Title: Strong Gravity

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Collection: Strong Gravity (2021/2022)

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Abstract: This course will introduce some advanced topics in general relativity related to describing gravity in the strong field and dynamical regime. Topics covered include properties of spinning black holes, black hole thermodynamics and energy extraction, how to define horizons in a dynamical setting, formulations of the Einstein equations as constraint and evolution equations, and gravitational waves and how they are sourced.

Gravitational Waves

Weak field regime

$$g_{ab} = \eta_{ab} + h_{ab} \qquad |h_{ab}| << |\eta_{ab}|$$

$$g^{ab} = \eta^{ab} - h^{ab}$$

Use harmonic gauge: $\Box x^a = 0$

$$\eta^{cd}\partial_c\partial_d h_{ab} = -16\pi\left(T_{ab} - \frac{1}{2}T g_{ab}\right)$$

$$\Box\left(h_{ab} - \frac{1}{2}\eta_{ab}h^{cd}\eta_{cd}\right) = -16\pi T_{ab}$$

call $\bar{h}_{ab}$

$$\Box x^a = 0 \iff \partial_a \bar{h}^{ab} = 0 \quad \text{(Lorenz gauge)}$$

In EM $A^a \to A^a + \partial_a \phi$ $\quad \partial_a \partial^a \phi = 0$

$\partial A^a = 0$ unchanged

Here: $x^a \to x^a + \xi^a$ with $\Box \xi^a = 0$

keep harmonic gauge

Propagation of GWs

In wavezone use 4 gauge DOF to set

$$h^{TT}_{0i} = 0 \quad \text{and} \quad \eta^{ab} h^{TT}_{ab} = 0 \quad \text{(traceless)}$$

$$\bar{h}_{ab} = h^{TT}_{ab}$$

Propagation of GWs

In wavezone use 4 gauge DOF to set

$$h^{TT}_{0i} = 0 \quad \text{and} \quad \eta^{as} h^{TT}_{as} = 0 \ \text{(traceless)}$$

$$\bar{h}_{as} = h^{TT}_{as}$$

Lorenz condition $0 = \partial_a h^{a0}_{TT} = -\partial_0 h^{00}_{TT}$

In vacuum $\partial_\alpha \partial^\alpha h^{TT}_{00} = -16\pi \cancel{T_{00}}^{\to 0 \text{ (vacuum)}} = 0$

$h^{00} = 0$ initially $\Rightarrow$ $h_{00} = 0$ for all time

$h^{TT}_{0a} = 0$

$\Box h^{TT}_{ij} = 0$

$h^{TT}_{ij} = A_{ij} \exp(i k_a x^a)$: Plane waves

A_{ij} ↑ constant

$$\Box^{TT} = \eta^{ab}(ik_a)(ik_b) = -k_a k^a h_{ij}^{TT}$$

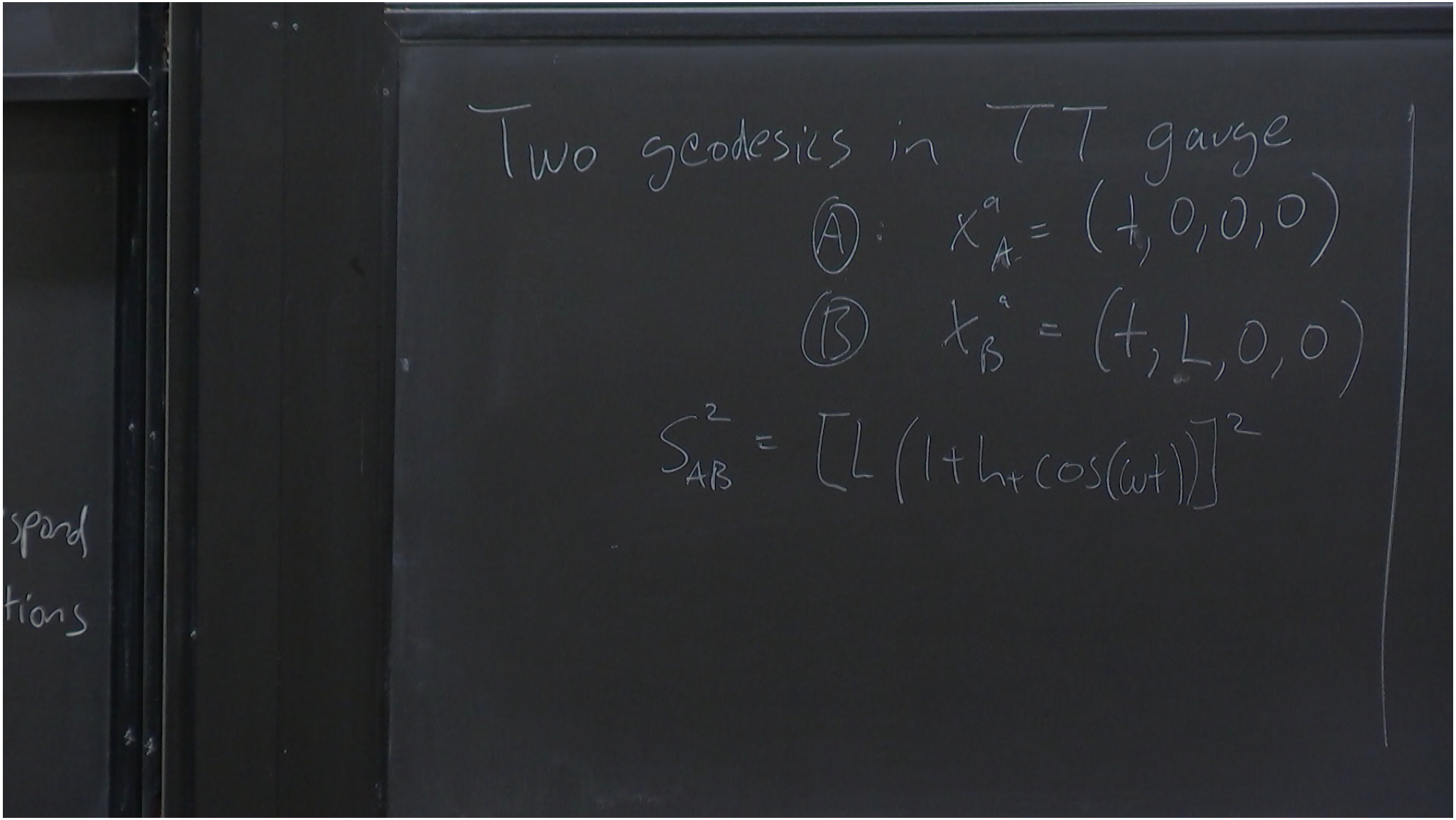
$= 0 \iff k^a$ is null

$$ik^a h^{ab}_{TT} = 0 \quad \text{(transverse)}$$

$(0,0,\omega)$ (propagates in z-direction)

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} A^{iz} = 0 \quad , \quad A^i_i = 0$$

$h_+, h_\times$ correspond to polarizations of GWs


Two geodesics in TT gauge
A: $x^a_A = (t, 0, 0, 0)$
B: $x^a_B = (t, L, 0, 0)$
$S^2_{AB} = [L(1 + h_+ \cos(\omega t))]^2$

Two geodesics in TT gauge

Ⓐ: $x^a_A = (t, 0, 0, 0)$

Ⓑ: $x^a_B = (t, L, 0, 0)$

$$S^2_{AB} = [L(1 + h_+ \cos(\omega t))]^2$$

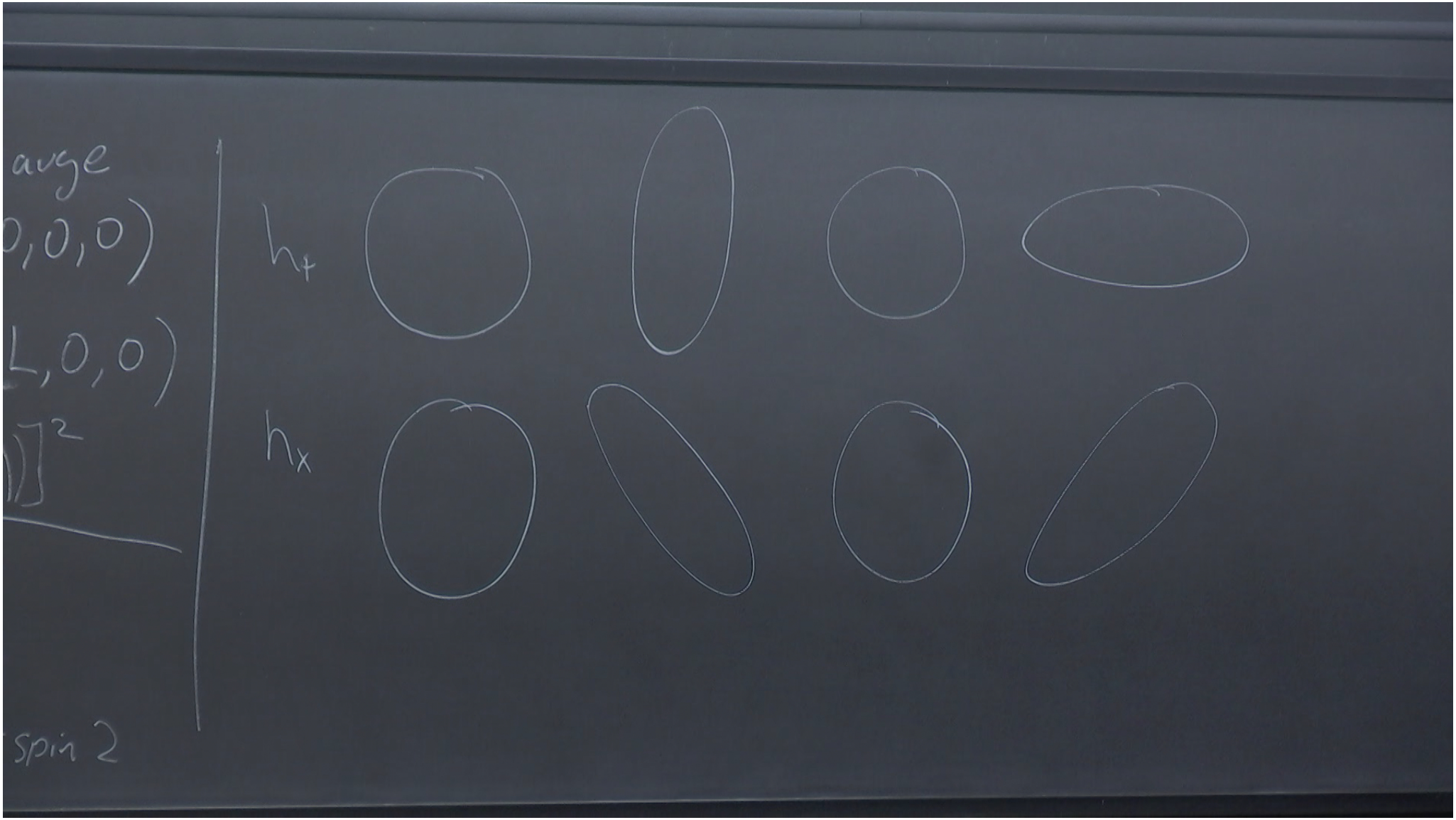
h_+

$h_\times$

Rotate by ϕ about z-axis

$$h_\times \pm i h_+ \to (h_\times \pm i h_+) e^{\mp 2i\phi}$$

spin 2


auge
0,0,0)
L,0,0)
$]^2$
h_+
h_x
spin 2

Generation of GWs in Weak field Regime

$$\Box \bar{h}_{ab} = -16\pi T_{ab}$$

Green function: $\Box G(x^a - \bar{x}^a) = \delta^4(x^a - \bar{x}^a)$

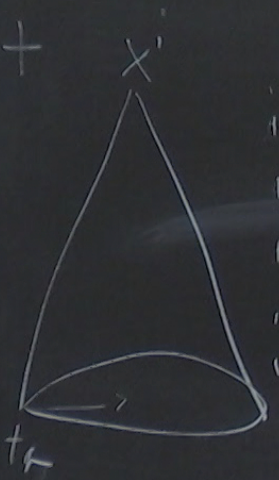
$$G(x^a - \bar{x}^a) = \frac{-1}{4\pi |x^i - \bar{x}^i|} \delta\left[|x^i - \bar{x}^i| - (x^0 - \bar{x}^0) \right] \Theta(x^0 - \bar{x}^0)$$

$$\Theta(x^0 - \bar{x}^0) = \begin{cases} 1 & \text{when } x^0 - \bar{x}^0 > 0 \\ 0 & \text{otherwise} \end{cases}$$

(for retarded soluti

$$\bar{h}_{ab}(x^a) = -16\pi \int G(x^a - \bar{x}^a)\, T_{ab}(\bar{x}^a)\, d^4\bar{x}$$

$$= 4 \int \frac{T_{ab}(t - |x^i - \bar{x}^i|, \bar{x}^i)}{|x^i - \bar{x}^i|}\, d^3\bar{x}$$

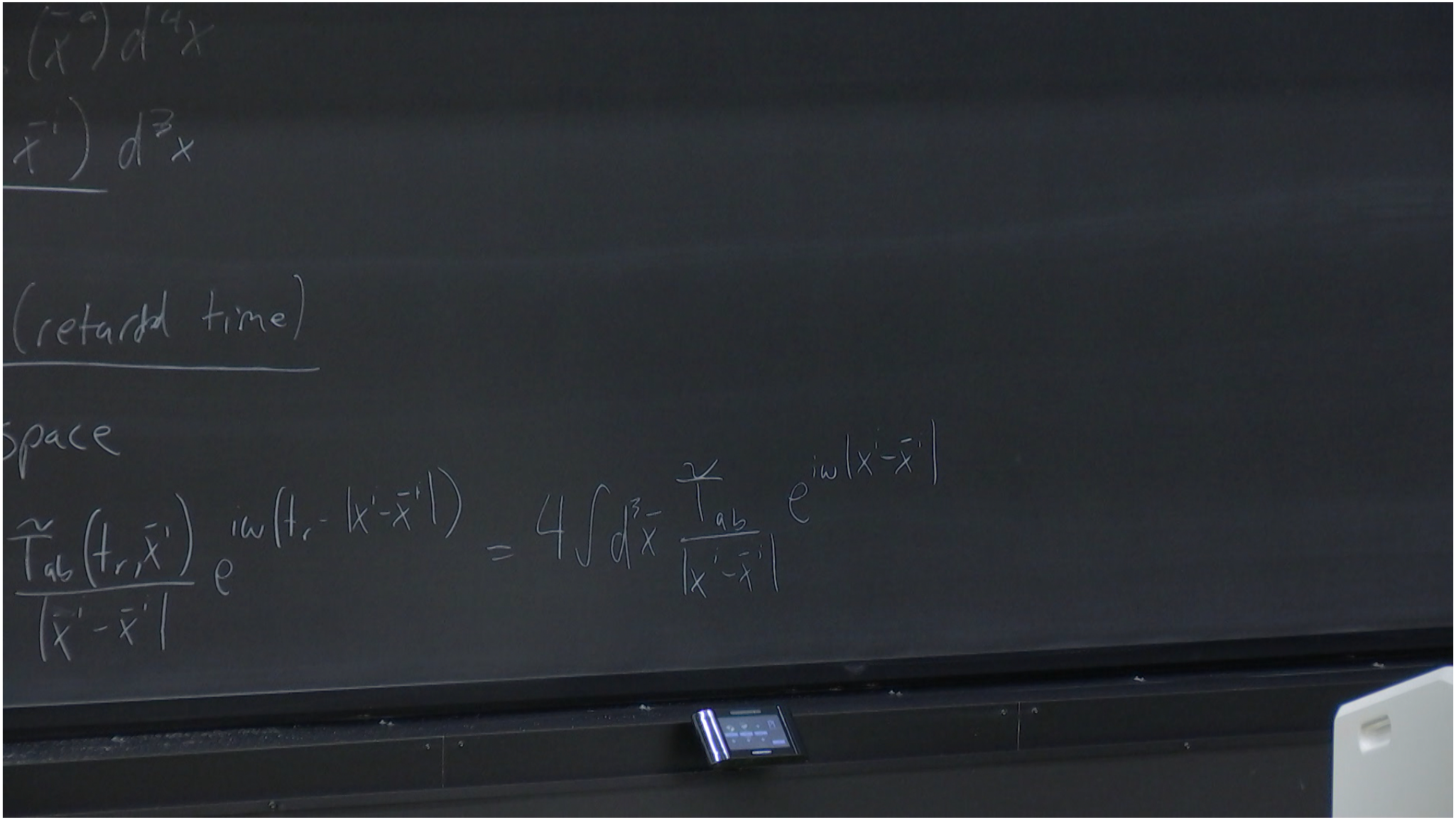


$t_r = t - |x^i - \bar{x}^i|$ (retarded time)

Transform to Fourier space

$$\tilde{\bar{h}} = \int dt\, e^{i\omega t}\, \bar{h}_{ab}$$

$$= 4 \int dt_r \int d^3\bar{x}\, \frac{\tilde{T}_{ab}(t_r, \bar{x}^i)}{|x^i - \bar{x}^i|}\, e^{i\omega(t_r - |x^i - \bar{x}^i|)} =$$


$(\bar{x}^a)\, d^4x$
$\bar{x}^i)\, d^3x$
(retarded time)
space
$\frac{\tilde{T}_{ab}(t_r, \bar{x}^i)}{|x^i - \bar{x}^i|} e^{i\omega(t_r - |x^i - \bar{x}^i|)} = 4\int d^3\bar{x}\, \frac{T_{ab}}{|x^i - \bar{x}^i|} e^{i\omega|x^i - \bar{x}^i|}$

$$\Theta(x^0-\bar{x}^0) = \begin{cases} 1 & \text{when } x^0-\bar{x}^0 > 0 \\ 0 & \text{otherwise} \end{cases}$$ (for retarded solution)

$$h_{ab}(x^a) = -16\pi \int G(x^a-\bar{x}^a)\, T_{ab}(\bar{x}^a)\, d^4\bar{x}$$

$$= 4 \int \frac{T_{ab}(t-|x'-\bar{x}'|, \bar{x}')}{|x'-\bar{x}'|}\, d^3\bar{x}$$



$t_r = t - |x'-\bar{x}'|$ (retarded time)

Transform to Fourier space

$$\hat{h} = \int dt\, e^{i\omega t}\, h_{ab}$$

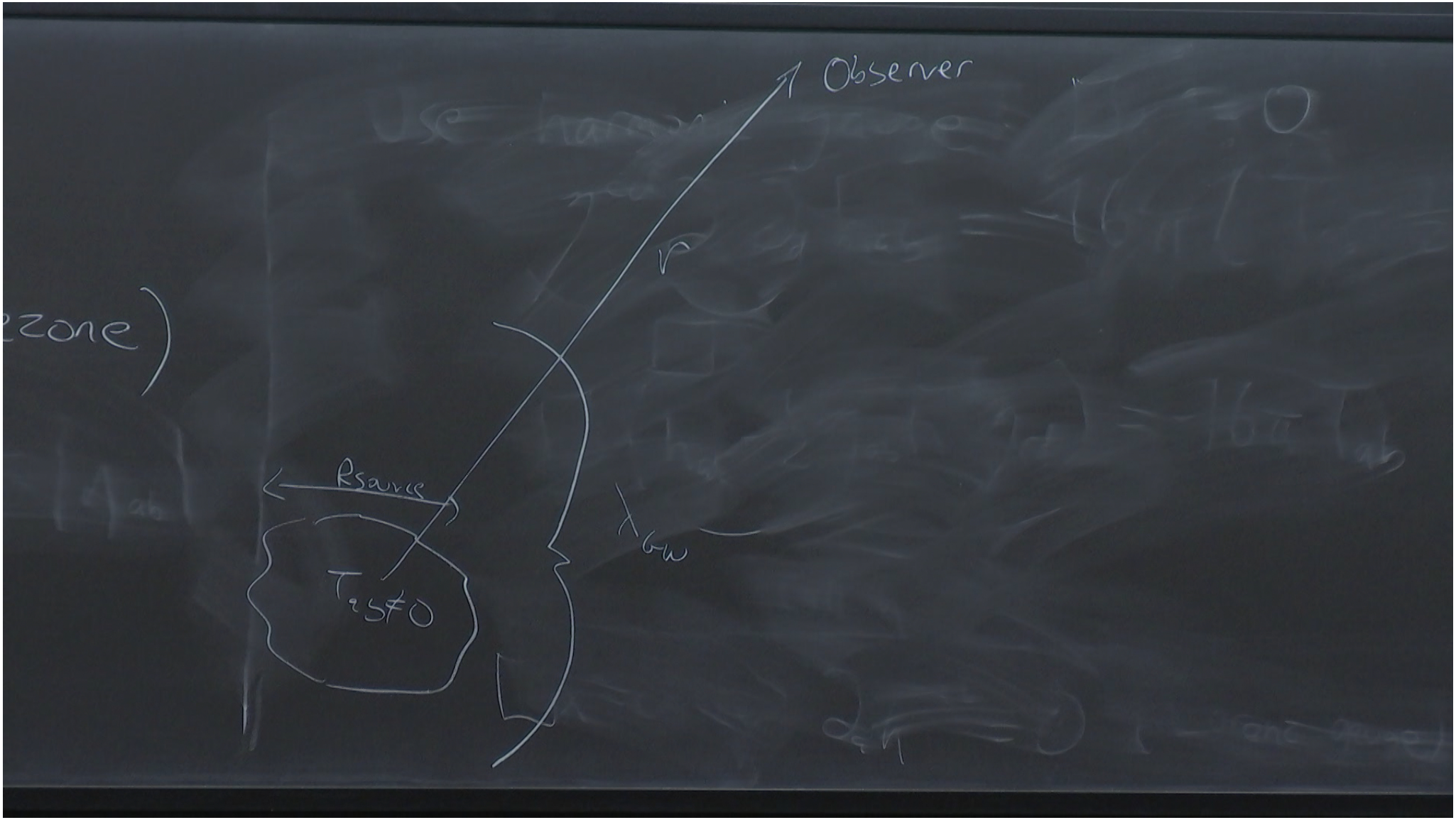
$$= 4 \int dt_r \int d^3\bar{x}\, \frac{T_{ab}(t_r, \bar{x}')}{|x'-\bar{x}'|}\, e^{i\omega(t_r + |x'-\bar{x}'|)} = 4 \int d^3\bar{x}\, \frac{\hat{T}_{ab}}{|x'-\bar{x}'|}\, e^{i\omega|x'-\bar{x}'|}$$

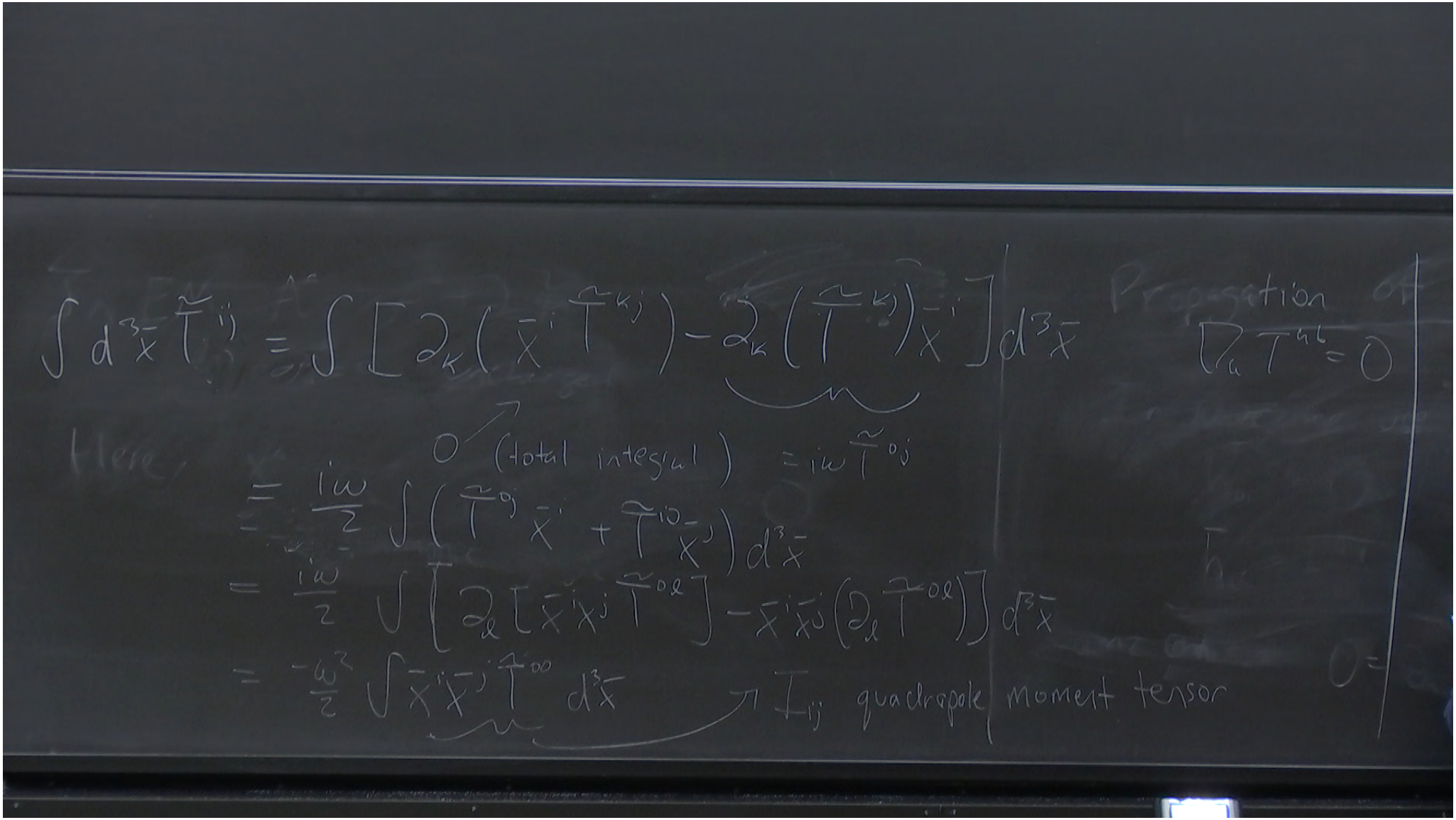
$\tilde{h}^{0a} = \frac{i}{\omega} \partial_i \tilde{h}^{ia}$ Waves

Assume $r \gg \frac{1}{\omega} = \lambda_{GW}$ (wavezone)

$R_{source} \ll \lambda_{GW} = \omega^{-1}$

$$\frac{e^{i\omega|\vec{x}'-\vec{x}'|}}{|\vec{x}'-\vec{x}'|} \approx \frac{e^{i\omega r}}{r}$$


Observer
r
Rsource
λ_{GW}
$T_{zz} \neq 0$


$\int d^3\bar{x}\,\tilde{T}^{ij} = \int [\partial_k(\bar{x}^i \tilde{T}^{kj}) - \partial_k(\tilde{T}^{kj})\bar{x}^i]\, d^3\bar{x}$
Propagation of
$\partial_\mu T^{\mu\nu} = 0$
0 (total integral)
$= i\omega \tilde{T}^{0j}$
$= \frac{i\omega}{2} \int (\tilde{T}^{0j}\bar{x}^i + \tilde{T}^{i0}\bar{x}^j)\, d^3\bar{x}$
$= \frac{i\omega}{2} \int [\partial_\ell[\bar{x}^i\bar{x}^j \tilde{T}^{0\ell}] - \bar{x}^i\bar{x}^j(\partial_\ell \tilde{T}^{0\ell})]\, d^3\bar{x}$
$= -\frac{\omega^2}{2} \int \bar{x}^i\bar{x}^j \tilde{T}^{00}\, d^3\bar{x}$
I_{ij} quadrupole moment tensor

Hence

$$\tilde{h}_{ij} = -2\omega^2 \frac{e^{i\omega r}}{r} \tilde{I}_{ij}$$

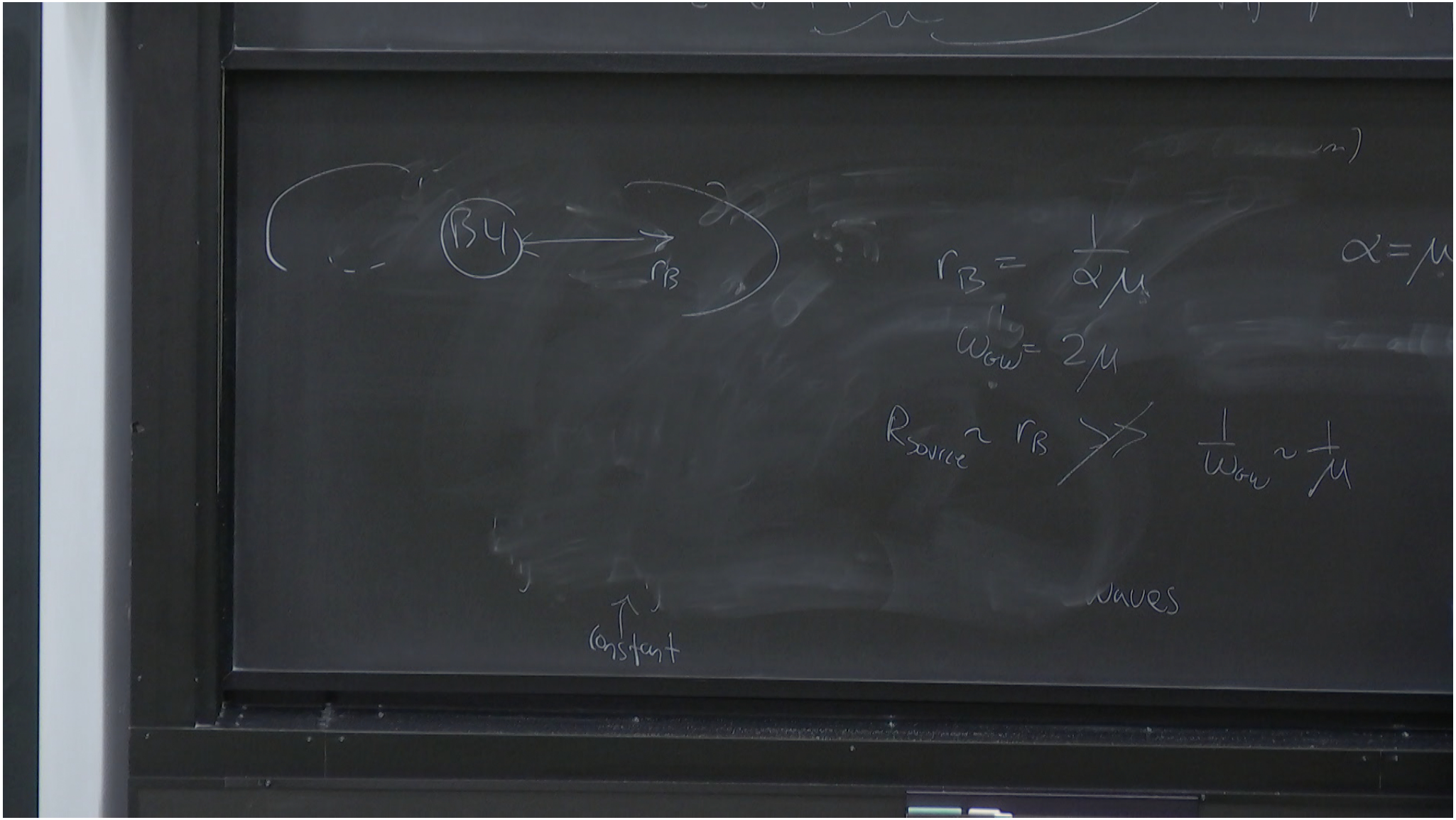
$$h_{ij} = \frac{2}{r} \ddot{I}_{ij}(t-r)$$

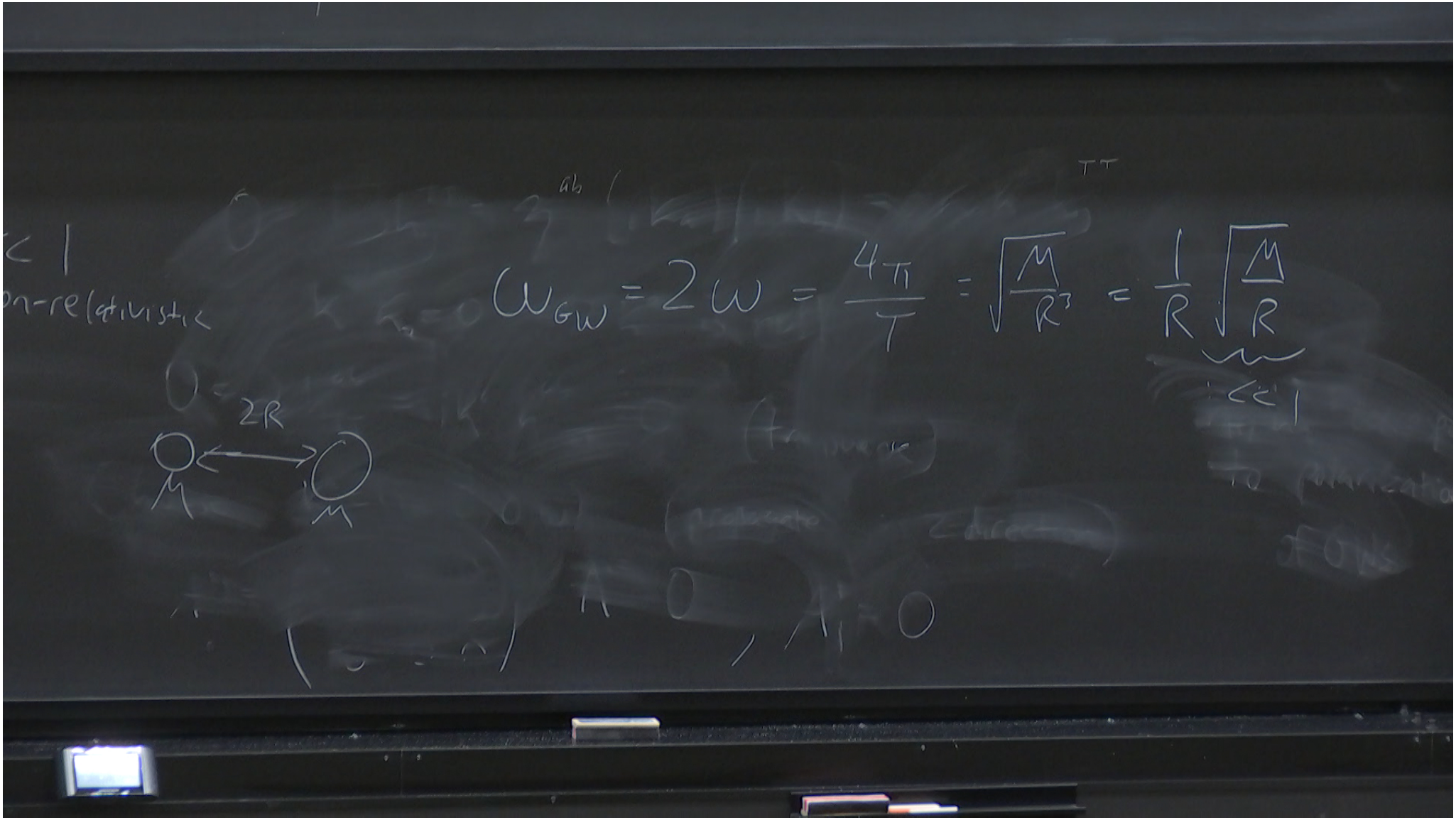
Lower order moments of $\rho = T^{00}$

Monopole: $\int d^3x \, \rho(t, x') = M(t)$

Dipole radiation?

$$\int d^3x \, \rho x' = D'(t)$$


BH
r_B
$r_B = \frac{1}{\alpha \mu}$
$\omega_{GW} = 2\mu$
$R_{source} \sim r_B \gg \frac{1}{\omega_{GW}} \sim \frac{1}{\mu}$
waves
constant


$\omega_{GW} = 2\omega = \frac{4\pi}{T} = \sqrt{\frac{M}{R^3}} = \frac{1}{R}\sqrt{\frac{M}{R}}$
$\ll 1$
2R
M
M

Energy & Angular momentum in GWs

$$g_{ab} = \eta_{ab} + h^{(1)}_{ab} + h^{(2)}_{ab}$$

$h^{(2)}_{ab}$: second order

$$R_{ab} = R^{(0)}_{ab} + R^{(1)}_{ab} + R^{(2)}_{ab}$$

($R^{(0)}_{ab} \to 0$)

First order $\quad G^{(1)}_{ab}[h^{(1)}_{cd}] = 0$

Second order $\quad G^{(1)}_{ab}[h^{(2)}_{cd}] + G^{(2)}_{ab}[h^{(1)}_{cd}] = 0$

GWs

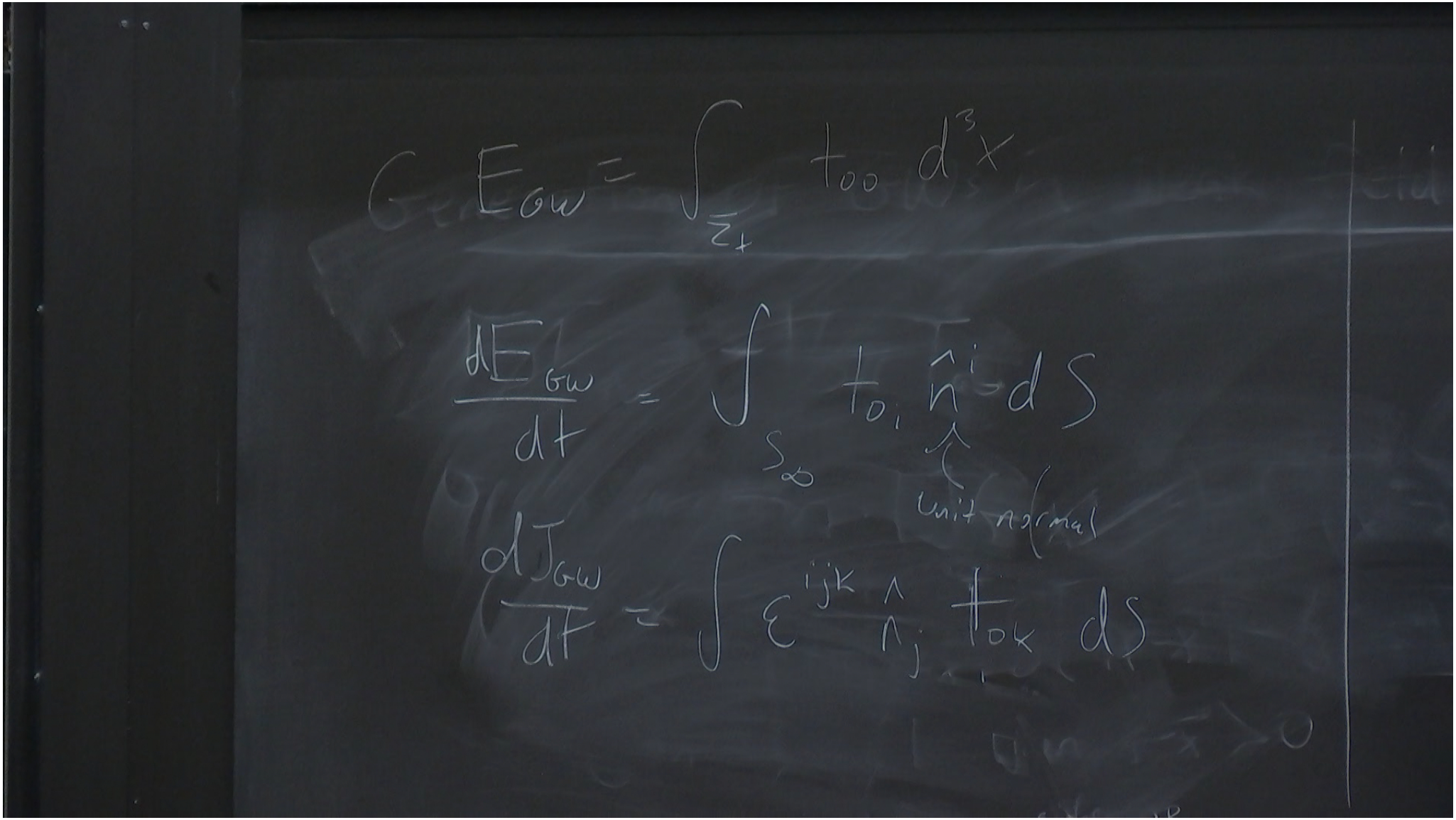
$$G^{(1)}_{ab}[h^{(2)}_{cd}] = 8\pi t_{ab} := -G^{(2)}_{ab}[h^{(1)}_{cd}]$$

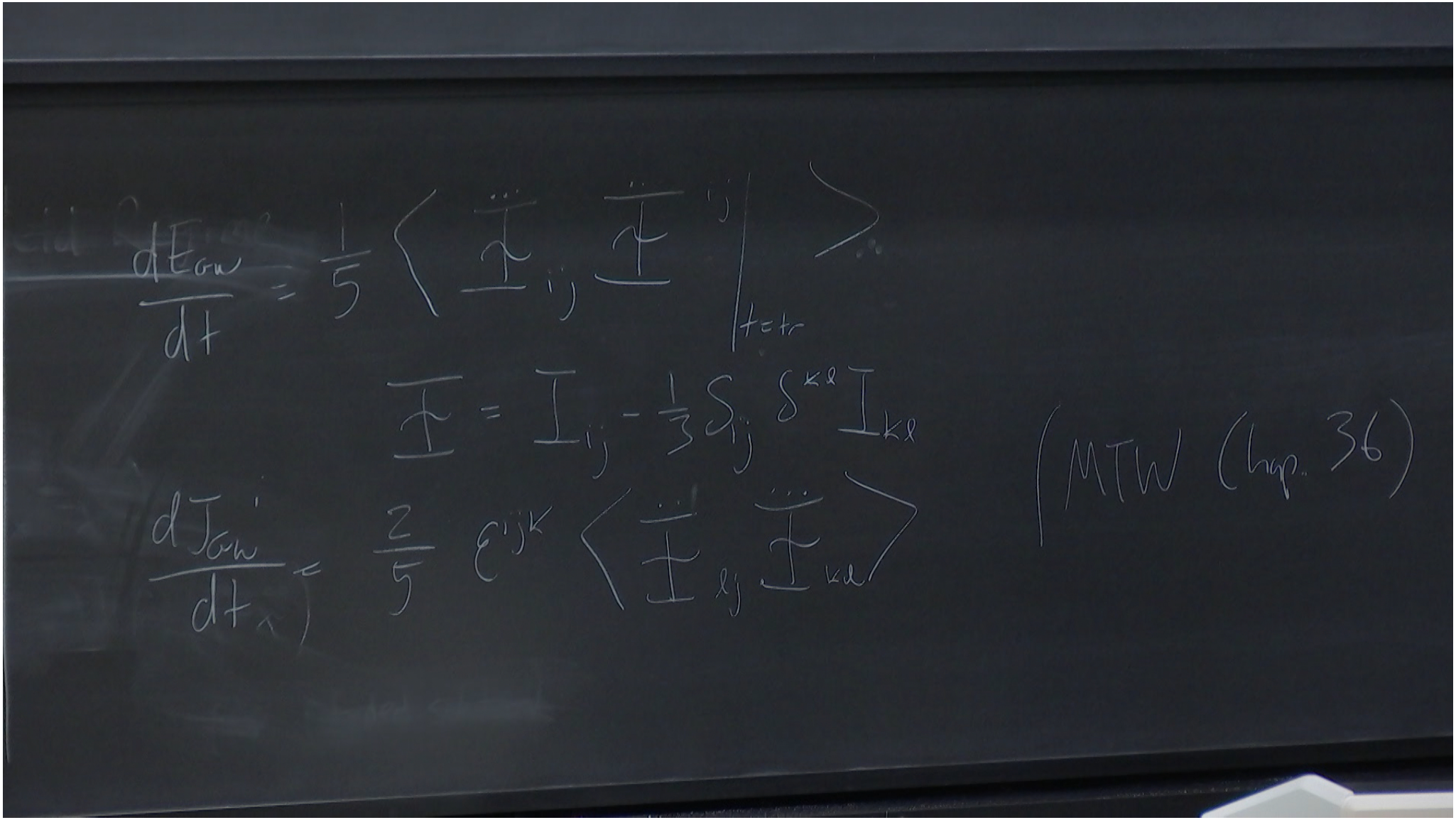
Caution: not gauge invariant

Get around averaging over several GW wavelengths

$$t_{ab} = \frac{1}{32\pi} \langle (\partial_a h^{TT}_{cd})(\partial_b h^{cd}_{TT}) \rangle$$

$$\langle \partial_a X \rangle = 0, \quad \langle X \partial_a Y \rangle = -\langle Y \partial_a X \rangle$$


$E_{GW} = \int_{\Sigma_t} t_{00}\, d^3x$
$\frac{dE_{GW}}{dt} = \int_{S_\infty} t_{0i}\, \hat{n}^i\, dS$
unit normal
$\frac{dJ_{GW}}{dt} = \int \varepsilon^{ijk}\, \hat{n}_j\, t_{0k}\, dS$


$\frac{dE_{GW}}{dt} = \frac{1}{5}\left\langle \dddot{\mathcal{I}}_{ij}\,\dddot{\mathcal{I}}^{ij} \right\rangle_{t=tr}$
$\mathcal{I}_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}\,\delta^{kl} I_{kl}$
$\frac{dJ^{i}_{GW}}{dt} = \frac{2}{5}\,\varepsilon^{ijk}\left\langle \ddot{\mathcal{I}}_{lj}\,\dddot{\mathcal{I}}_{kl} \right\rangle$
(MTW (chap. 36)