigma_B} H_*(A|B)_{\rho|\sigma}$
$H^\varepsilon_*(A|B)_\rho = \max/\min_{\hat\rho\approx_\varepsilon\rho} H_*(A|B)_{\hat\rho}$
Duality: $H_*(A|B) = -H_*(A|C)$ for pure ρ_{ABC}
Parameter ranges: $\alpha \geqslant 0$, $0 \leqslant \varepsilon \leqslant 1$
FOOTNOTES
REFERENCES


complexity
$H_h^{r,\eta}(\rho)$
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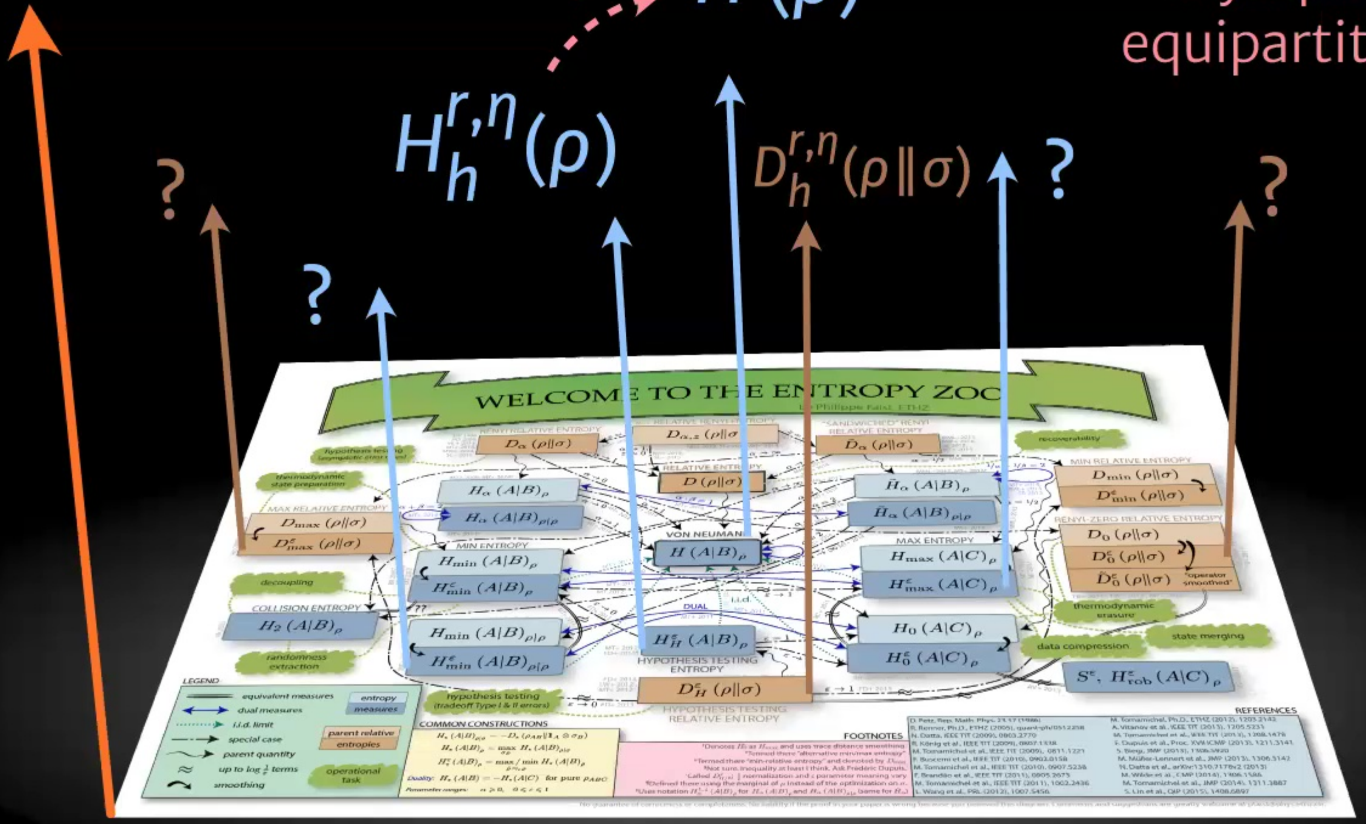


relative entropy ✓

conditional entropy ?

mutual information ?

complexity

asymptotic equipartition ?

? $H^{r}(\rho)$

$H_h^{r,\eta}(\rho)$

$D_h^{r,\eta}(\rho\|\sigma)$

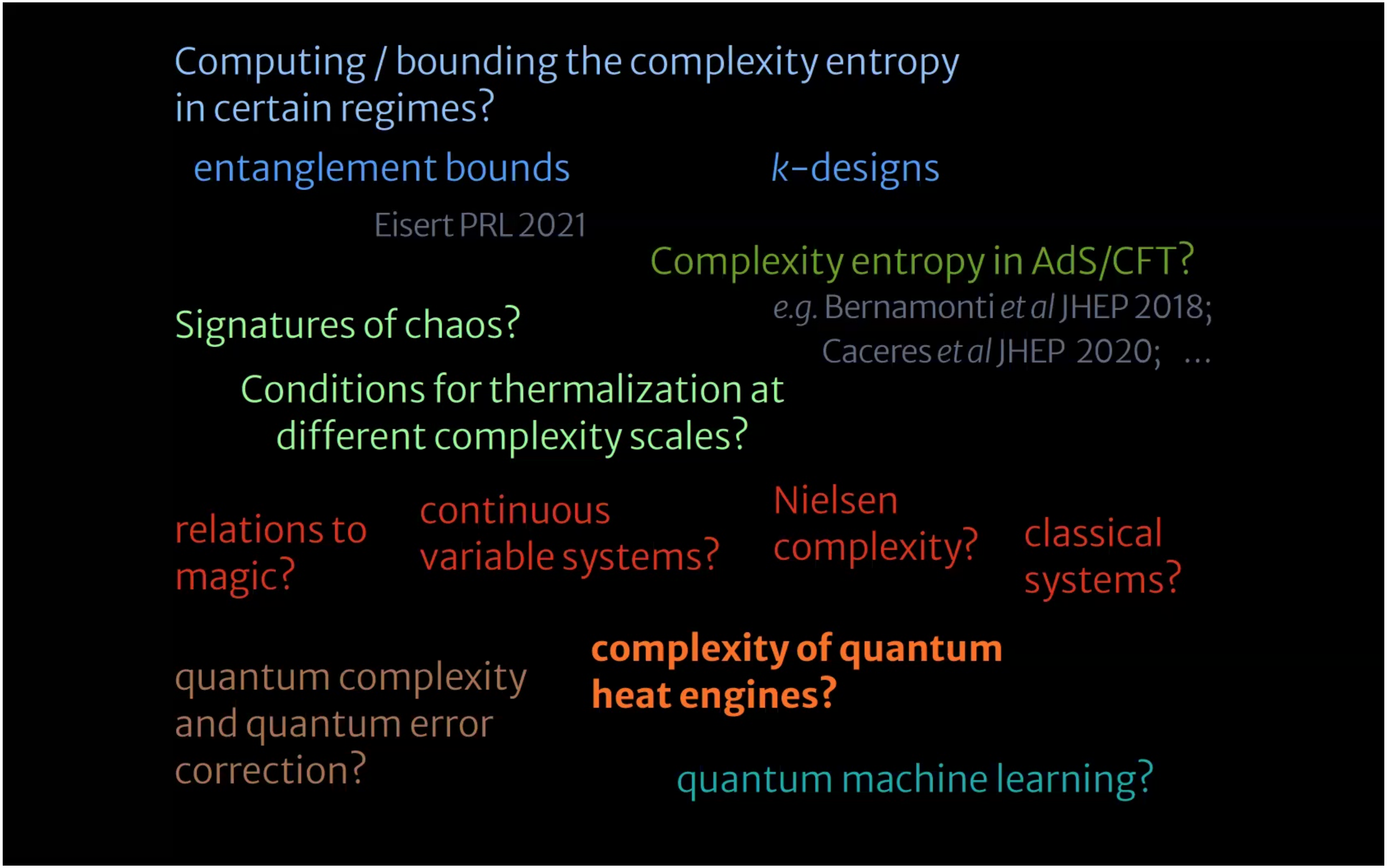
?

?

?

?

Gell-Mann & Hartle PRA 2007; Chen+ 1704.07309; …



Computing / bounding the complexity entropy in certain regimes?

entanglement bounds

k−designs

Eisert PRL 2021

Complexity entropy in AdS/CFT?

e.g. Bernamonti *et al* JHEP 2018; Caceres *et al* JHEP 2020; …

Signatures of chaos?

Conditions for thermalization at different complexity scales?

relations to magic?

continuous variable systems?

Nielsen complexity?

classical systems?

quantum complexity and quantum error correction?

complexity of quantum heat engines?

quantum machine learning?