

Title: Quantum error correction meets continuous symmetries: fundamental trade-offs and case studies

Speakers: Sisi Zhou

Series: Colloquium

Date: April 13, 2022 - 2:00 PM

URL: <https://pirsa.org/22040103>

Abstract: Quantum error correction and symmetries are two key notions in quantum information and physics. The competition between them has fundamental implications in fault-tolerant quantum computing, many-body physics and quantum gravity. We systematically study the competition between quantum error correction and continuous symmetries associated with a quantum code in a quantitative manner. We derive various forms of trade-off relations between the quantum error correction inaccuracy and three types of symmetry violation measures. We introduce two frameworks for understanding and establishing the trade-offs based on the notions of charge fluctuation and gate implementation error. From the perspective of fault-tolerant quantum computing, we demonstrate fundamental limitations on transversal logical gates. We also analyze the behaviors of two near-optimal codes: a parametrized extension of the thermodynamic code, and quantum Reed-Muller codes.

Zoom Link: TBD

Quantum error correction meets continuous symmetries: fundamental trade-offs and case studies

Sisi Zhou
(Caltech)

Joint work with [Zi-Wen Liu](#) (Perimeter Institute)

[arXiv:2111.06355](#) & [arXiv:2111.06360](#)

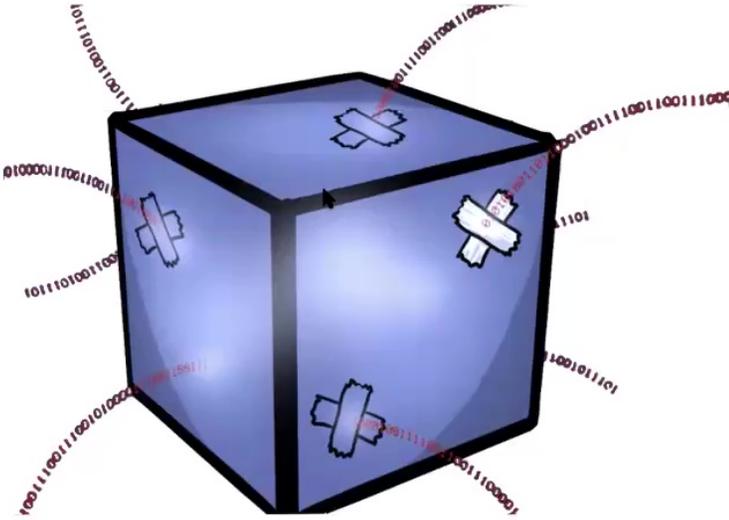
2022/04/13 Perimeter Colloquium





Sisi Zhou

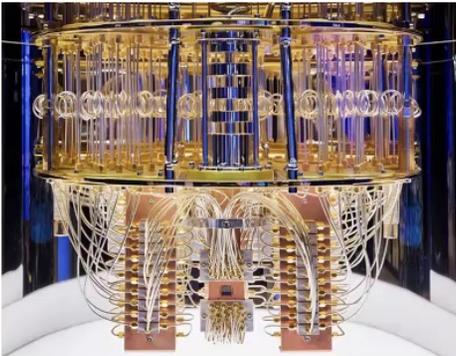
Noise is one of the biggest enemies of quantum computer.



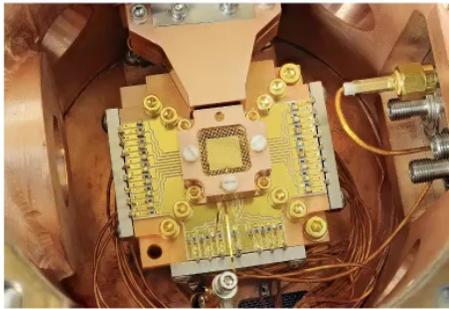


Sisi Zhou

Noise is one of the biggest enemies of quantum computer.



The innards of an IBM quantum computer. Image Credit: IBM

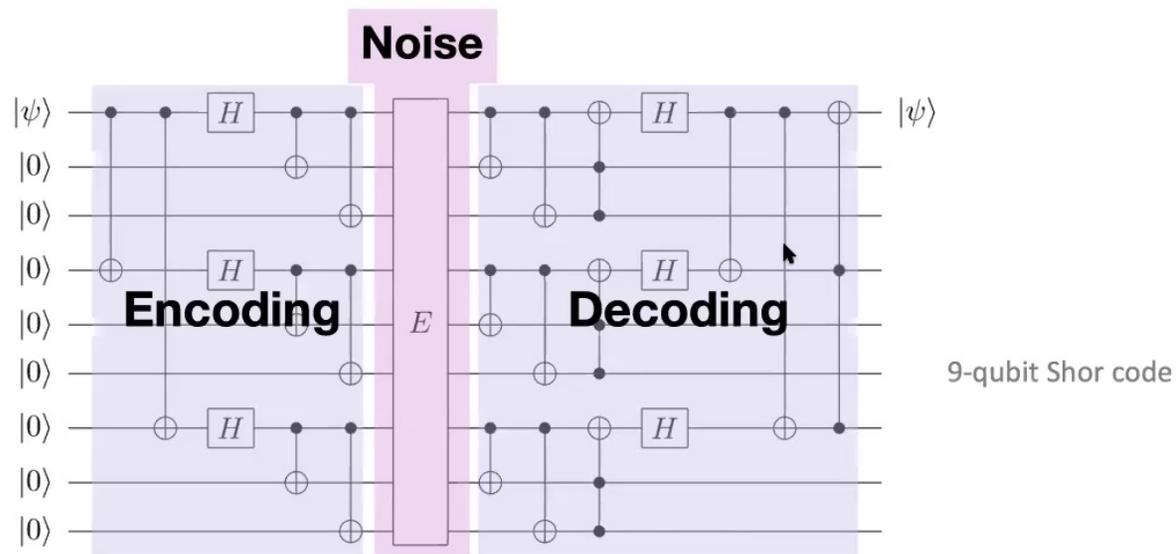


Chip ion trap for quantum computing at NIST. Image Credit: Y. Colombe/NIST



Sisi Zhou

Quantum error correction protects quantum information from noise, where **logical qubits** are encoded in a large number of **physical qubits**.



Quantum error correction

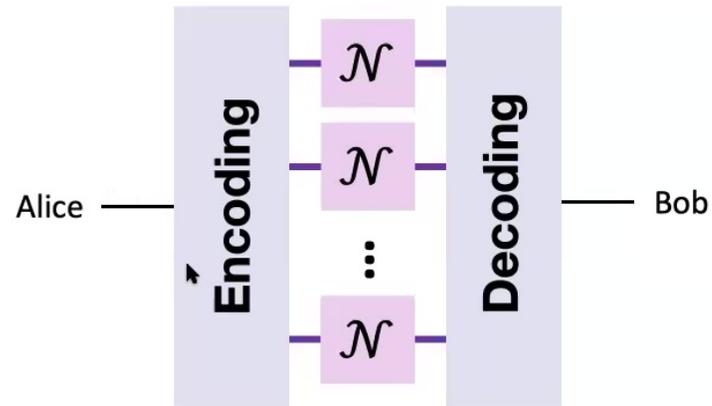
- Quantum computing



Sisi Zhou

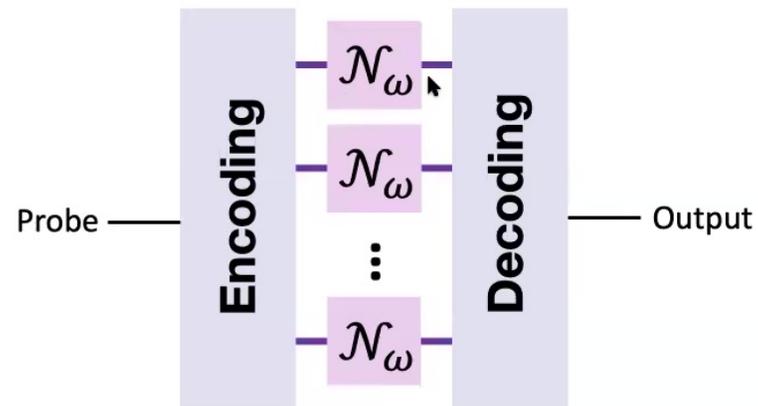
Quantum error correction

- Quantum computing
- Quantum communication



Quantum error correction

- Quantum computing
- Quantum communication
- Quantum sensing



Quantum error correction

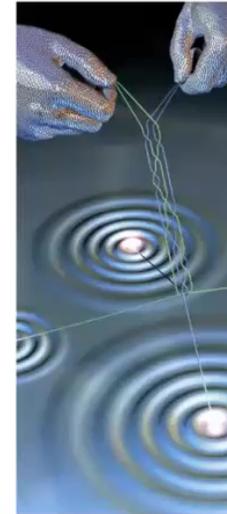
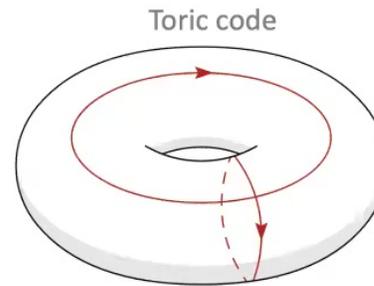
- Quantum computing
- Quantum communication
- Quantum sensing

- Condensed matter physics:

Topological phases, eigenstate thermalization hypothesis, etc.

[Kitaev '97]

[Brandao *et al.* '19]



Topological quantum computing



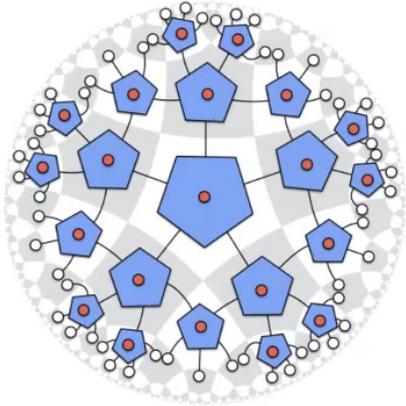


Quantum error correction

- Quantum computing
- Quantum communication
- Quantum sensing

- Condensed matter physics:
Topological phases, eigenstate thermalization hypothesis, etc.
[Kitaev '97] [Brandao et al. '19]

- Quantum gravity:
Holographic codes, black hole information problem, etc.
[Almheiri et al. '14], [Pastawski et al. '15] [Hayden & Preskill '07] ...

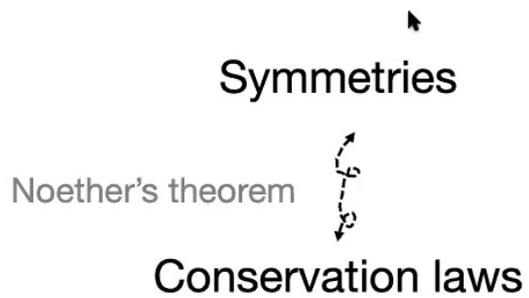


HaPPY code 5

Quantum error correction **with symmetries**



Sisi Zhou



Quantum error correction **with symmetries**



Sisi Zhou

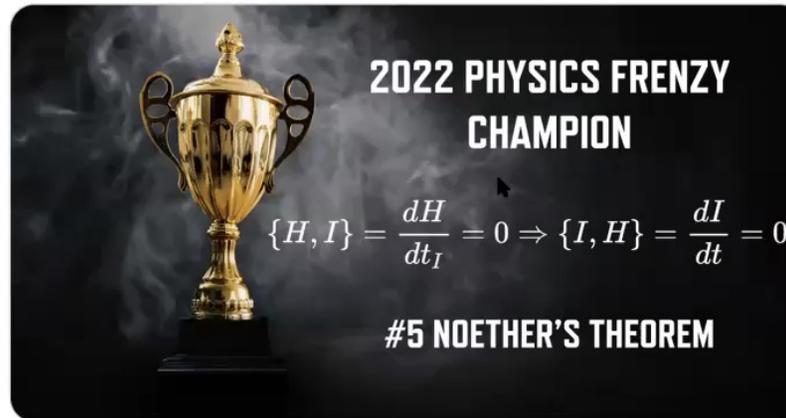
Perimeter Institute @Perimeter · Apr 5
The people have spoken! Noether's theorem defeated Maxwell's equations to win the Physics Frenzy Championship and claim the title of all-time greatest equation in physics. Thanks for voting and sharing your equation-related love over the last three weeks! hubs.ly/Q017q_KD0

Symmetries

Noether's theorem



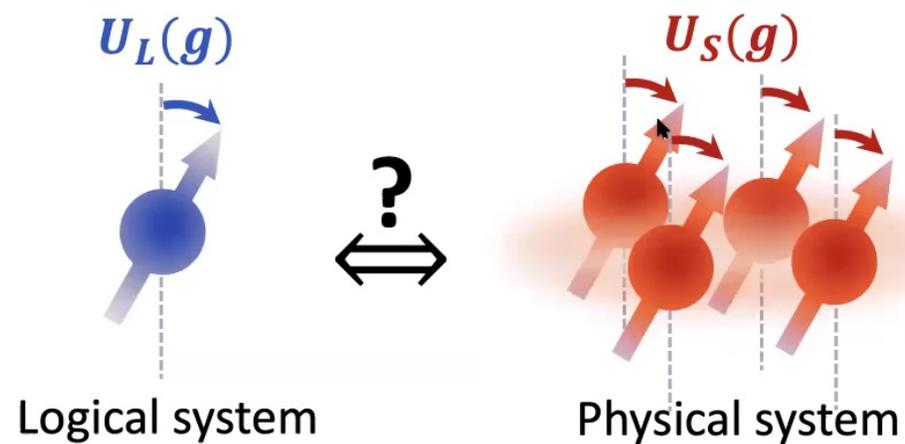
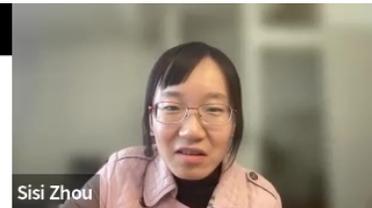
Conservation laws



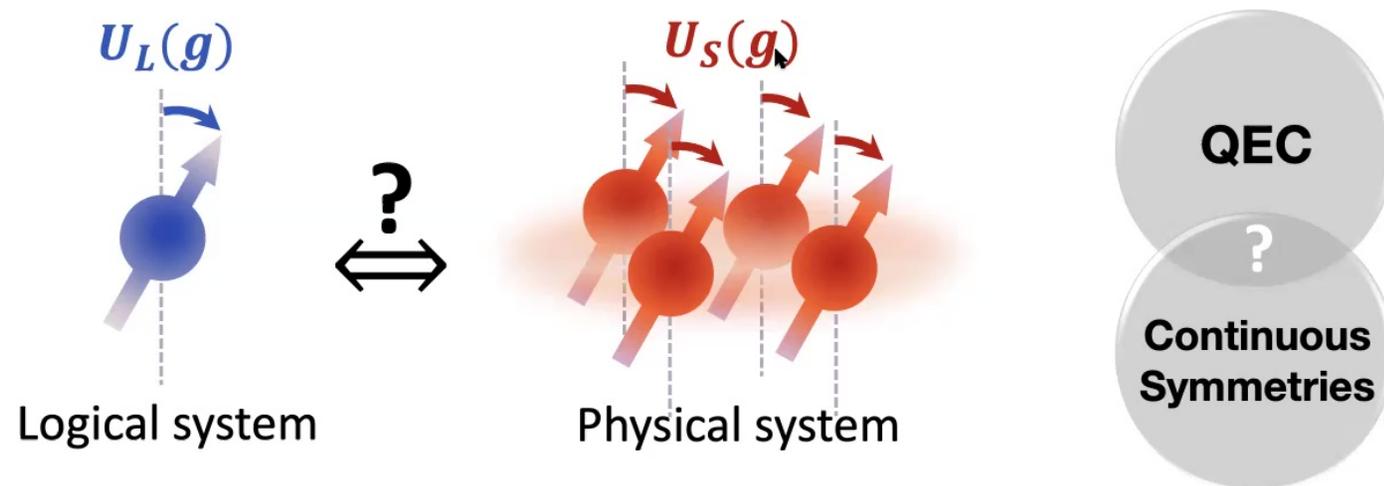
20 237 891



Quantum error correction **with symmetries**



Quantum error correction **with symmetries**



7



Setting: Quantum Information Processing



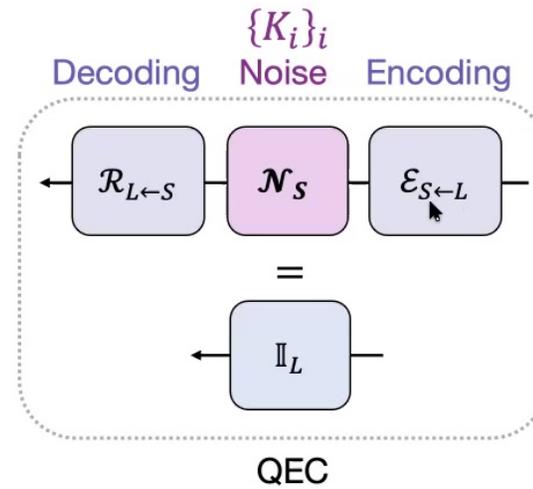
Quantum error correction

Logical state:

$$\sum_{\ell} \psi_{\ell} |\ell_L\rangle$$

Encoded physical state:

$$\sum_{\ell} \psi_{\ell} |(c_{\ell})_S\rangle$$



Sisi Zhou

Quantum error correction

Logical state:

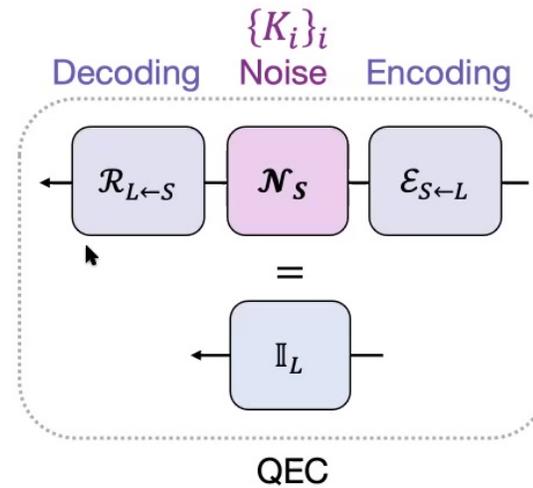
$$\sum_{\ell} \psi_{\ell} |\ell_L\rangle$$

Encoded physical state:

$$\sum_{\ell} \psi_{\ell} |(c_{\ell})_S\rangle$$

Corrupted state:

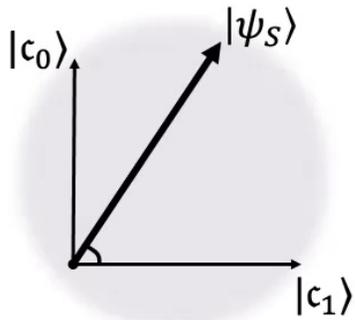
$$\{ \sum_{\ell} \psi_{\ell} K_i |(c_{\ell})_S\rangle \}_i$$





Sisi Zhou

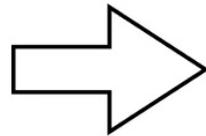
QEC condition



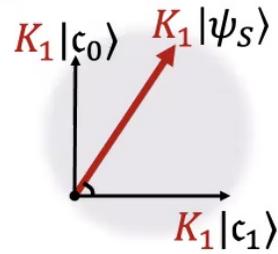
Codewords

$$\langle c_0 | K^\dagger K | c_1 \rangle = 0 \text{ (orthogonal)}$$

$$\langle c_0 | K^\dagger K | c_0 \rangle = \langle c_1 | K^\dagger K | c_1 \rangle \text{ (same scale)}$$



Noise: K



Corrupted codewords

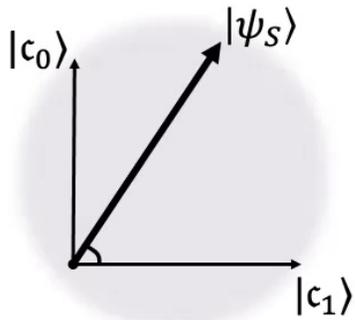
$$PK^\dagger KP \propto P$$

[Bennett et al. '96, Knill & Laflamme '97]



Sisi Zhou

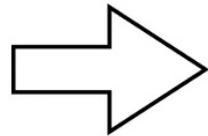
QEC condition



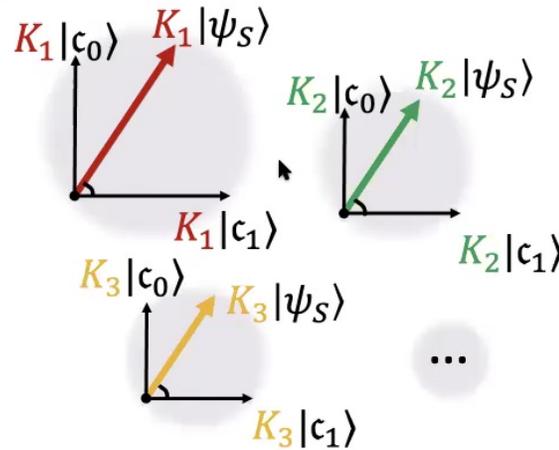
Codewords

$$\langle c_0 | K^\dagger K | c_1 \rangle = 0 \text{ (orthogonal)}$$

$$\langle c_0 | K^\dagger K | c_0 \rangle = \langle c_1 | K^\dagger K | c_1 \rangle \text{ (same scale)}$$



Noise: $\{K_i\}$



Corrupted codewords

$$P K_i^\dagger K_j P \propto P$$

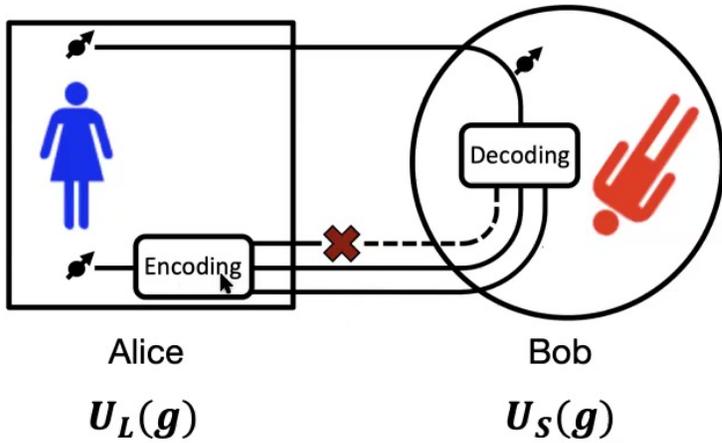
(QEC condition)

[Bennett et al. '96, Knill & Laflamme '97]



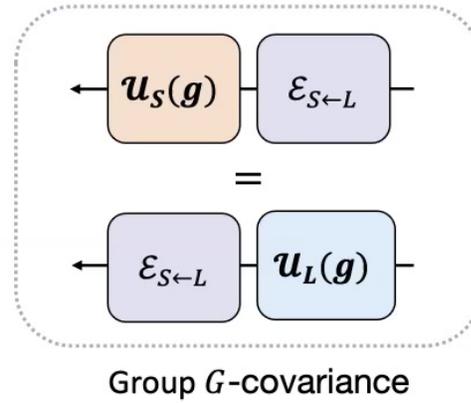
Sisi Zhou

Codes with symmetries: Covariant codes



Reference frame QEC

[Hayden et al. PRXQ'21]

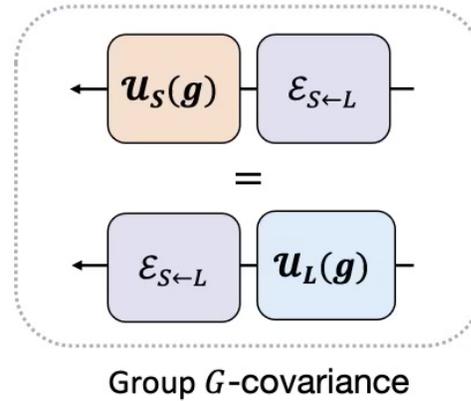
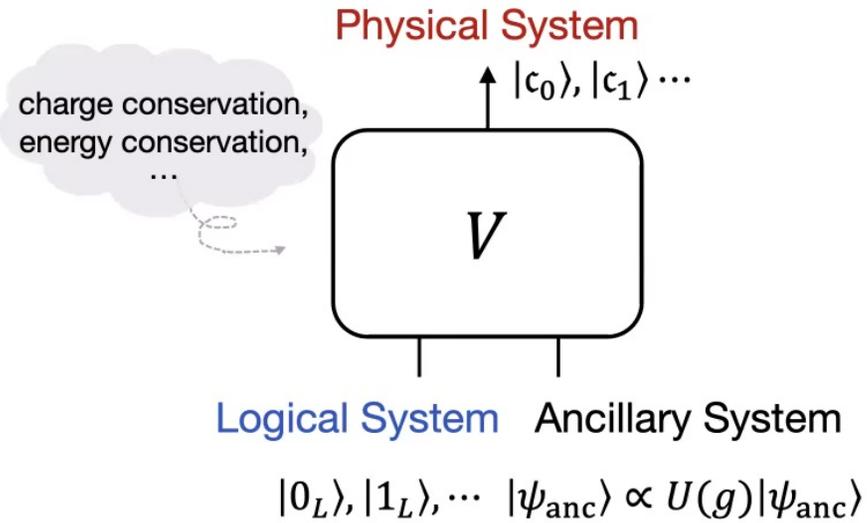


Group G -covariance



Sisi Zhou

Codes with symmetries: Covariant codes

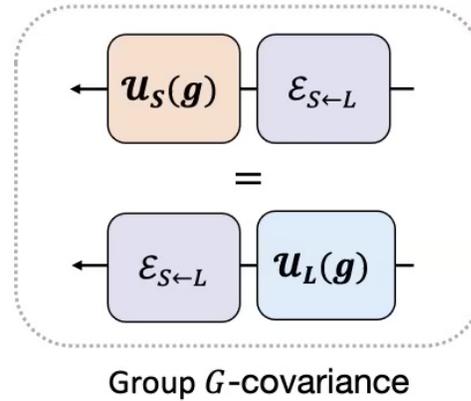
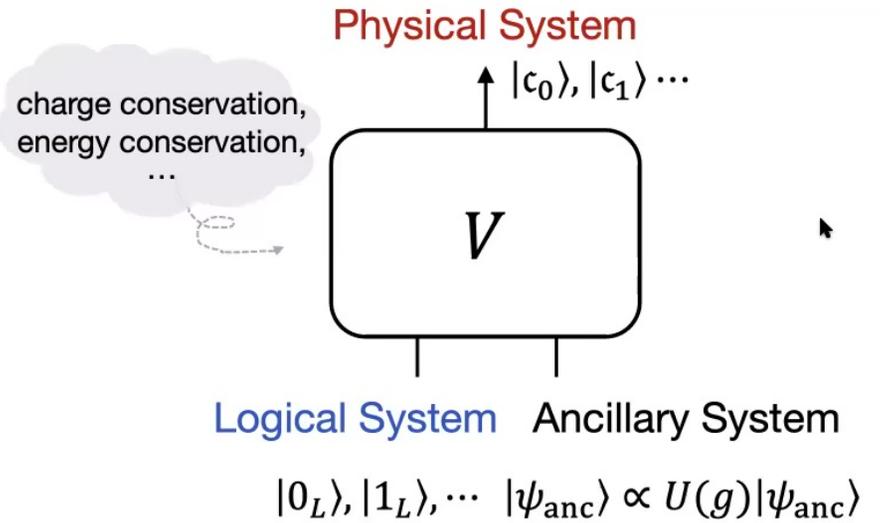


G -symmetric encoding: $U(g)V = VU(g)$



Sisi Zhou

Codes with symmetries: Covariant codes

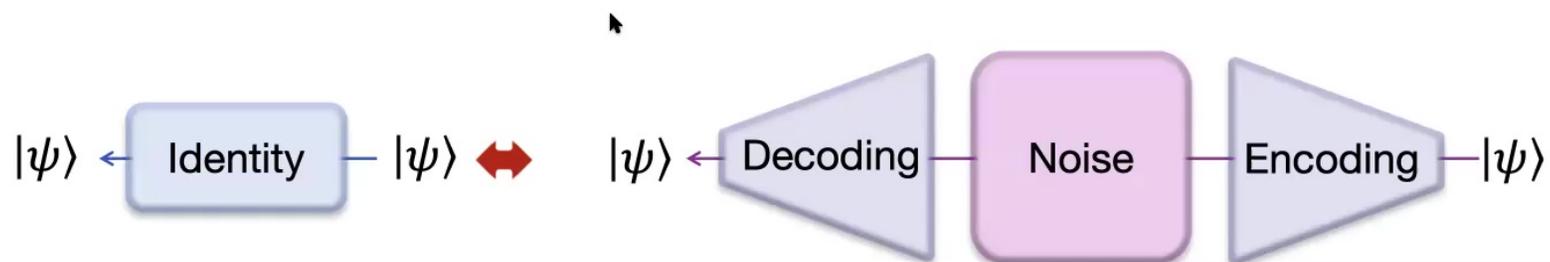


Quantum gravity

[Harlow & Ooguri. PRL'19 & CMP'21]
 [Yoshida PRD'19]

G -symmetric encoding: $U(g)V = VU(g)$

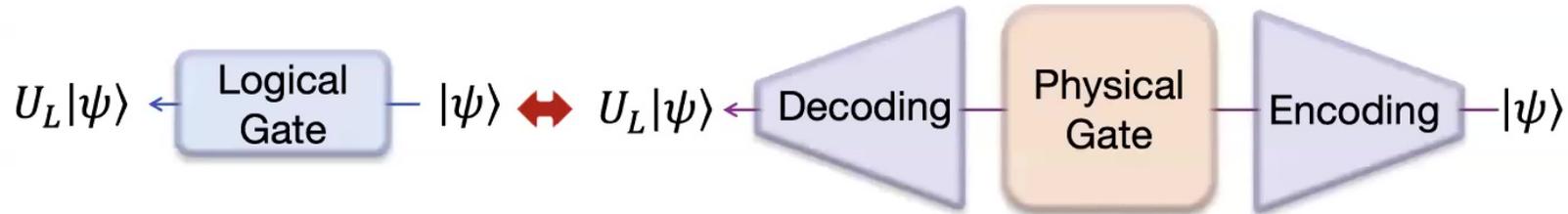
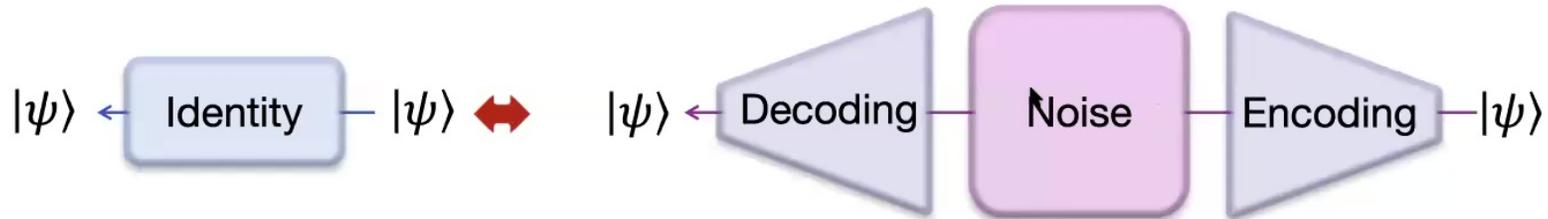
Good memory vs. Easy computing





Sisi Zhou

Good memory vs. Easy computing





Sisi Zhou

A toy example

Logical bit	Codeword	Corrupted codeword	Majority voting
0	n-bits { 0, 0, ⋮, 0 }	0, 1, ⋮, 0	0
1	n-bits { 1, 1, ⋮, 1 }	0, 1, ⋮, 1	1

Can correct errors fewer than $\lfloor n/2 \rfloor - 1$.

Error probability: $p \rightarrow p^{\lfloor n/2 \rfloor}$



Sisi Zhou

A toy example

Logical bit	Codeword	Corrupted codeword	Majority voting
0	n-bits { 0, 0, ⋮, 0 }	0, 1, ⋮, 0	0
1	n-bits { 1, 1, ⋮, 1 }	1, 0, ⋮, 1	1

Can correct errors fewer than $\lfloor n/2 \rfloor - 1$.

Error probability: $p \rightarrow p^{\lfloor n/2 \rfloor}$



Sisi Zhou

A toy example

Logical bit	Codeword	Logical NOT Gate	Majority voting
0	0	1	1
	0	1	
	⋮	⋮	
	0	1	
1	1	0	0
	1	0	
	⋮	⋮	
	1	0	

Logical NOT gate = Physical NOT gates on each bit.

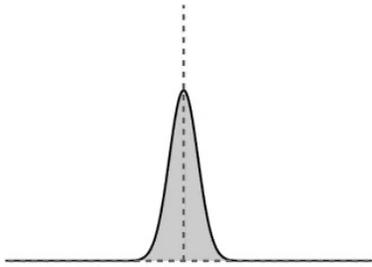


Sisi Zhou

A toy example

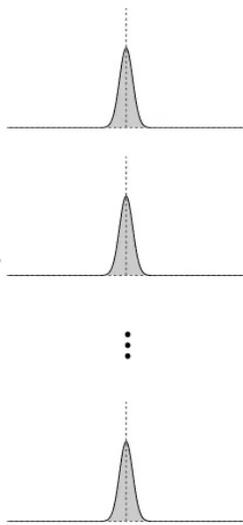
Logical x

$$x \in \mathbb{R}$$



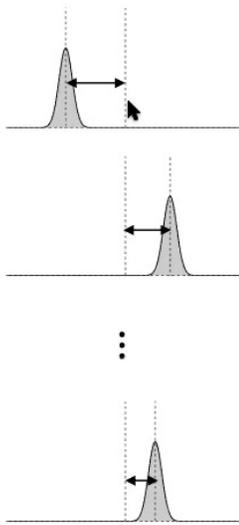
Codeword

$$(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$$



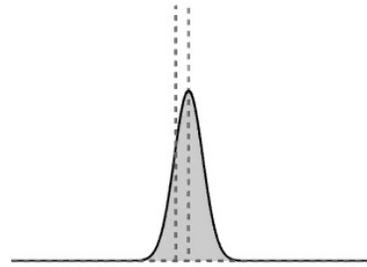
Corrupted codeword

$$Y_n = x_n + Z_n$$



Averaging

$$Y = x + Z_n/n$$



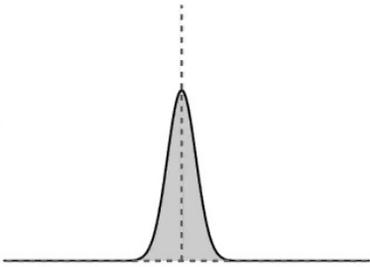


Sisi Zhou

A toy example

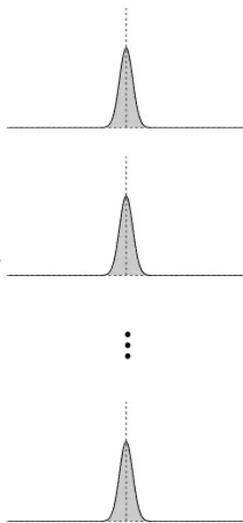
Logical x

$$x \in \mathbb{R}$$



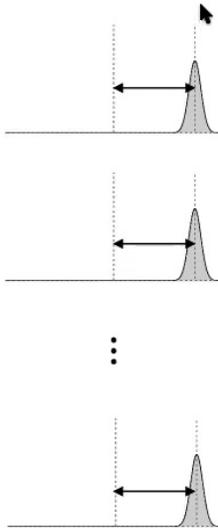
Codeword

$$(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$$



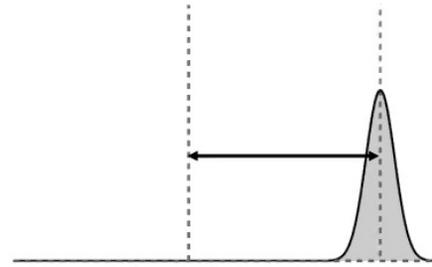
Logical shift

$$Y_n = x_n + \Delta$$



Averaging

$$Y = x + \Delta$$



Logical shift =
Physical shift on each x .

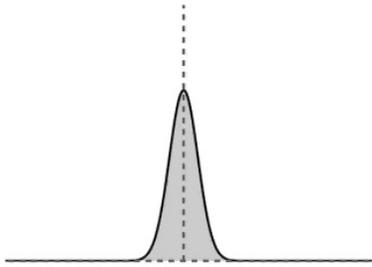


Sisi Zhou

A toy example

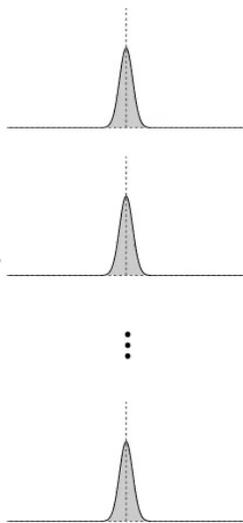
Logical x

$$x \in \mathbb{R}$$



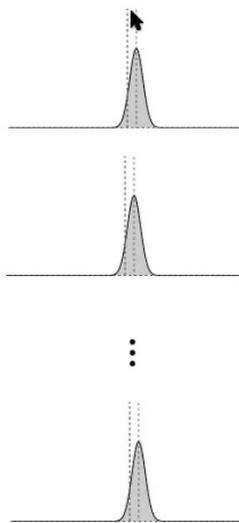
Codeword

$$(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$$



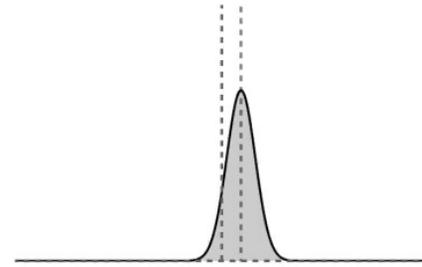
Logical shift

$$Y_n = x_n + \Delta$$



Averaging

$$Y = x + \Delta$$



When $\Delta \lesssim \sigma/\sqrt{n}$, gate and noise are **indistinguishable!!**

Good memory vs. Easy computing



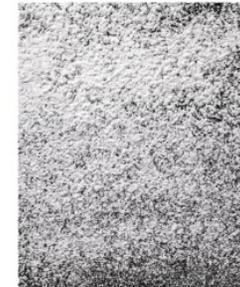
QEC condition ↗



Gate



Good memory vs.
Easy computing



QEC condition

Noise



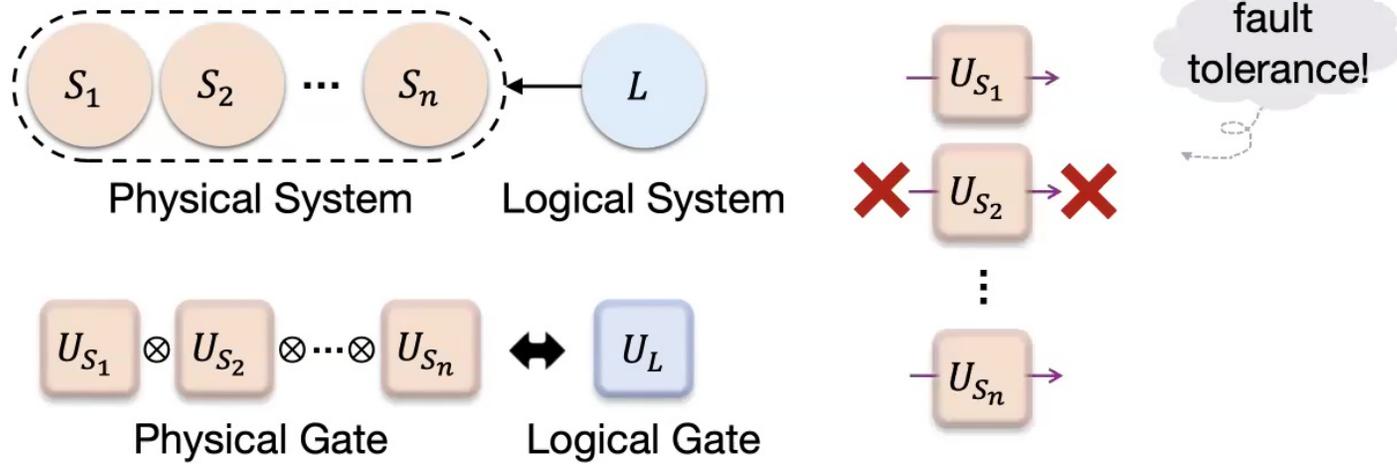
Sisi Zhou



Sisi Zhou

Eastin-Knill Theorem

For any quantum code that corrects **local errors**, the set of **transversal logical gates** is not **universal**. [Eastin & Knill '09]

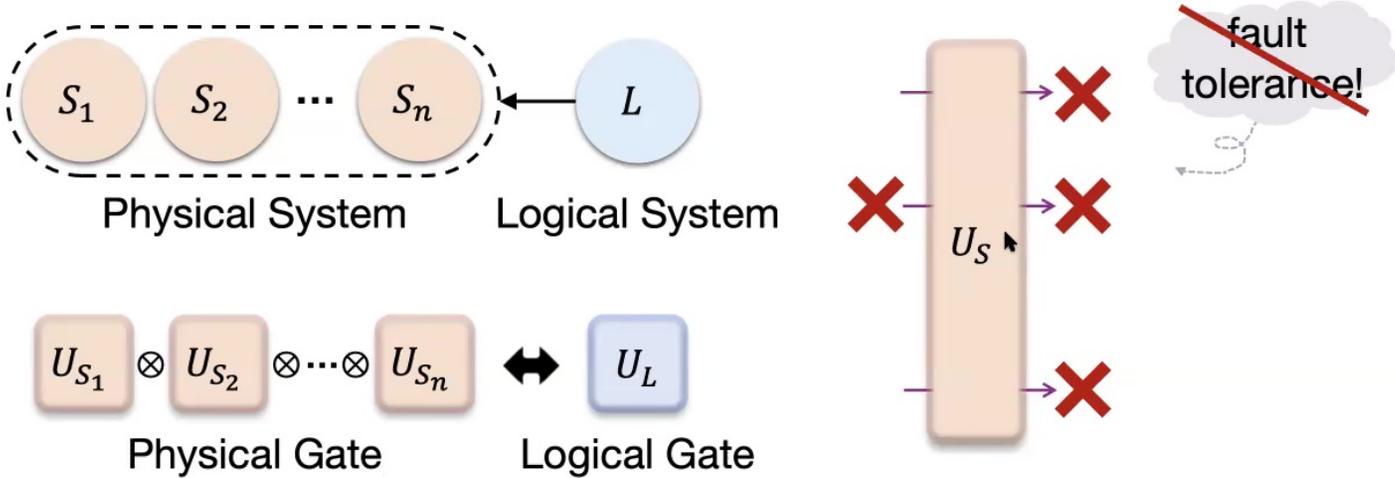




Sisi Zhou

Eastin-Knill Theorem

For any quantum code that corrects **local errors**, the set of **transversal logical gates** is not **universal**. [Eastin & Knill '09]

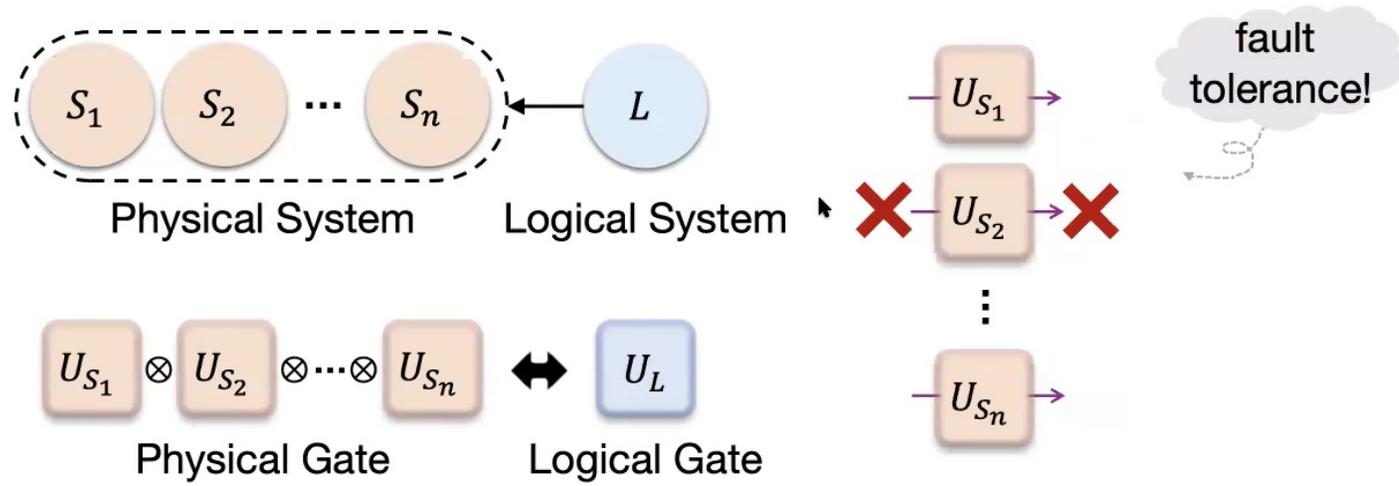




Sisi Zhou

Eastin-Knill Theorem

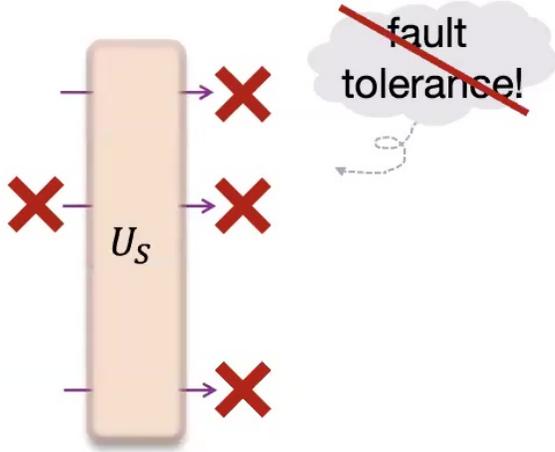
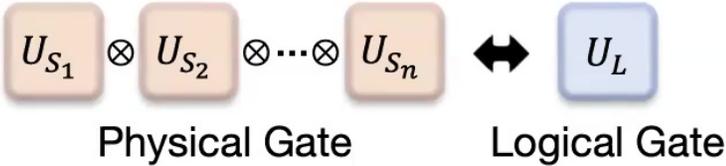
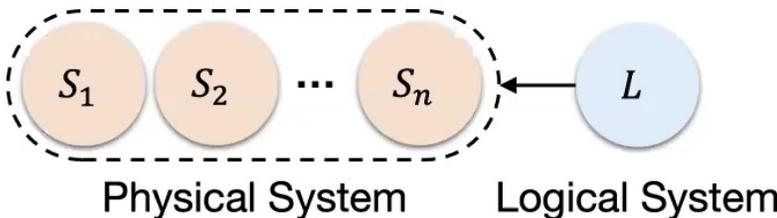
For any quantum code that corrects **local errors**, the set of **transversal logical gates** is not **universal**. [Eastin & Knill '09]





Eastin-Knill Theorem

For any quantum code that corrects **local errors**, the set of **transversal logical gates** is not **universal**. [Eastin & Knill '09]





$$PK_i^\dagger K_j P \propto P$$



QEC condition



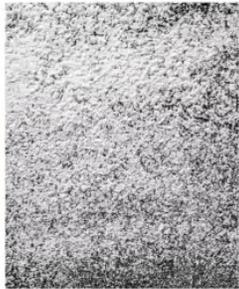
Sisi Zhou

Gate $\{e^{iH\theta}\}$,
for some local H

$$HP = PHP \propto P \Rightarrow e^{iH\theta} P \propto P.$$



$$PK_i^\dagger K_j P \propto P$$



QEC condition

Both gate and noise
act as the identity in
the code subspace.

Local noise E

$$PEP \propto P, \text{ for local operators } E.$$

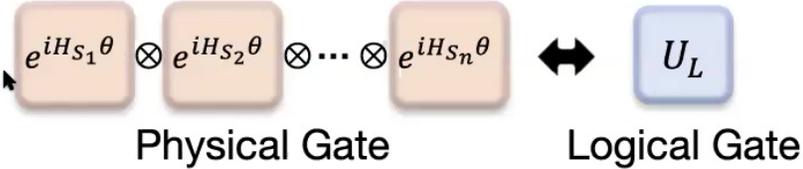
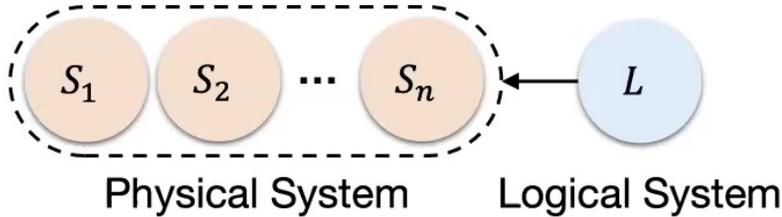




Sisi Zhou

Eastin-Knill Theorem

For any quantum code that corrects **local errors**, the set of **transversal logical gates** is not universal. [Eastin & Knill 09]

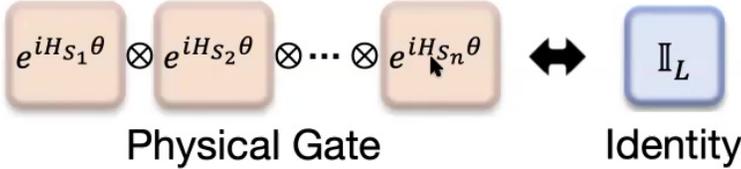
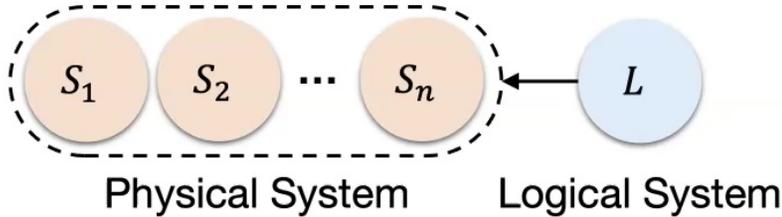




Sisi Zhou

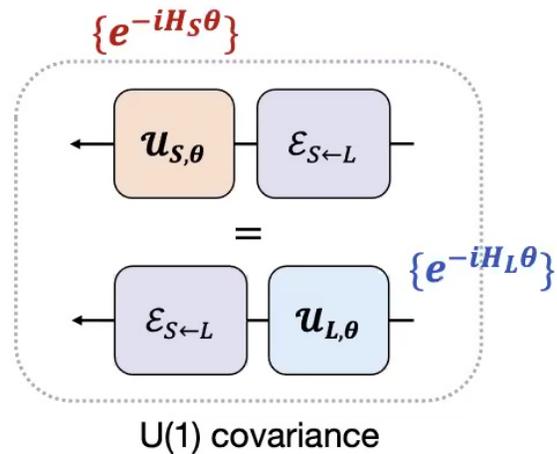
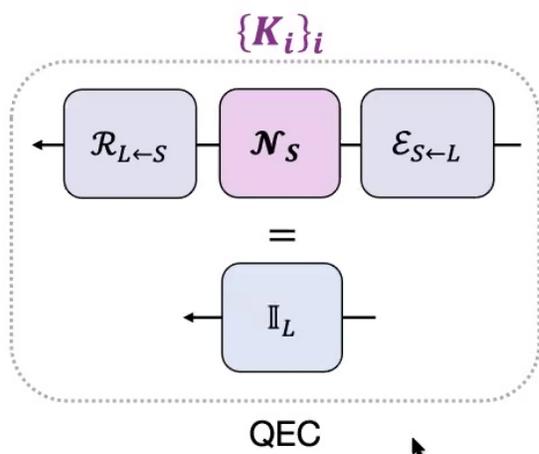
Eastin-Knill Theorem

For any quantum code that corrects **local errors**, the set of **transversal logical gates** is not universal. [Eastin & Knill 09]



The connected component of the identity as a **subgroup** of the entire group of transversal logical gates **acts as the identity** in the code subspace.

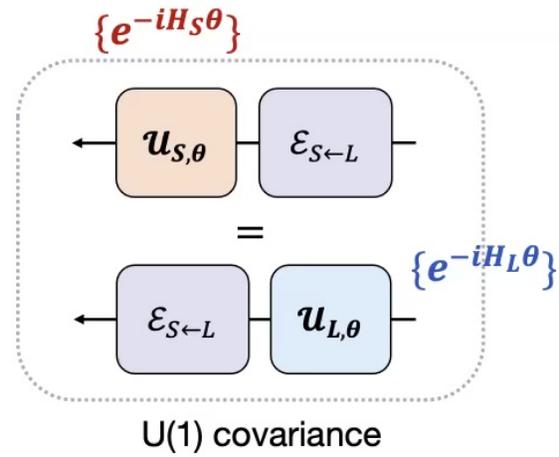
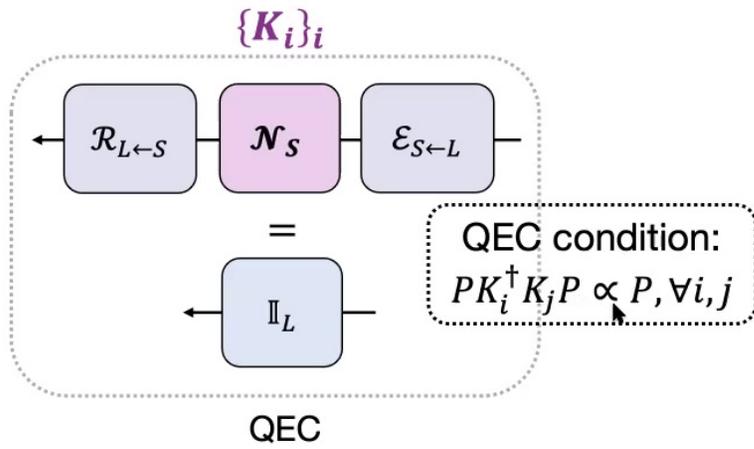
QEC vs. Continuous symmetries





Sisi Zhou

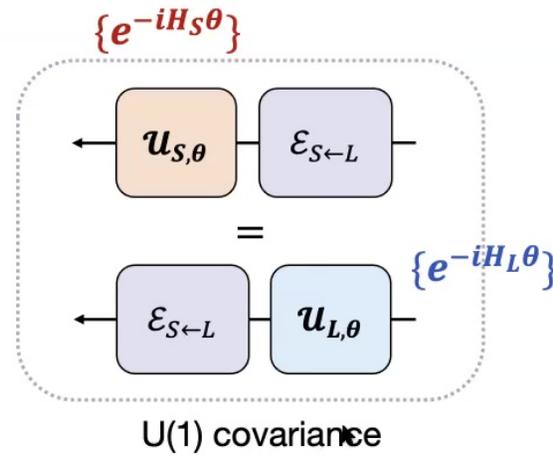
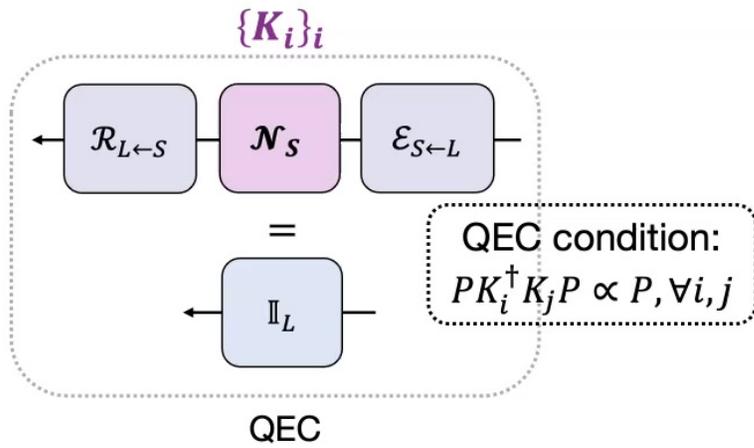
QEC vs. Continuous symmetries





Sisi Zhou

QEC vs. Continuous symmetries

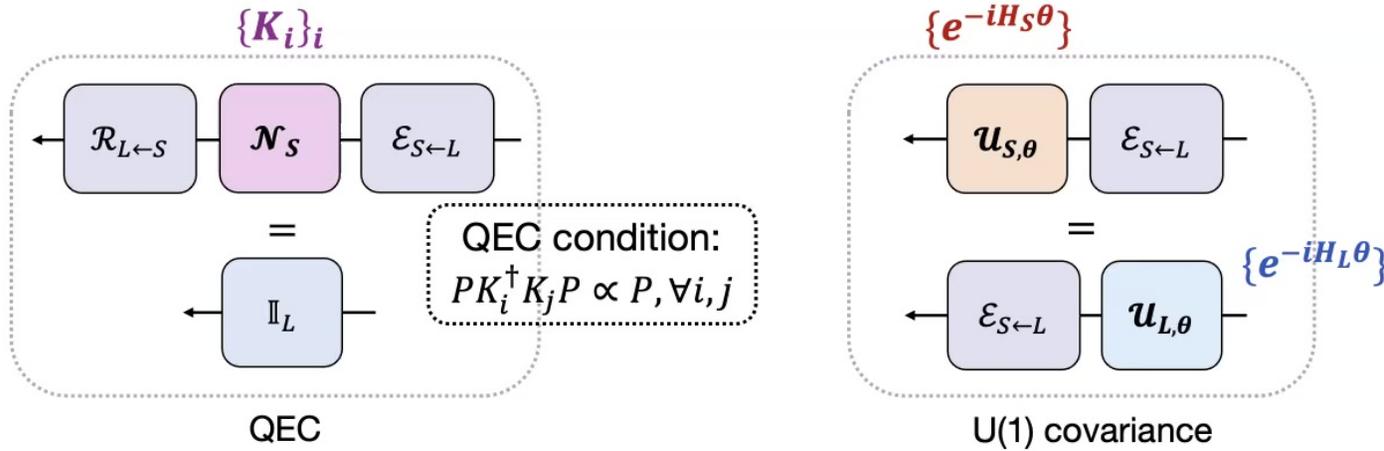


When $H_S \in \text{span}\{K_i^\dagger K_j\}$, H_L acts as the identity.



Sisi Zhou

QEC vs. Continuous symmetries

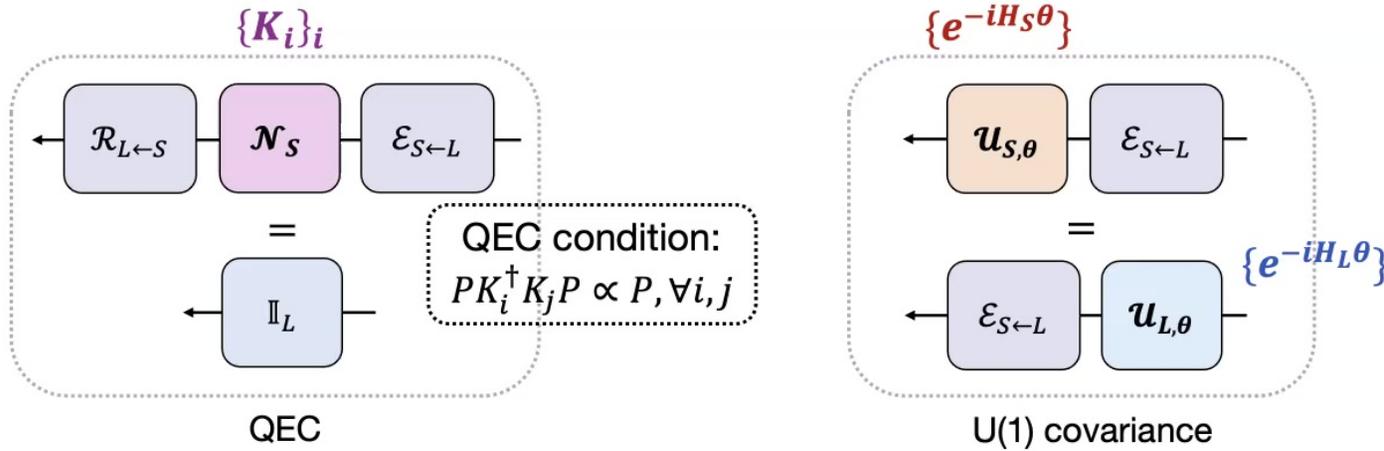


When $H_S \in \text{span}\{K_i^\dagger K_j\}$, H_L acts as the identity.
 (the Hamiltonian-in-Kraus-Span (HKS) condition)



Sisi Zhou

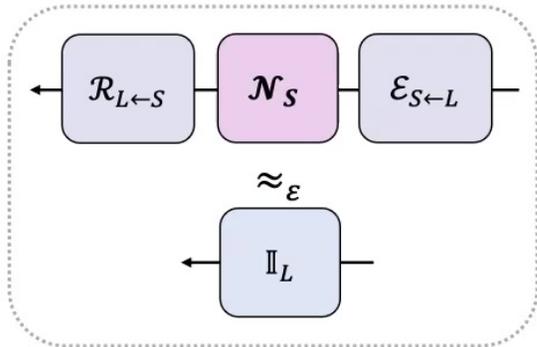
QEC vs. Continuous symmetries



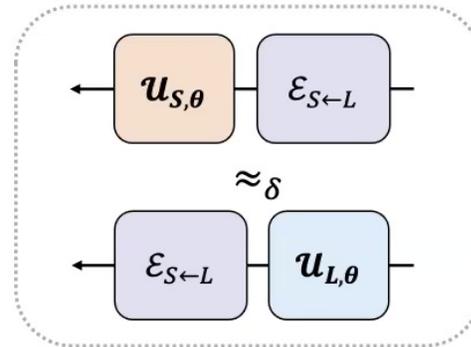
When $H_S \in \text{span}\{K_i^\dagger K_j\}$, H_L acts as the identity.
 (the Hamiltonian-in-Kraus-Span (HKS) condition)
 Charge



Approximate QEC vs. Approximate symmetries



Approximate QEC



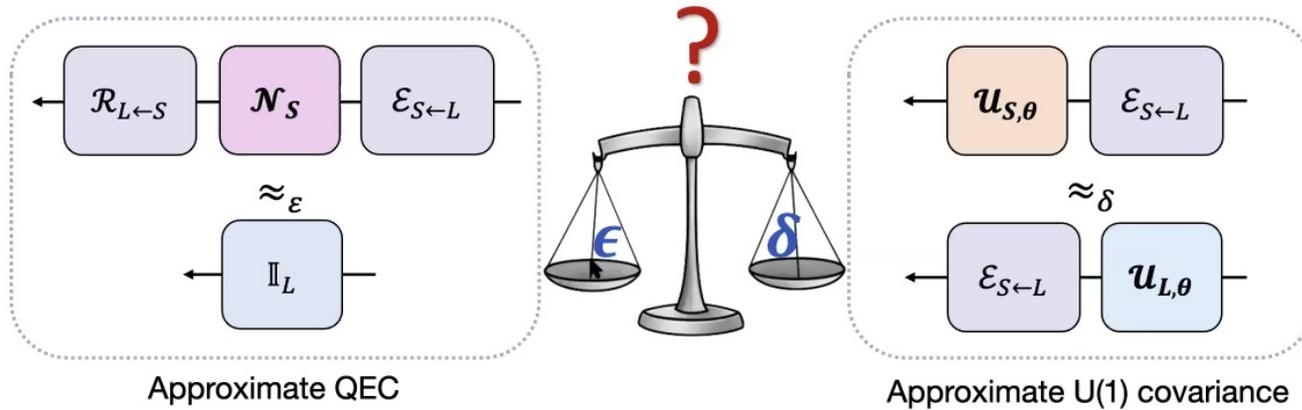
Approximate U(1) covariance

ϵ : Distance measure between $\mathcal{R}_{L \leftarrow S} \circ \mathcal{N}_S \circ \mathcal{E}_{S \leftarrow L}$ and \mathbb{I}_L

δ : Distance measure between $\mathcal{U}_{S,\theta} \circ \mathcal{E}_{S \leftarrow L}$ and $\mathcal{E}_{S \leftarrow L} \circ \mathcal{U}_{L,\theta}$



Approximate QEC vs. Approximate symmetries



ϵ : Distance measure between $\mathcal{R}_{L \leftarrow S} \circ \mathcal{N}_S \circ \mathcal{E}_{S \leftarrow L}$ and \mathbb{I}_L

δ : Distance measure between $\mathcal{U}_{S, \theta} \circ \mathcal{E}_{S \leftarrow L}$ and $\mathcal{E}_{S \leftarrow L} \circ \mathcal{U}_{L, \theta}$



Sisi Zhou

QEC and symmetry measures

QEC inaccuracy: $\epsilon := \min_{\mathcal{R}_{L \leftarrow S}} D(\mathcal{R}_{L \leftarrow S} \circ \mathcal{N}_S \circ \mathcal{E}_{S \leftarrow L}, \mathbb{I}_L)$.

$D(\cdot, \cdot)$: Purified distance.

(Group-) **Global covariance violation:** $\delta_G := \max_{\theta} D(\mathcal{U}_{S, \theta} \circ \mathcal{E}_{S \leftarrow L}, \mathcal{E}_{S \leftarrow L} \circ \mathcal{U}_{L, \theta})$.

(Group-) **Local covariance violation:**

$F(\cdot)$: Quantum Fisher information (QFI).

$$\delta_P := \sqrt{2\partial_{\theta}^2 D^2(\mathcal{U}_{S, \theta} \circ \mathcal{E}_{S \leftarrow L}, \mathcal{E}_{S \leftarrow L} \circ \mathcal{U}_{L, \theta})|_{\theta=0}} = \sqrt{F(\mathcal{U}_{S, \theta} \circ \mathcal{E}_{S \leftarrow L} \circ \mathcal{U}_{L, \theta}^{\dagger})|_{\theta=0}}$$

Charge conservation violation: $\delta_C := \Delta(H_L - \mathcal{E}_{L \leftarrow S}^{\dagger}(H_S))$. $\Delta(A) = \lambda_{\max}(A) - \lambda_{\min}(A)$.

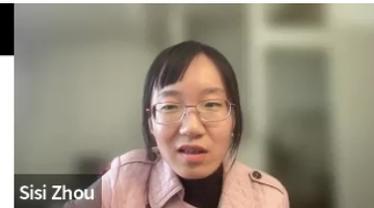
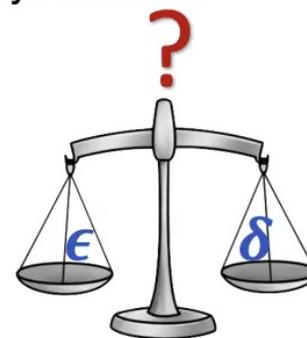
[Faist et al. PRX'20]

Main Results



Main results

- Various trade-off relations between QEC and continuous symmetries.





Sisi Zhou

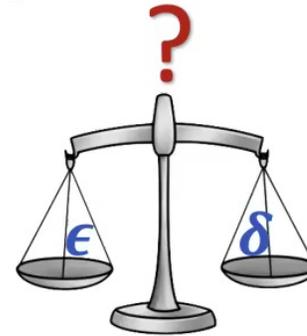
Main results

- Various trade-off relations between QEC and continuous symmetries.

$$\delta \gtrsim \sqrt{\frac{\Delta H_L - 2\epsilon \mathfrak{F}(\mathcal{N}_S, H_S)}{\Delta H_S}}, \quad \epsilon + \delta \gtrsim \frac{\Delta H_L}{\sqrt{4\mathfrak{F}(\mathcal{N}_S, H_S)}}, \quad \dots$$

: Strengths of Hamiltonians.
 $\lambda_{\max}(H) - \lambda_{\min}(H)$

: Related to the HKS condition.
 : QFI of $\mathcal{N}_S \circ \mathcal{U}_{S,\theta}$.





Sisi Zhou

Main results

- Various trade-off relations between QEC and continuous symmetries.

$$\delta \gtrsim \sqrt{\frac{\Delta H_L - 2\epsilon \mathfrak{F}(\mathcal{N}_S, H_S)}{\Delta H_S}}, \quad \epsilon + \delta \gtrsim \frac{\Delta H_L}{\sqrt{4\mathfrak{F}(\mathcal{N}_S, H_S)}}, \quad \dots$$

: Strengths of Hamiltonians.
 $\lambda_{\max}(H) - \lambda_{\min}(H)$

: Related to the HKS condition.

: QFI of $\mathcal{N}_S \circ \mathcal{U}_{S,\theta}$.



- Code examples that nearly attain the bounds.

Previous works:

- Lower bounds on the QEC inaccuracy ϵ when $\delta = 0$.
- Code construction that nearly attains the lower bounds.

[Faist *et al.* PRX'20]
 [Woods & Alhambra, Quantum'20]
 [Kubica & Demkowicz-Dobrzański PRL'20]
 [SZ *et al.* Quantum'21]
 [Yang *et al.* arXiv 2007.09154]



Sisi Zhou

Behavior of the trade-off relations

$$\delta \gtrsim \sqrt{\frac{\Delta H_L - 2\epsilon \mathcal{O}(n)_S}{\mathcal{O}(n)}} \quad \epsilon + \delta \gtrsim \frac{\Delta H_L}{\sqrt{4\mathcal{O}(n^2)_S}}$$

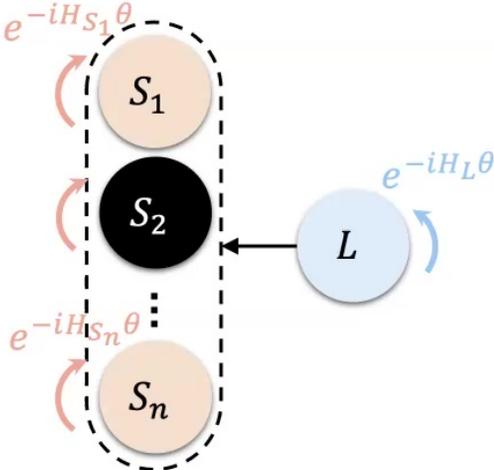
For sufficiently small δ ,

$$\epsilon = \Omega\left(\frac{1}{n}\right)$$

For sufficiently small ϵ ,

$$\delta = \Omega\left(\frac{1}{\sqrt{n}}\right)$$

$$\delta = \Omega\left(\frac{1}{n}\right)$$



Random local noise:

Noise acts on one subsystem chosen uniformly at random.

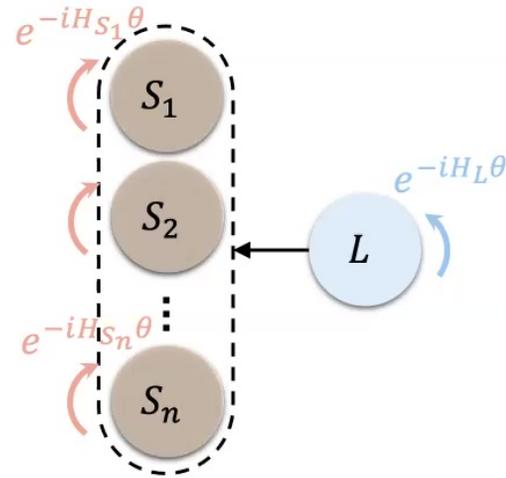


Sisi Zhou

Behavior of the trade-off relations

$$\delta \gtrsim \sqrt{\frac{\Delta H_L - 2\epsilon \mathfrak{F}(\mathcal{N}_S, H_S)}{\Delta H_S}}$$

$$\epsilon + \delta \gtrsim \frac{\Delta H_L}{\sqrt{4\mathfrak{F}(\mathcal{N}_S, H_S)}}$$



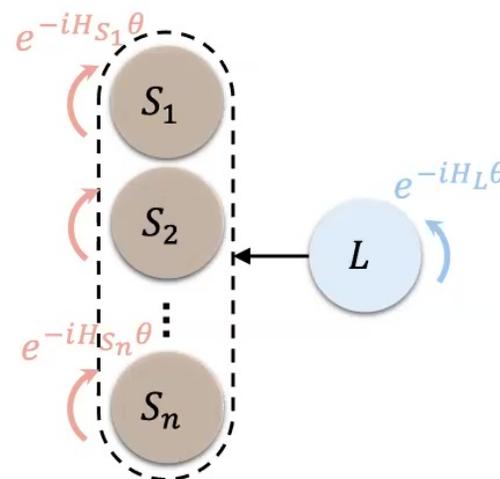
I.I.D. noise:

Noise acts on every subsystem with a fixed probability.

Behavior of the trade-off relations

$$\delta \gtrsim \sqrt{\frac{\Delta H_L - 2\epsilon \mathcal{S}(\mathcal{O}(n)_S)}{\mathcal{O}(n)}}$$

$$\epsilon + \delta \gtrsim \frac{\Delta H_L}{\sqrt{4\mathcal{S}(\mathcal{O}(n)_S)}}$$



I.I.D. noise:

Noise acts on every subsystem with a fixed probability.



Sisi Zhou



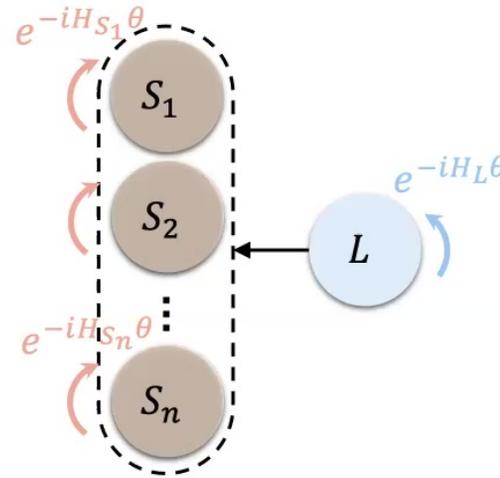
Sisi Zhou

Behavior of the trade-off relations

$$\delta \gtrsim \sqrt{\frac{\Delta H_L - 2\epsilon \mathcal{O}(n) I_S}{\mathcal{O}(n)}} \quad \epsilon + \delta \gtrsim \frac{\Delta H_L}{\sqrt{4\mathcal{O}(n) I_S}}$$

For sufficiently small δ ,

$$\epsilon = \Omega\left(\frac{1}{n}\right) \quad \epsilon = \Omega\left(\frac{1}{\sqrt{n}}\right) \uparrow$$



I.I.D. noise:

Noise acts on every subsystem with a fixed probability.



Behavior of the trade-off relations

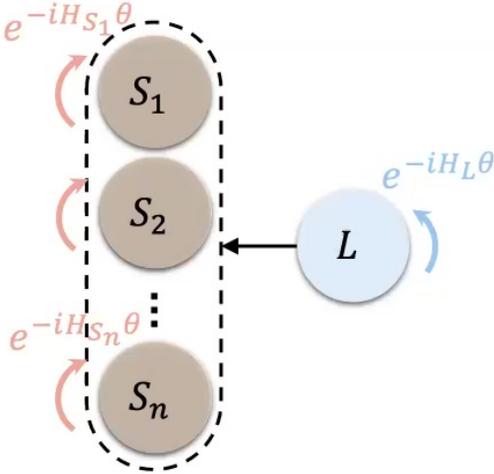
$$\delta \gtrsim \sqrt{\frac{\Delta H_L - 2\epsilon \mathcal{O}(n) I_S}{\mathcal{O}(n)}} \quad \epsilon + \delta \gtrsim \frac{\Delta H_L}{\sqrt{4\mathcal{O}(n) I_S}}$$

For sufficiently small δ ,

$$\epsilon = \Omega\left(\frac{1}{n}\right) \quad \epsilon = \Omega\left(\frac{1}{\sqrt{n}}\right) \uparrow$$

For sufficiently small ϵ ,

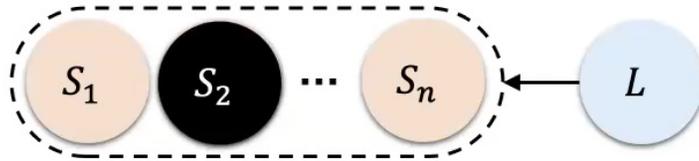
$$\delta = \Omega\left(\frac{1}{\sqrt{n}}\right) -$$



I.I.D. noise:

Noise acts on every subsystem with a fixed probability.

Limitation on transversal logical gates



Consider a QEC code that corrects local errors and admits a transversal implementation $V_S = \otimes_{l=1}^n e^{-i2\pi H_{S_l}/D}$ of the logical gate $V_L = e^{-i2\pi H_L/D}$, where D is an integer, and $H_{L,S}$ have integer eigenvalues and have constant scalings.

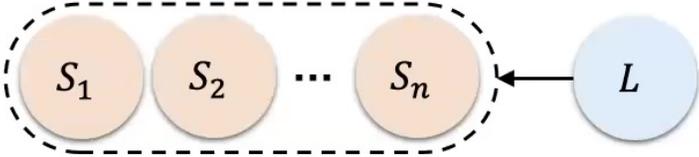
31





Sisi Zhou

Limitation on transversal logical gates



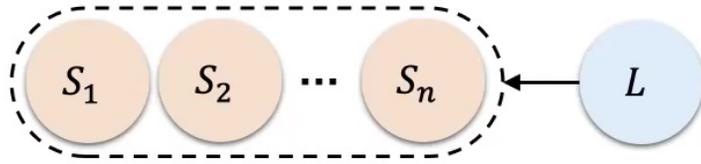
$$\delta \approx \Omega(1/\sqrt{n})$$

Consider a QEC code that corrects local errors and admits a transversal implementation $V_S = \otimes_{l=1}^n e^{-i2\pi H_{S_l}/D}$ of the logical gate $V_L = e^{-i2\pi H_L/D}$, where D is an integer, and $H_{L,S}$ have integer eigenvalues and have constant scalings.



Sisi Zhou

Limitation on transversal logical gates



$$\exp\left(\frac{-i2\pi H_{S_1}}{D}\right) \otimes \dots \otimes \exp\left(\frac{-i2\pi H_{S_n}}{D}\right) = \exp\left(\frac{-i2\pi H_L}{D}\right)$$

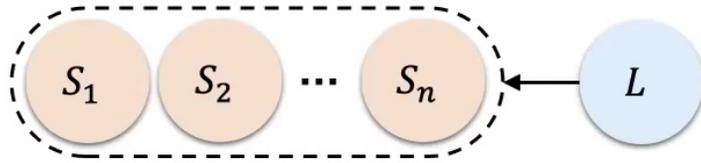
$$\delta = \Omega(1/\sqrt{n})$$

Consider a QEC code that corrects local errors and admits a transversal implementation $V_S = \otimes_{l=1}^n e^{-i2\pi H_{S_l}/D}$ of the logical gate $V_L = e^{-i2\pi H_L/D}$, where D is an integer, and $H_{L,S}$ have integer eigenvalues and have constant scalings.



Sisi Zhou

Limitation on transversal logical gates



$$\exp\left(\frac{-i2\pi H_{S_1}}{D}\right) \otimes \dots \otimes \exp\left(\frac{-i2\pi H_{S_n}}{D}\right) = \exp\left(\frac{-i2\pi H_L}{D}\right) \implies \begin{aligned} \delta &= O(n/D), \\ D &= O(n^{3/2}), \\ \delta &= \Omega(1/\sqrt{n}) \end{aligned}$$

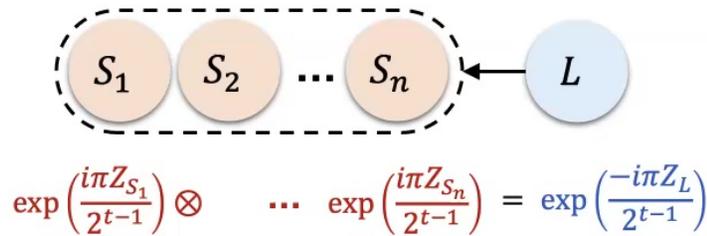
$$\exp(i\theta H_{S_1}) \otimes \dots \otimes \exp(i\theta H_{S_n}) \approx_{\delta} \exp(i\theta H_L)$$

Consider a QEC code that corrects local errors and admits a transversal implementation $V_S = \otimes_{l=1}^n e^{-i2\pi H_{S_l}/D}$ of the logical gate $V_L = e^{-i2\pi H_L/D}$, where D is an integer, and $H_{L,S}$ have integer eigenvalues and have constant scalings.



Sisi Zhou

Example: Quantum Reed-Muller code



Consider the $[[n = 2^t - 1, 1, 3]]$ quantum Reed-Muller code:

$$|c_0\rangle = \frac{1}{\sqrt{2^t}} \left(\sum_{\mathbf{x} \in \text{RM}(1,t)} |\mathbf{x}\rangle \right), \quad |c_1\rangle = \frac{1}{\sqrt{2^t}} \left(\sum_{\mathbf{x} \in \text{RM}(1,t)} |\mathbf{1} + \mathbf{x}\rangle \right).$$

The code is **exactly error-correcting** against single-qubit errors, and is **approximately covariant**:

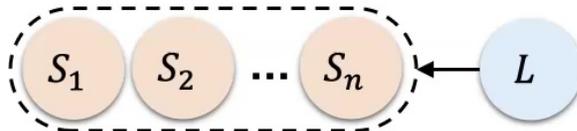
$$\epsilon = 0, \quad \delta \approx \frac{2}{\sqrt{n}} \gtrsim \frac{1}{\sqrt{n}}.$$

0	0 0 0 0 0 0 0 0	c ₀ ⟩
v ₃	0 0 0 0 1 1 1 1	
v ₂	0 0 1 1 0 0 1 1	
v ₁	0 1 0 1 0 1 0 1	
v ₂ + v ₃	0 0 1 1 1 1 0 0	
v ₁ + v ₃	0 1 0 1 1 0 1 0	
v ₁ + v ₂	0 1 1 0 0 1 1 0	
v ₁ + v ₂ + v ₃	0 1 1 0 1 0 0 1	
1	1 1 1 1 1 1 1 1	
1 + v ₃	1 1 1 1 0 0 0 0	
1 + v ₂	1 1 0 0 1 1 0 0	
1 + v ₁	1 0 1 0 1 0 1 0	
1 + v ₂ + v ₃	1 1 0 0 0 0 1 1	
1 + v ₁ + v ₃	1 0 1 0 0 1 0 1	
1 + v ₁ + v ₂	1 0 0 1 1 0 0 1	
1 + v ₁ + v ₂ + v ₃	1 0 0 1 0 1 1 0	

|c₁⟩ = X^{⊗n}|c₀⟩

t = 3

Example: Modified thermodynamic code



[Brandao *et al.* PRL'19, Faist *et al.* PRX'21]

Thermodynamic code: a spin chain with the total charge $H_S = -\sum_{l=1}^n Z_{S_l}$.

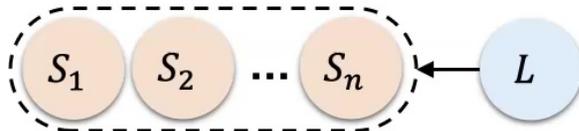
$$|c_0\rangle = |m\rangle_{\text{Dicke}} \propto \sum_{j \in \{0,1\}^n, H_S|j\rangle=m|j\rangle} |j\rangle, \quad |c_1\rangle = |-m\rangle_{\text{Dicke}} \propto \sum_{j \in \{0,1\}^n, H_S|j\rangle=-m|j\rangle} |j\rangle.$$





Sisi Zhou

Example: Modified thermodynamic code



$$\exp(i\theta Z_{S_1}) \otimes \dots \otimes \exp(i\theta Z_{S_n}) = \exp(-i\theta Z_L), \forall \theta$$

[Brandao et al. PRL'19, Faist et al. PRX'21]

Thermodynamic code: a spin chain with the total charge $H_S = -\sum_{l=1}^n Z_{S_l}$.

$$|c_0\rangle = |m\rangle_{\text{Dicke}} \propto \sum_{\substack{j \in \{0,1\}^n \\ H_S|j\rangle = m|j\rangle}} |j\rangle, \quad |c_1\rangle = |-m\rangle_{\text{Dicke}} \propto \sum_{\substack{j \in \{0,1\}^n \\ H_S|j\rangle = -m|j\rangle}} |j\rangle.$$

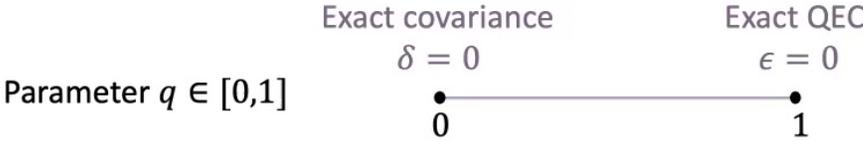
The code is **exactly covariant**, and **approximately error-correcting** against single-qubit erasure errors:

$$\epsilon \approx \frac{m}{2n} \gtrsim \frac{m}{2n}, \quad \delta = 0.$$



Sisi Zhou

Example: Modified thermodynamic code



Consider a spin chain where the total charge $H_S = -\sum_{l=1}^n Z_{S_l}$.

$$|c_0^q\rangle \propto \sqrt{n}|m\rangle_{\text{Dicke}} + \sqrt{qm}|-n\rangle_{\text{Dicke}}, \quad |c_1^q\rangle \propto \sqrt{n}|-m\rangle_{\text{Dicke}} + \sqrt{qm}|n\rangle_{\text{Dicke}}.$$

The code transits smoothly from an exactly covariant code to an exactly error-correcting code when q increases from 0 to 1:

$$\epsilon \approx \frac{(1-q)m}{2n} \gtrsim \frac{(1-4q)m}{2n}, \quad \delta \approx \frac{\sqrt{4qm}}{\sqrt{n}} \gtrsim \frac{\sqrt{qm}}{\sqrt{n}}.$$



Proof Techniques



Proof technique: Charge fluctuation

Charge fluctuation $\chi := \langle c_0 | H_S | c_0 \rangle - \langle c_1 | H_S | c_1 \rangle$.

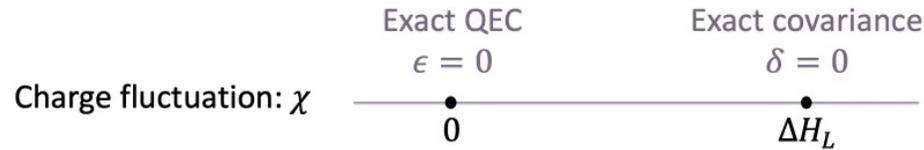
$|c_0\rangle$ and $|c_1\rangle$ correspond to the largest and the smallest eigenvalues of H_L .





Sisi Zhou

Proof technique: Charge fluctuation



Charge fluctuation $\chi := \langle c_0 | H_S | c_0 \rangle - \langle c_1 | H_S | c_1 \rangle$.

$|c_0\rangle$ and $|c_1\rangle$ correspond to the largest and the smallest eigenvalues of H_L .

When the code is exactly error-correcting ($\epsilon = 0$),

The QEC condition: $PK_i^\dagger K_j P \propto P$,

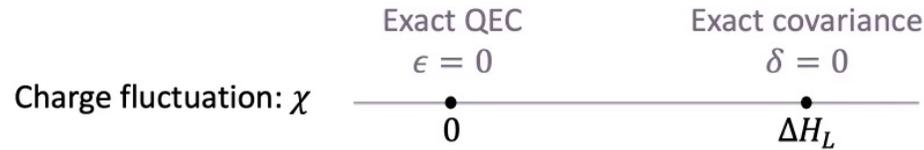
The HKS condition: $H_S \in \text{span}\{K_i^\dagger K_j, \forall i, j\}$,

$$\Rightarrow PH_S P \propto P, \quad \Rightarrow \chi = 0.$$



Sisi Zhou

Proof technique: Charge fluctuation



Charge fluctuation $\chi := \langle c_0 | H_S | c_0 \rangle - \langle c_1 | H_S | c_1 \rangle$.

$|c_0\rangle$ and $|c_1\rangle$ correspond to the largest and the smallest eigenvalues of H_L .

When the code is exactly covariant ($\delta = 0$) and the encoding isometry is W ,

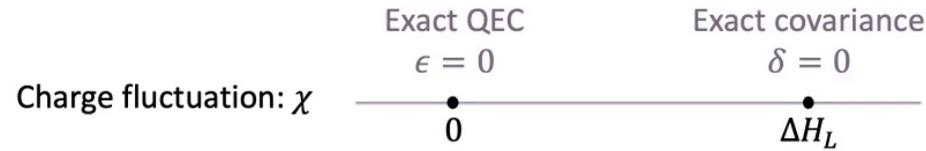
$$\exists v \in \mathbb{R}, \quad W^\dagger H_S W = H_L - v \mathbb{I}_L, \quad \text{[Faist et al. PRX'20]}$$

$$\Rightarrow \chi = \langle 0_L | W^\dagger H_S W | 0_L \rangle - \langle 1_L | W^\dagger H_S W | 1_L \rangle = \Delta H_L.$$



Sisi Zhou

Proof technique: Charge fluctuation



Charge fluctuation $\chi := \langle c_0 | H_S | c_0 \rangle - \langle c_1 | H_S | c_1 \rangle$.

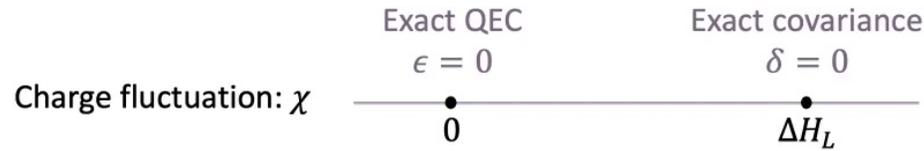
$|c_0\rangle$ and $|c_1\rangle$ correspond to the largest and the smallest eigenvalues of H_L .

$$\delta \gtrsim \sqrt{\frac{|\Delta H_L - \chi|}{\Delta H_S}}$$



Sisi Zhou

Proof technique: Charge fluctuation



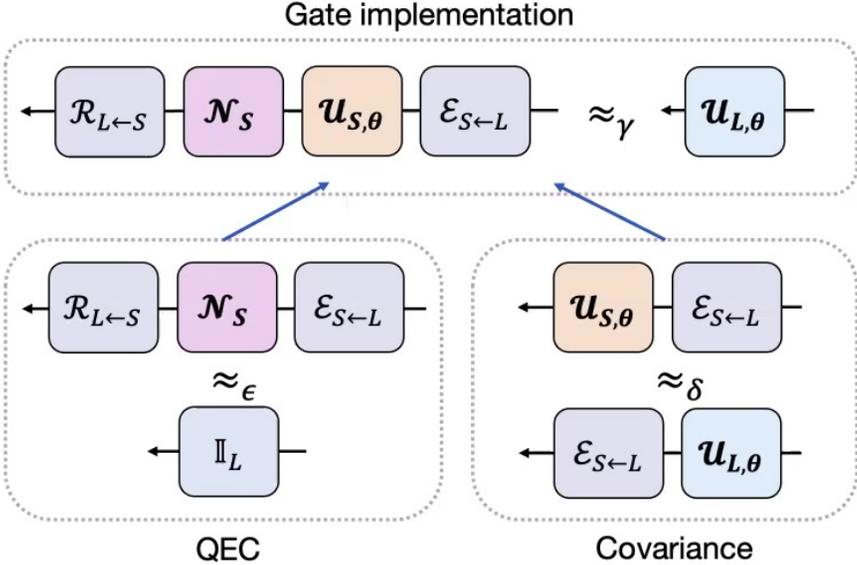
Charge fluctuation $\chi := \langle c_0 | H_S | c_0 \rangle - \langle c_1 | H_S | c_1 \rangle$.

$|c_0\rangle$ and $|c_1\rangle$ correspond to the largest and the smallest eigenvalues of H_L .

$$\delta \gtrsim \sqrt{\frac{\Delta H_L - 2\epsilon\tilde{\mathfrak{J}}}{\Delta H_S}} \iff \delta \gtrsim \sqrt{\frac{|\Delta H_L - \chi|}{\Delta H_S}} + |\chi| \leq 2\epsilon\tilde{\mathfrak{J}}$$



Proof technique: Gate implementation error

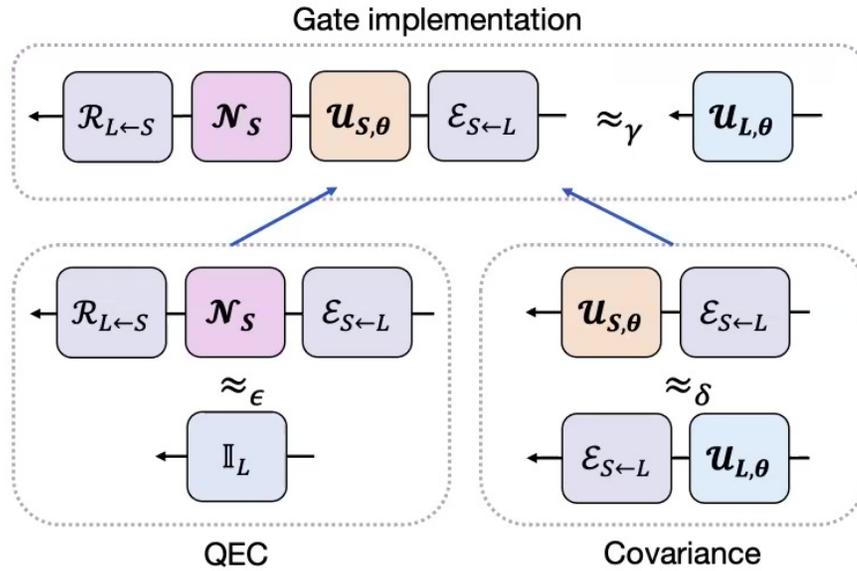


γ : Distance measure between $\mathcal{R}_{L \leftarrow S} \circ \mathcal{N}_S \circ \mathcal{U}_{S, \theta} \circ \mathcal{E}_{S \leftarrow L}$ and $\mathcal{U}_{L, \theta}$



Sisi Zhou

Proof technique: Gate implementation error



$$\delta + \epsilon \geq \gamma$$

$$+$$

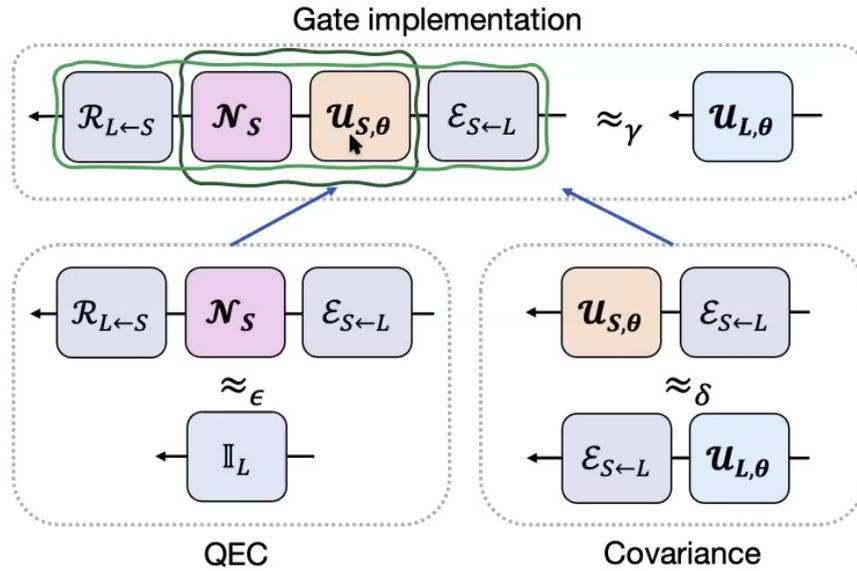
$$\gamma \geq \frac{\Delta H_L}{\sqrt{4\mathfrak{F}}}$$

γ : Distance measure between $\mathcal{R}_{L \leftarrow S} \circ \mathcal{N}_S \circ \mathcal{U}_{S, \theta} \circ \mathcal{E}_{S \leftarrow L}$ and $\mathcal{U}_{L, \theta}$



Sisi Zhou

Proof technique: Gate implementation error



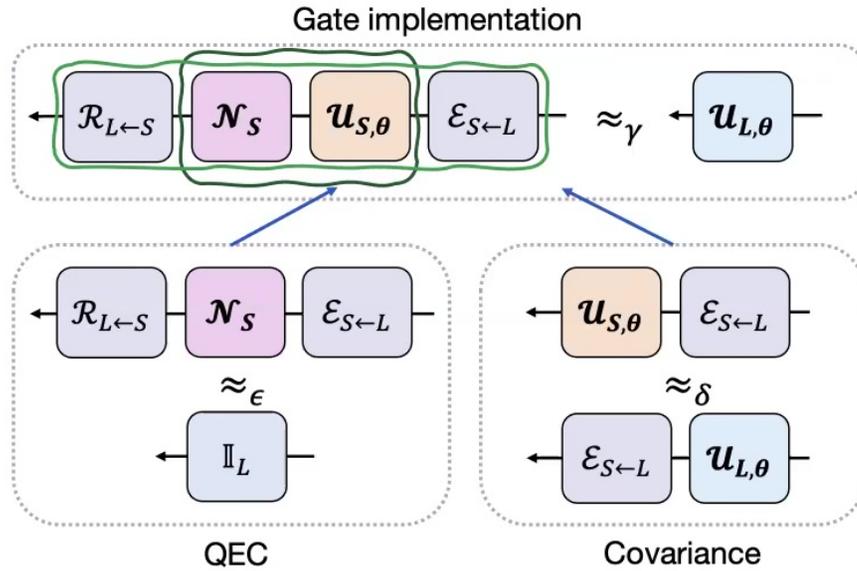
$$\begin{aligned}
 & \delta + \epsilon \geq \gamma \\
 & \quad + \\
 & \gamma \gtrsim \frac{\Delta H_L}{\sqrt{4\mathfrak{F}}} \quad \mathfrak{F} \gtrsim \frac{(\Delta H_L)^2}{4\gamma^2}
 \end{aligned}$$

γ : Distance measure between $\mathcal{R}_{L \leftarrow S} \circ \mathcal{N}_S \circ \mathcal{U}_{S, \theta} \circ \mathcal{E}_{S \leftarrow L}$ and $\mathcal{U}_{L, \theta}$



Sisi Zhou

Proof technique: Gate implementation error



$$\delta + \epsilon \gtrsim \frac{\Delta H_L}{\sqrt{4\mathfrak{F}}}$$

$$\Updownarrow$$

$$\delta + \epsilon \geq \gamma$$

$$+$$

$$\gamma \gtrsim \frac{\Delta H_L}{\sqrt{4\mathfrak{F}}}$$

$$\mathfrak{F} \gtrsim \frac{(\Delta H_L)^2}{4\gamma^2}$$

γ : Distance measure between $\mathcal{R}_{L \leftarrow S} \circ \mathcal{N}_S \circ \mathcal{U}_{S, \theta} \circ \mathcal{E}_{S \leftarrow L}$ and $\mathcal{U}_{L, \theta}$

Summary and outlook

- Tradeoff relations between QEC and continuous symmetries.
- The relations are near-optimal in certain cases, shown by construction.
- Other types of tradeoff relations based on different symmetry measures; Other proof techniques based on quantum metrology, quantum resource theory, etc.
- Application in fault-tolerant quantum computation.
- Potential physical applications in quantum gravity (AdS/CFT, black hole information problem), many-body physics, etc.

Thank you!

