Title: Strong Gravity

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Collection: Strong Gravity (2021/2022)

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Abstract: This course will introduce some advanced topics in general relativity related to describing gravity in the strong field and dynamical regime. Topics covered include properties of spinning black holes, black hole thermodynamics and energy extraction, how to define horizons in a dynamical setting, formulations of the Einstein equations as constraint and evolution equations, and gravitational waves and how they are sourced.

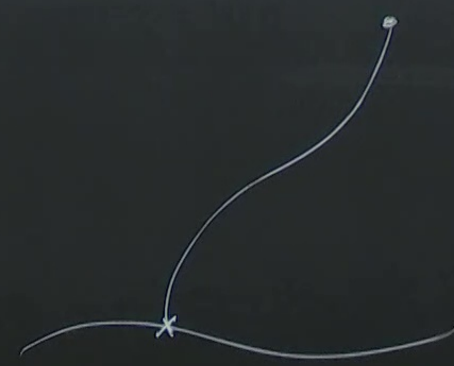
Well posedness (Hadamard)

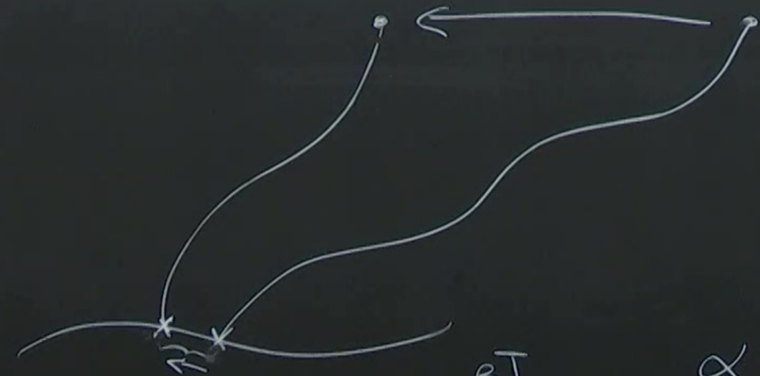
if

i) Exists a soln

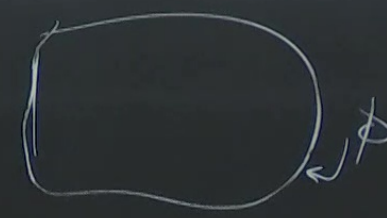
ii) the soln is unique

iii) (Must depend continuously on the "configuration" data)





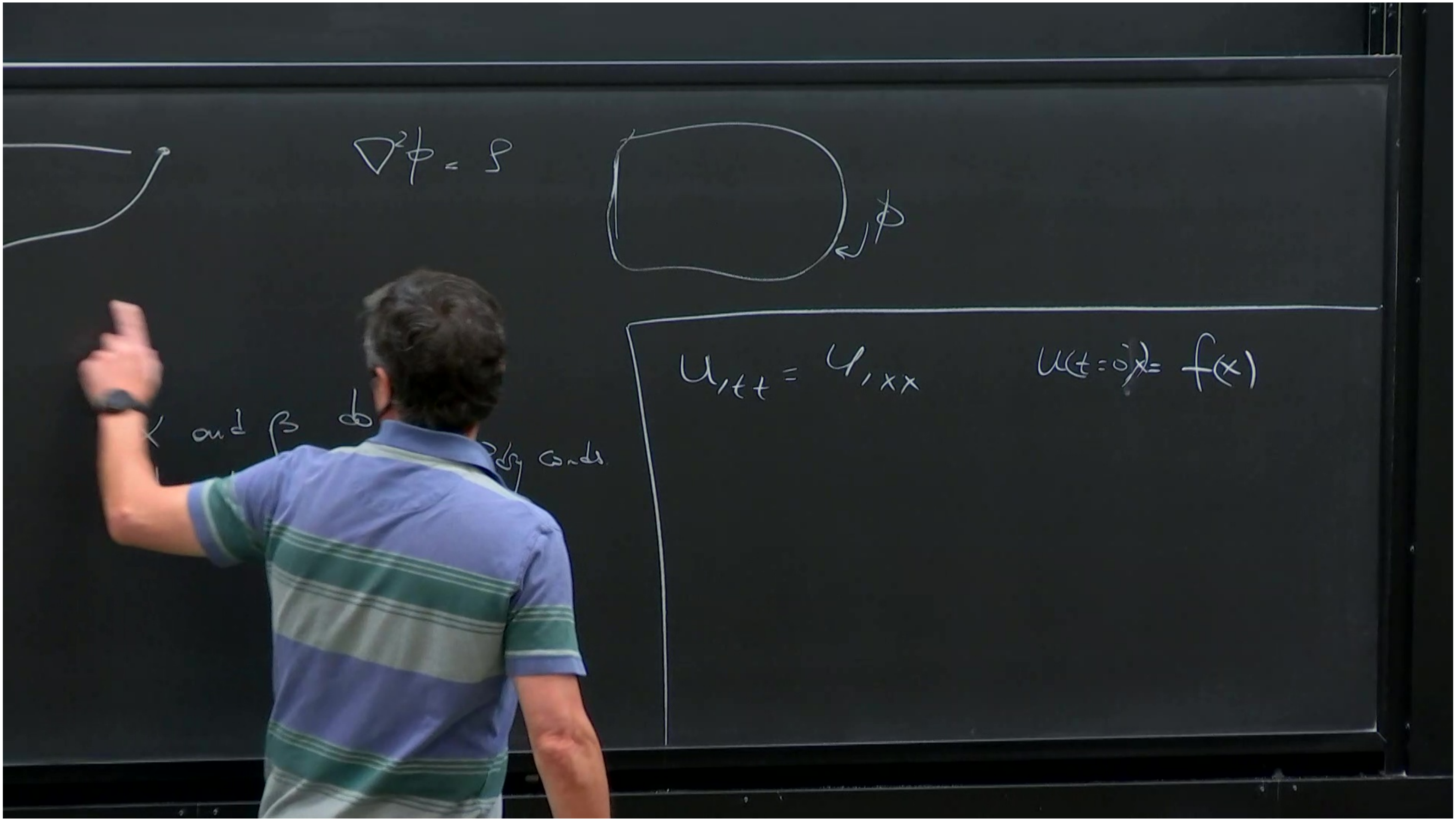
$\nabla^2 \phi = \rho$



the)

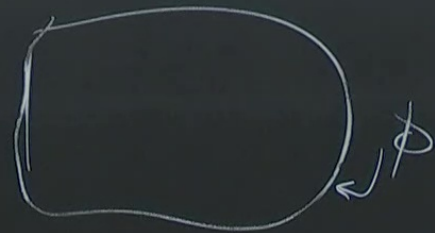
$$\|u(T)\| \leq \|u(0)\|\, \alpha\, e^{\beta T}$$

α and β don't depend on Initial / Bdy conds



$\nabla^2\phi = \rho$
ϕ
$u_{,tt} = u_{,xx}$
$u(t=0,x) = f(x)$
and β
bdy conds

$\nabla^2 \phi = \rho$

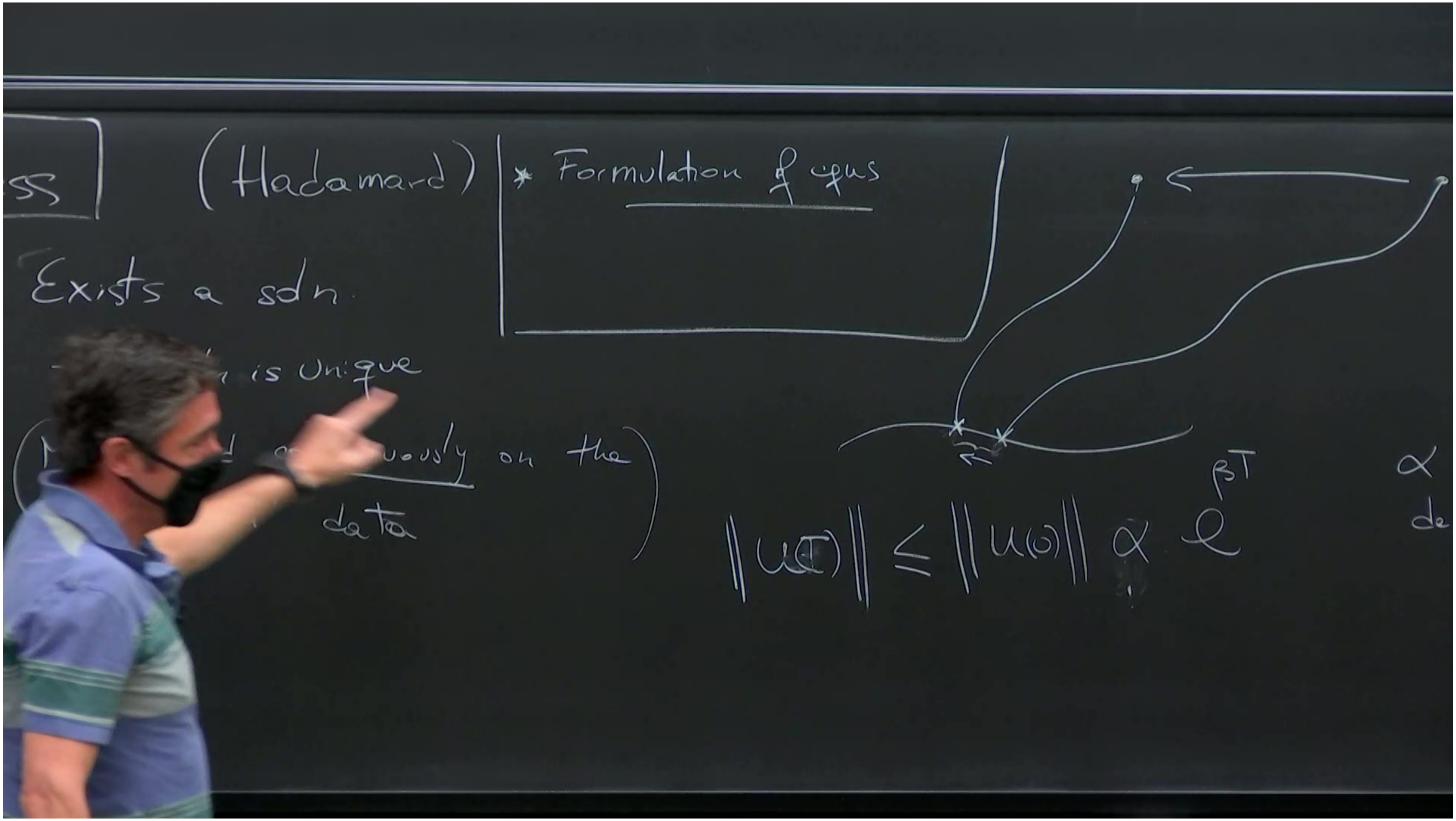


α and β don't depend on Initial / Bdry conds

$u_{,tt} = u_{,xx}$

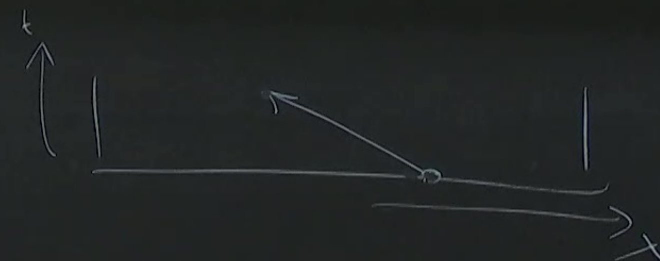
$u(t=0, x) = f(x)$

$u_{,t}$?


(Hadamard)
* Formulation of eqns
Exists a soln
is Unique
on the
data
||u(T)|| ≤ ||u(0)|| α e^{βT}
α
de

1) <u>Advection Eqn</u> $u_{,t} = \lambda\, u_{,x}$ is $\lambda \gtrless 0$ Real? Complex?

(i) $u_{,t} = u_{,x} \longrightarrow f(t+x)$



- Characteristics

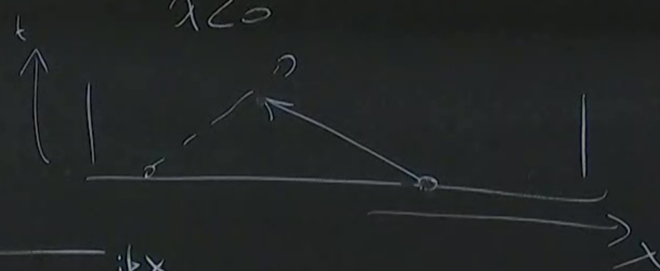
$df \quad = \quad df$

Hyperbolic problems

- Parabolic problems $\partial_t T = k \partial_x^2 T$
- Elliptic problems. $\nabla^2 \phi = \rho$

1) Advection Eqn $\quad u_{,t} = \lambda u_{,x} \quad$ is $\lambda \gtrless 0$ Real? complex

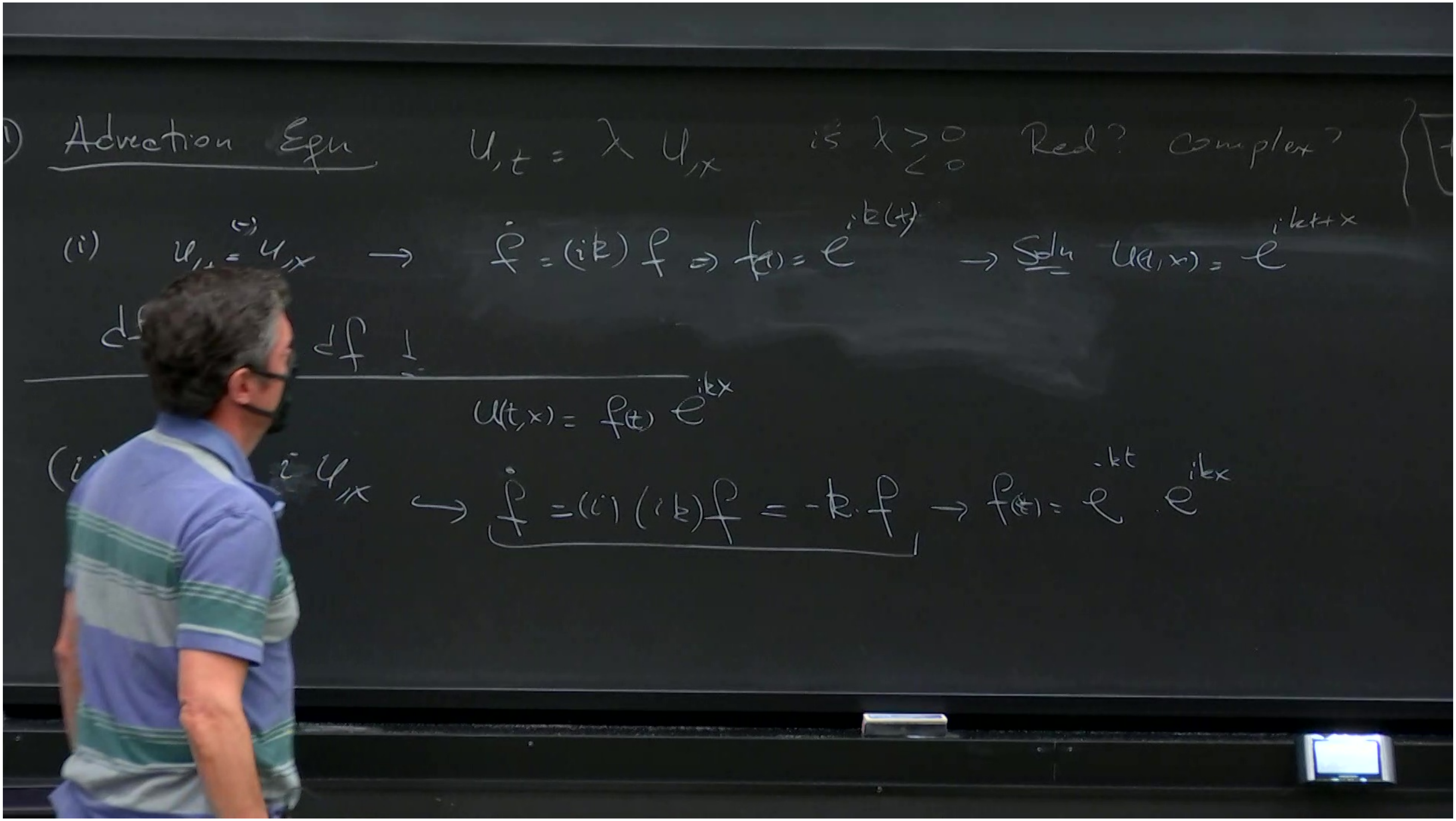
(i) $u_{,t} \overset{(-)}{=} u_{,x} \longrightarrow f(t \overset{(-)}{+} x)$ $\quad \lambda < 0$ $\quad$ - Characteristi



$df \; 1 = df \; 1$

$u(t,x) = f(t)\, e^{ikx}$

(ii) $u_{,t} = i\, u_{,x} \longrightarrow \underbrace{\dot f = (i)(ik) f = -k \cdot f}_{} \rightarrow f(t) = e^{-kt} \cdot e^{ikx}$



Advection Eqn
u,t = λ u,x
is λ > 0, < 0
Real? complex?
(i) u,t = u,x → ḟ = (ik) f ⇒ f(t) = e^{ik(t)} → Soln u(t,x) = e^{ikt+x}
u(t,x) = f(t) e^{ikx}
ḟ = (i)(ik) f = -k f → f(t) = e^{-kt} e^{ikx}

is $\lambda \geq 0$, < 0 Real? complex?

$\boxed{\text{Hyperbolic problems}}$

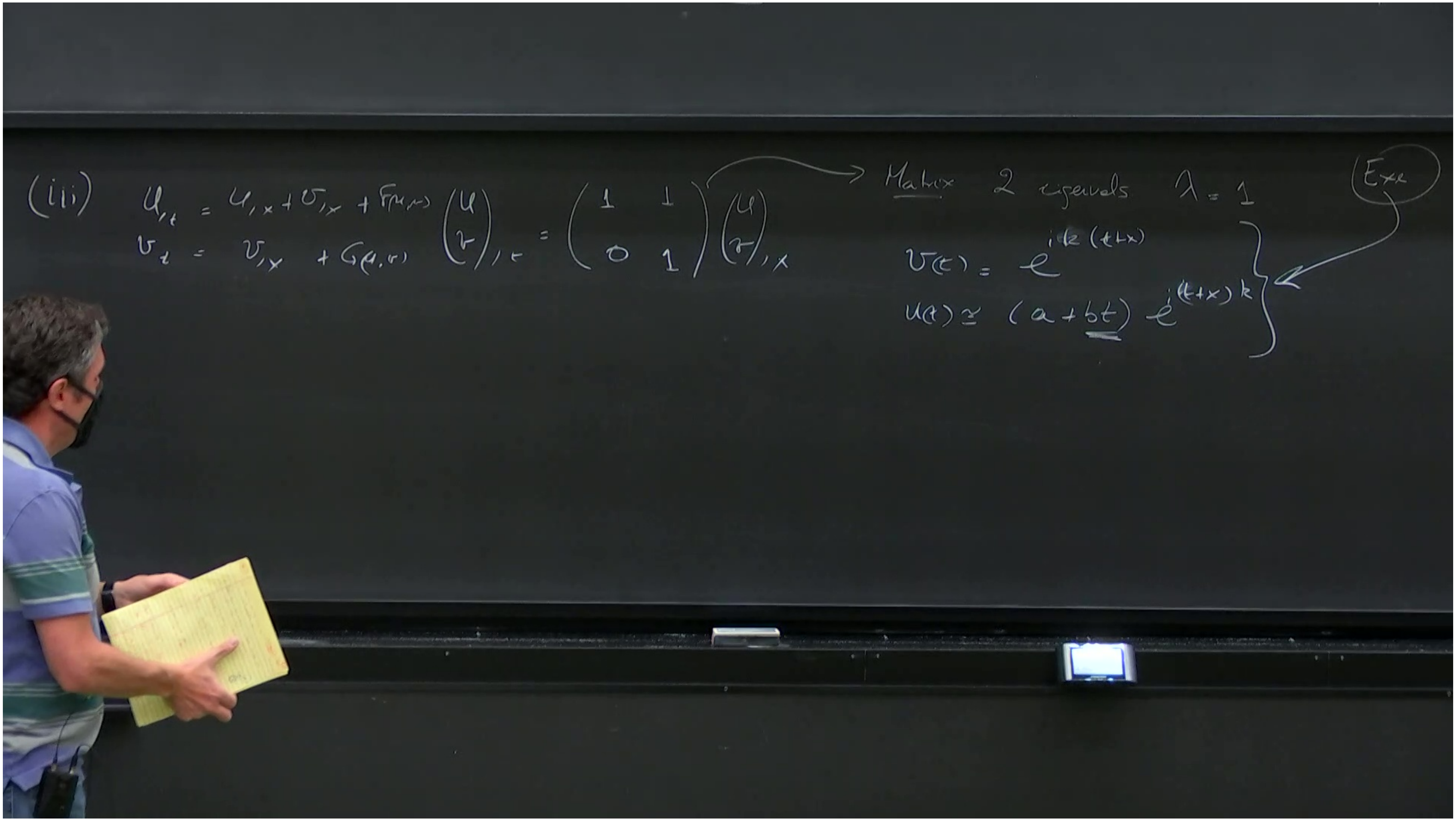
- Parabolic problems $\partial_t T =$
- Elliptic problems $\nabla^2 \phi =$

$e^{ik(t)} \rightarrow$ Soln $u(t,x) = e^{ik(t+x)} \Rightarrow |u(t)| \leq |u(0)|$

$u(t=0,x) = e^{ikx}$

take $k<0$

$-k \cdot f \rightarrow u(t,x) = e^{-kt} \cdot e^{ikx}$ $\quad \|u(t)\| \overset{?}{\leq} e^{-kt} |u(0)|$


(iii) $u_{,t} = u_{,x} + v_{,x} + F(u,v)$
$v_{t} = v_{,x} + G(u,v)$
$\begin{pmatrix} u \\ v \end{pmatrix}_{,t} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{,x}$
Matrix 2 eigenvals $\lambda = 1$
Exe
$v(t) = e^{ik(t+x)}$
$u(t) \simeq (a + bt)\, e^{i(t+x)k}$

→ Matrix 2 eigenvals $\lambda = 1$

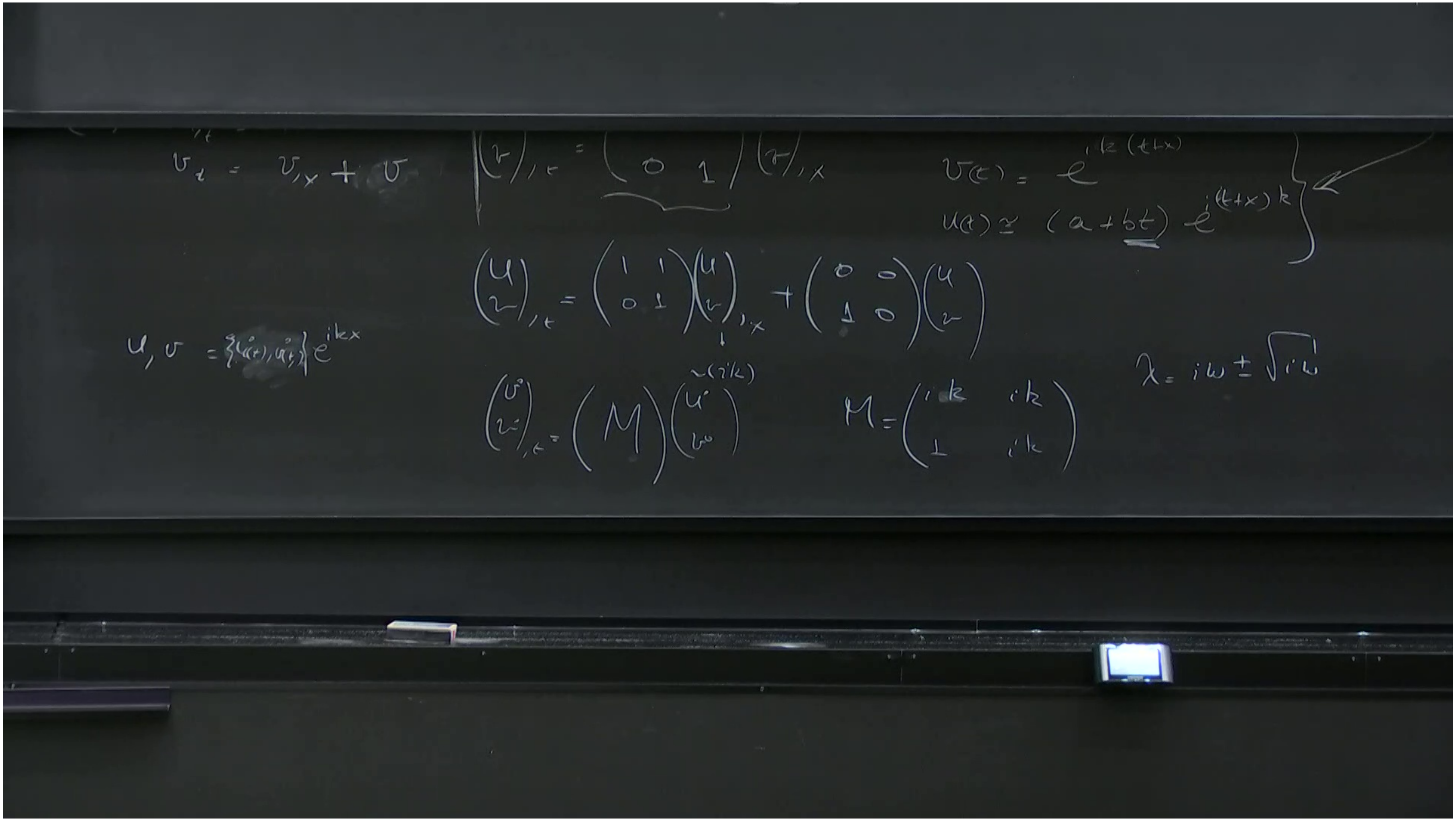
Exe

→ If λ's are not associated with a complete set of eigenvectors

x

$$v(t) = e^{ik(t+x)}$$

$$u(t) \simeq (a + bt)\, e^{i(t+x)k}$$



v_t = v_{,x} + v
u, v = {u(t), v(t)} e^{ikx}
(u v)_{,t} = (1 1; 0 1)(u v)_{,x} + (0 0; 1 0)(u v)
(u° v°)_{,t} = (M)(u° v°)
M = (ik ik; 1 ik)
v(t) = e^{ik(t+x)}
u(t) ≃ (a + bt) e^{i(t+x)k}
λ = iω ± √(iω)

A sufficient cond. for well posedness.

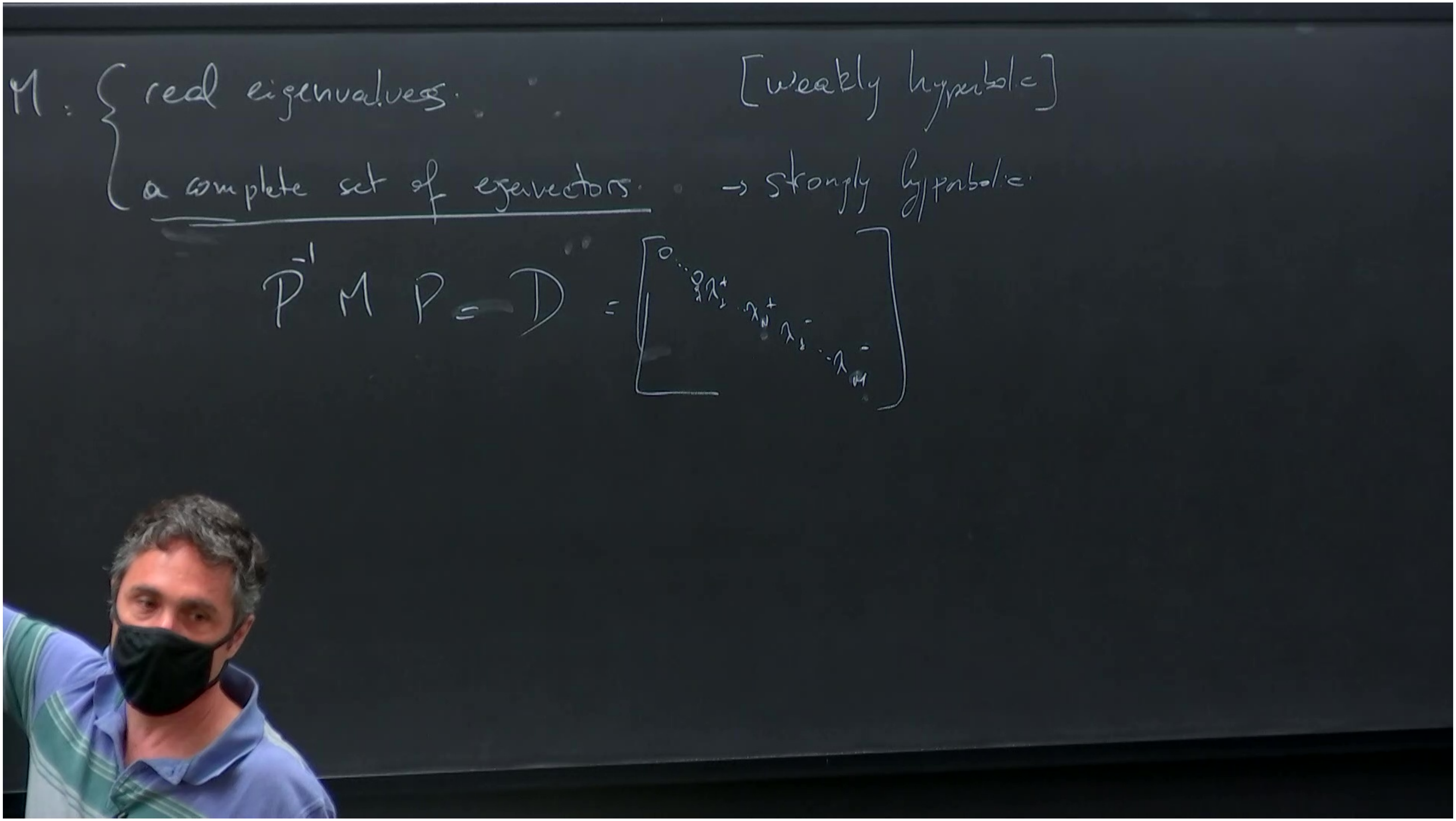
[assuming ID / BD is provided)

$$u_{,t} = M\, u_{,x} + B(u)$$

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}_{,t} \qquad \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}_{,x}$$

M : $\begin{cases} \text{real eigenvalues.} \\ \text{a complete set of eigenvectors.} \end{cases}$

[w


M : real eigenvalues
[weakly hyperbolic]
a complete set of eigenvectors → strongly hyperbolic
$P^{-1} M P = D =$
$\lambda_1^+ \cdots \lambda_N^+ \; \lambda_1^- \cdots \lambda_M^-$

A sufficient cond. for well posedness. [assuming ID / BD is provided)

$$U_{,t} = M\, U_{,x} + B(u)$$

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}_{,t} \qquad \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}_{,x}$$

$B(u)$: lower order terms.

M : { real eigenva... a complete se...

$P^{-1} M$

$$U_{,t} = M\, U_{,x} + B$$

to left by P^{-1}

$$P^{-1} U_{,t} = P^{-1} P \underbrace{P^{-1} M P}\, P^{-1} U_{,x} + P^{-1} B$$

$$P^{-1} U_{,t} = D\, P^{-1} U_{,x} + \text{l.o.t}$$

$P^{-1} U = V$

$$(P^{-1}U)_{,t} + \text{l.o.t} = D\left[(P^{-1}U)_{,x} + \text{l.o.t}\right] + \text{l.o.t}$$

$\Rightarrow V_{,t} =$

M: $\begin{cases} \text{real eigenvalues.} & \text{[weakly hyperbolic]} \\ \underline{\text{a complete set of eigenvectors.}} & \to \text{strongly hyperbolic} \end{cases}$

terms.

$$P^{-1} M P = D = \begin{bmatrix} 0 & & & & & & & \\ & \ddots & & & & & & \\ & & 0 & & & & & \\ & & & \lambda_1^+ & & & & \\ & & & & \ddots & & & \\ & & & & & \lambda_N^+ & & \\ & & & & & & \lambda_1^- & & \\ & & & & & & & \ddots & \\ & & & & & & & & \lambda_N^- \end{bmatrix}$$

$v^i_{,t} = 0$ for $i = 1 \dots q$

$v^{\pm l}_{,t} = \lambda^{\pm l} v^{\pm l}_{,x}$ $\quad l = 1 \dots N$

$= X$

$$\to \underbrace{V_{,t} = D V_{,x}} + \text{l.o.t}$$

l.o.t

$$\boxed{\phi_{,tt} = \phi_{,xx}}$$

$$g \equiv \phi_{,t}$$

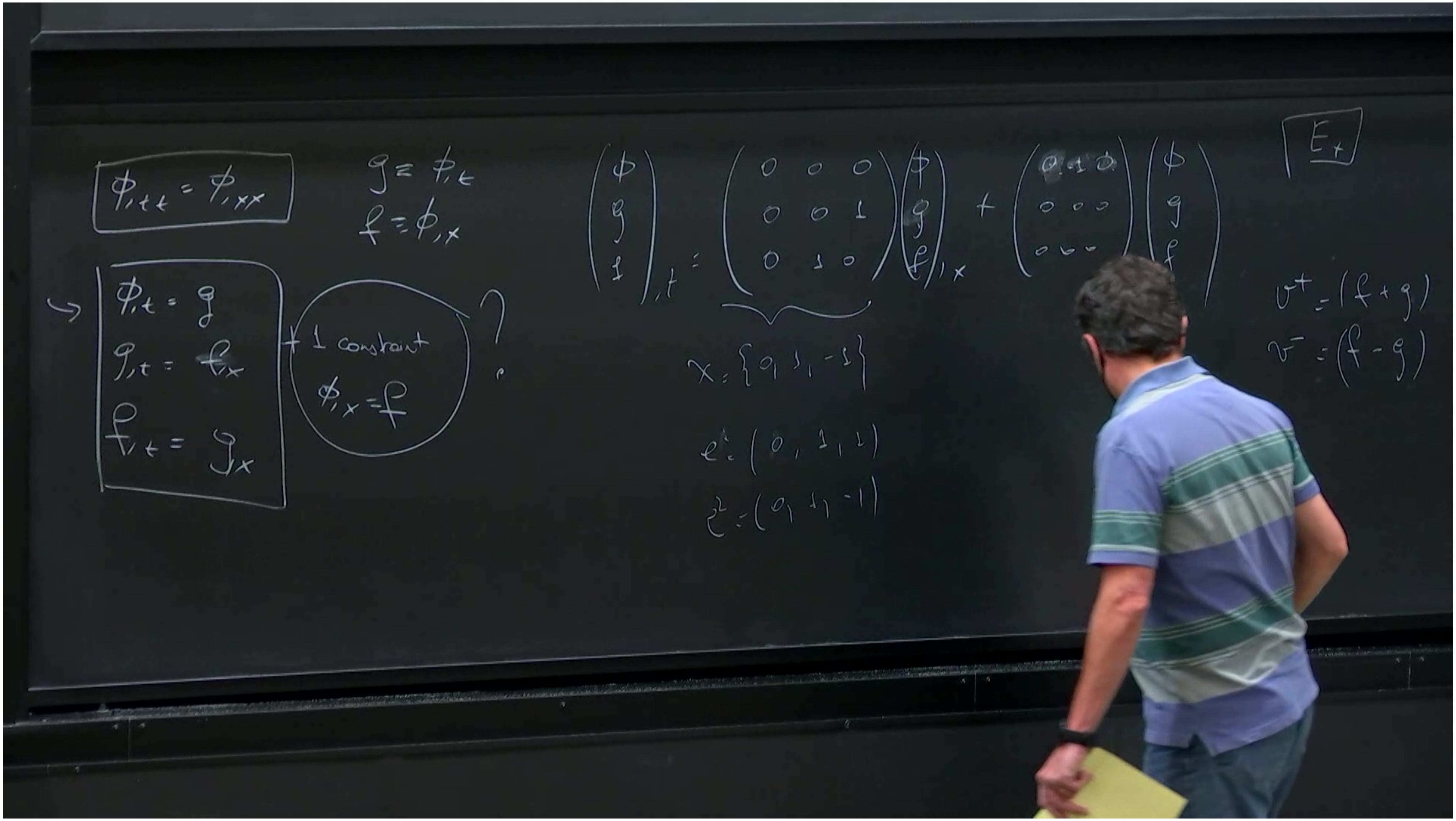
$$f \equiv \phi_{,x}$$

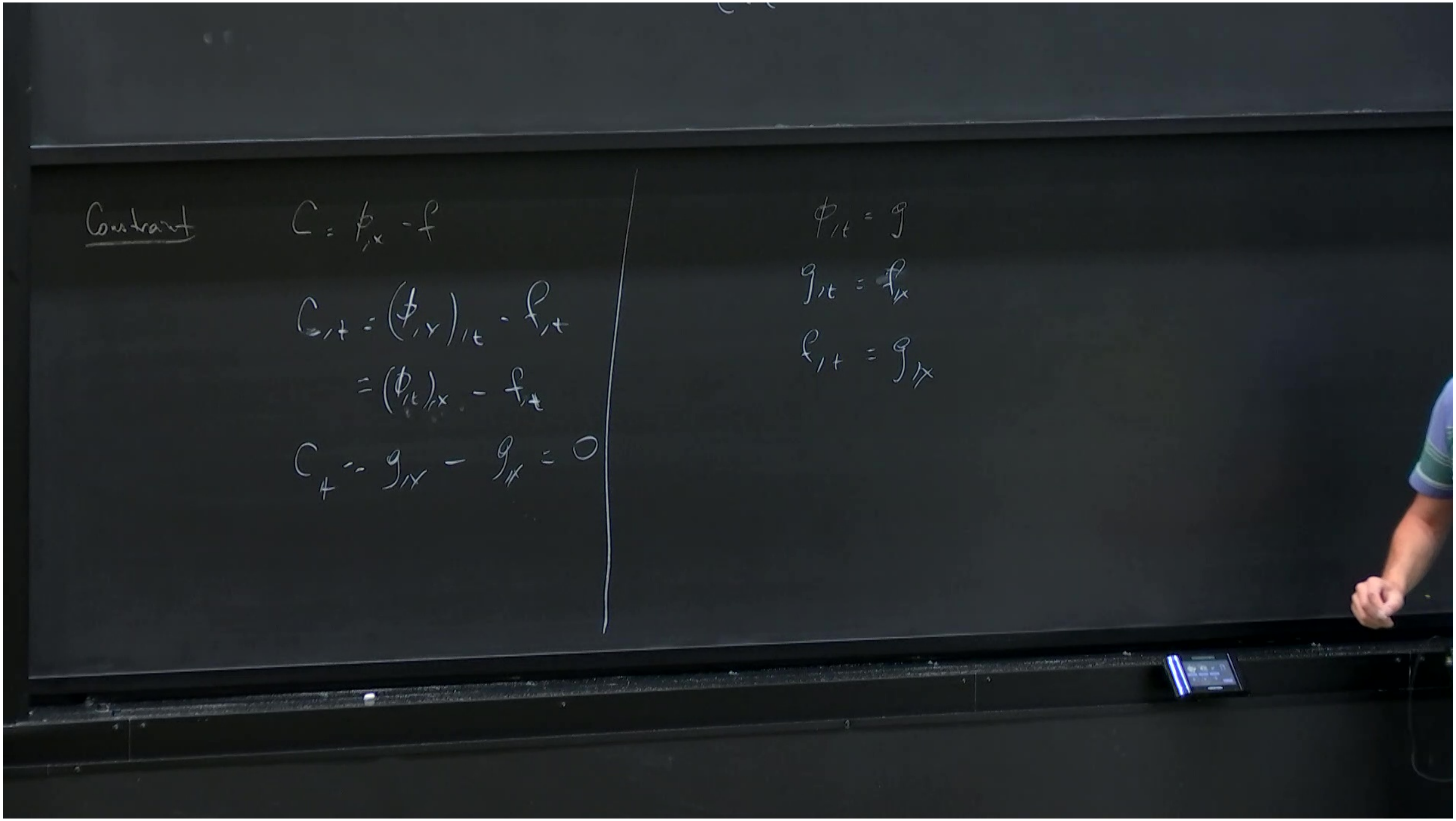
$$\hookrightarrow \boxed{\begin{aligned} \phi_{,t} &= g \\ g_{,t} &= f_{,x} \\ f_{,t} &= g_{,x} \end{aligned}}$$

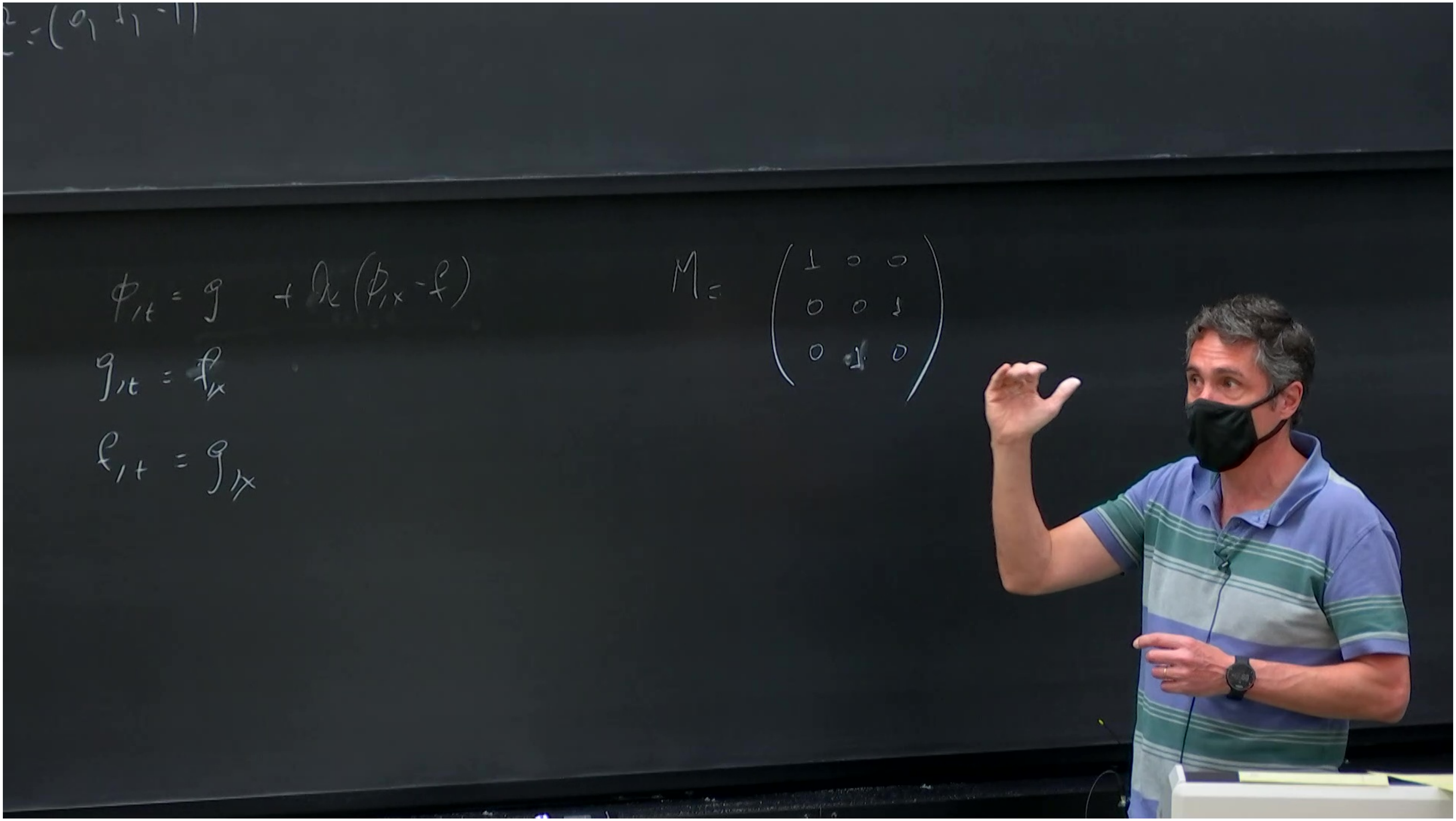
+ 1 constraint

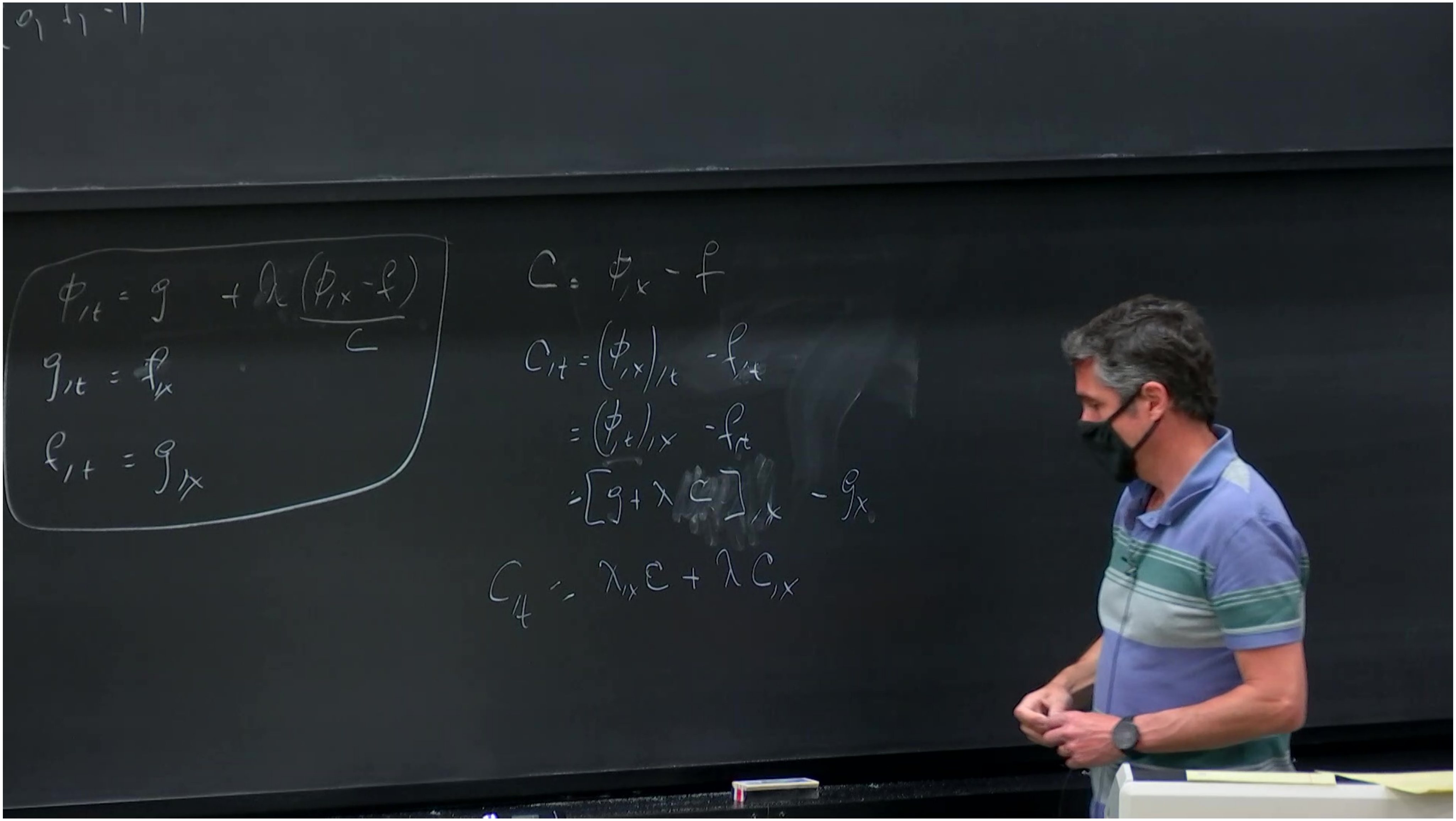
$$\phi_{,x} = f$$

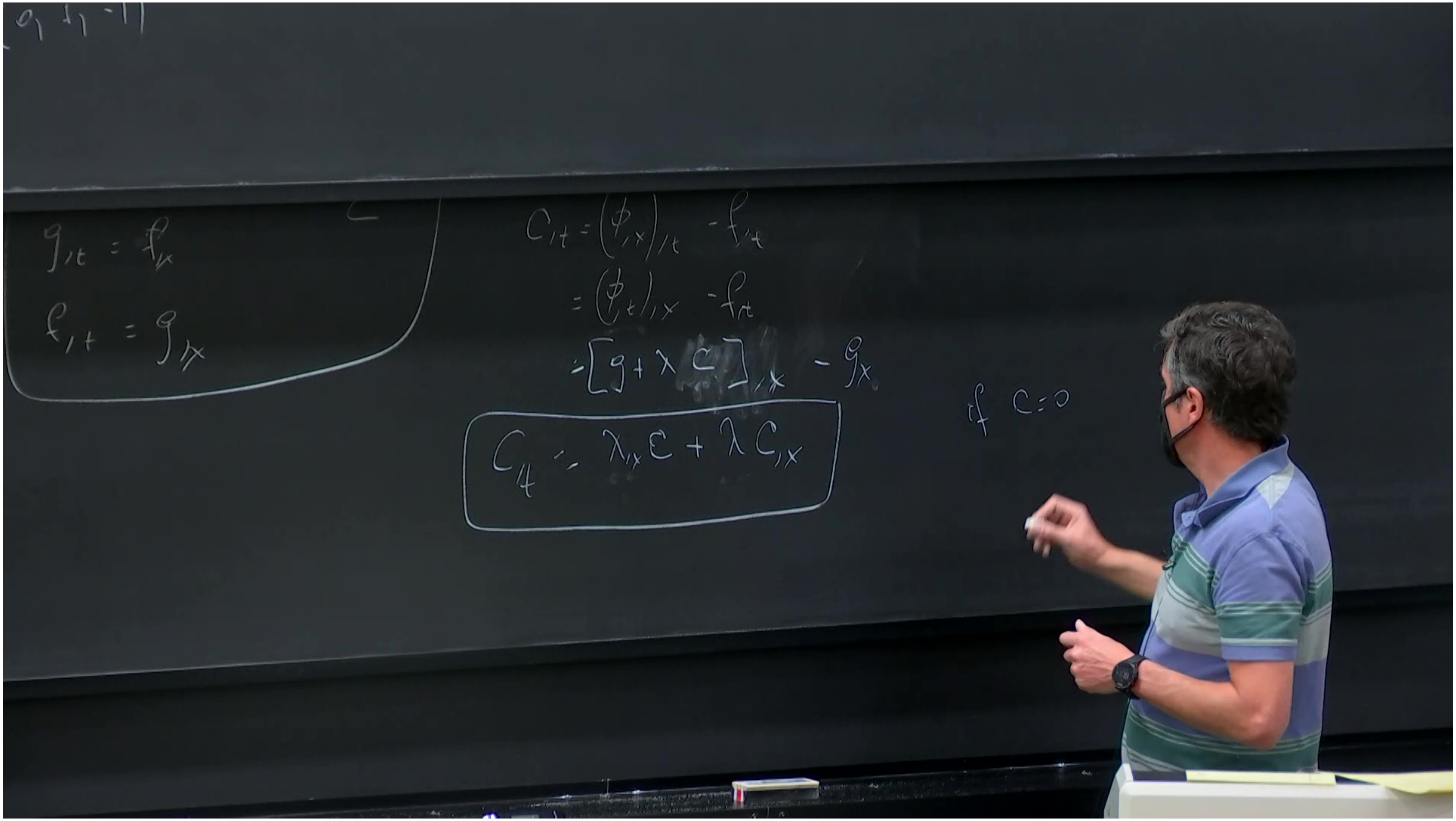
$$\begin{pmatrix} \phi \\ g \\ f \end{pmatrix}_{,t} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ g \\ f \end{pmatrix}_{,x} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ g \\ f \end{pmatrix}$$


φ,tt = φ,xx
g = φ,t
f = φ,x
φ,t = g
g,t = f,x
f,t = g,x
+1 constraint
φ,x = f
x = {0, 1, -1}
e¹ = (0, 1, 1)
e² = (0, 1, -1)
v+ = (f + g)
v- = (f - g)
E+


Constraint
$C = \phi_{,x} - f$
$C_{,t} = (\phi_{,x})_{,t} - f_{,t}$
$= (\phi_{,t})_{,x} - f_{,t}$
$C_{,t} = g_{,x} - g_{,x} = 0$
$\phi_{,t} = g$
$g_{,t} = f_{,x}$
$f_{,t} = g_{,x}$


$\phi_{,t} = g + \lambda(\phi_{,x} - f)$
$g_{,t} = f_{,x}$
$f_{,t} = g_{,x}$
$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$


$\phi_{,t} = g + \lambda \underbrace{(\phi_{,x} - f)}_{C}$
$g_{,t} = f_{,x}$
$f_{,t} = g_{,x}$
$C = \phi_{,x} - f$
$C_{,t} = (\phi_{,x})_{,t} - f_{,t}$
$= (\phi_{,t})_{,x} - f_{,t}$
$= [g + \lambda C]_{,x} - g_{,x}$
$C_{,t} = \lambda_{,x} C + \lambda C_{,x}$



$g_{,t} = f_{,x}$
$f_{,t} = g_{,x}$
$C_{,t} = (\phi_{,x})_{,t} - f_{,t}$
$= (\phi_{,t})_{,x} - f_{,t}$
$= [g + \lambda C]_{,x} - g_{,x}$
$\boxed{C_{,t} = \lambda_{,x} C + \lambda C_{,x}}$
if $C = 0$