Title: Strong Gravity

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Collection: Strong Gravity (2021/2022)

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Abstract: This course will introduce some advanced topics in general relativity related to describing gravity in the strong field and dynamical regime. Topics covered include properties of spinning black holes, black hole thermodynamics and energy extraction, how to define horizons in a dynamical setting, formulations of the Einstein equations as constraint and evolution equations, and gravitational waves and how they are sourced.

Foliation of spacetime

$$\mathcal{M} = \{\Sigma_t\}_{t\in\mathbb{R}}$$

$$\Sigma_t = \{x^a \in \mathcal{M} \mid t(x^a) = t\}$$

$$n_a = -K\,\nabla_a t$$

$$n_a = (-\underset{\text{lapse}}{\alpha}, \vec{0})$$

$$g_{ab}\, n^a n^b = -1$$

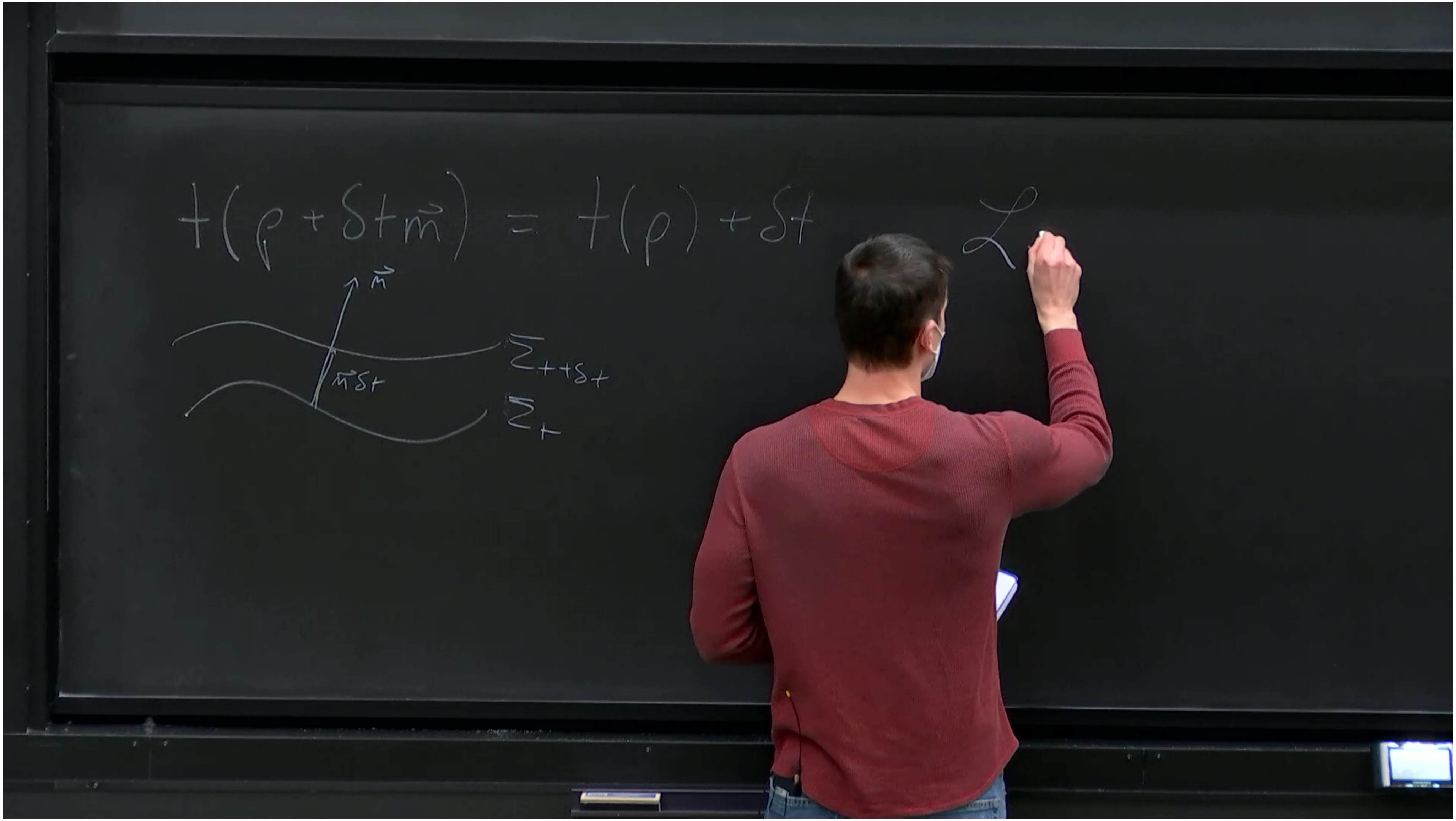
$$-1 = g^{tt}(n_t)^2 \iff \alpha = \sqrt{\frac{-1}{g^{tt}}}$$

$$n^a = \frac{1}{\alpha}(-1, -\underbrace{\beta^i}_{\text{shift vector}})$$

$$-\frac{\beta^i}{\alpha} = n_t\, g^{ti} \Rightarrow \beta^i = \alpha^2 g^{ti}$$

$$\alpha = \frac{d\tau}{dt}, \quad -\beta^i = \frac{dx^i}{dt}$$

Evolution operator: $m_a = \alpha\, n_a$, $\quad m^a \nabla_a t = 1$


$t(p + \delta t \vec{m}) = t(p) + \delta t$
$\vec{m}$
$\vec{m}\,\delta t$
$\Sigma_{t+\delta t}$
Σ_t
$\mathcal{L}$

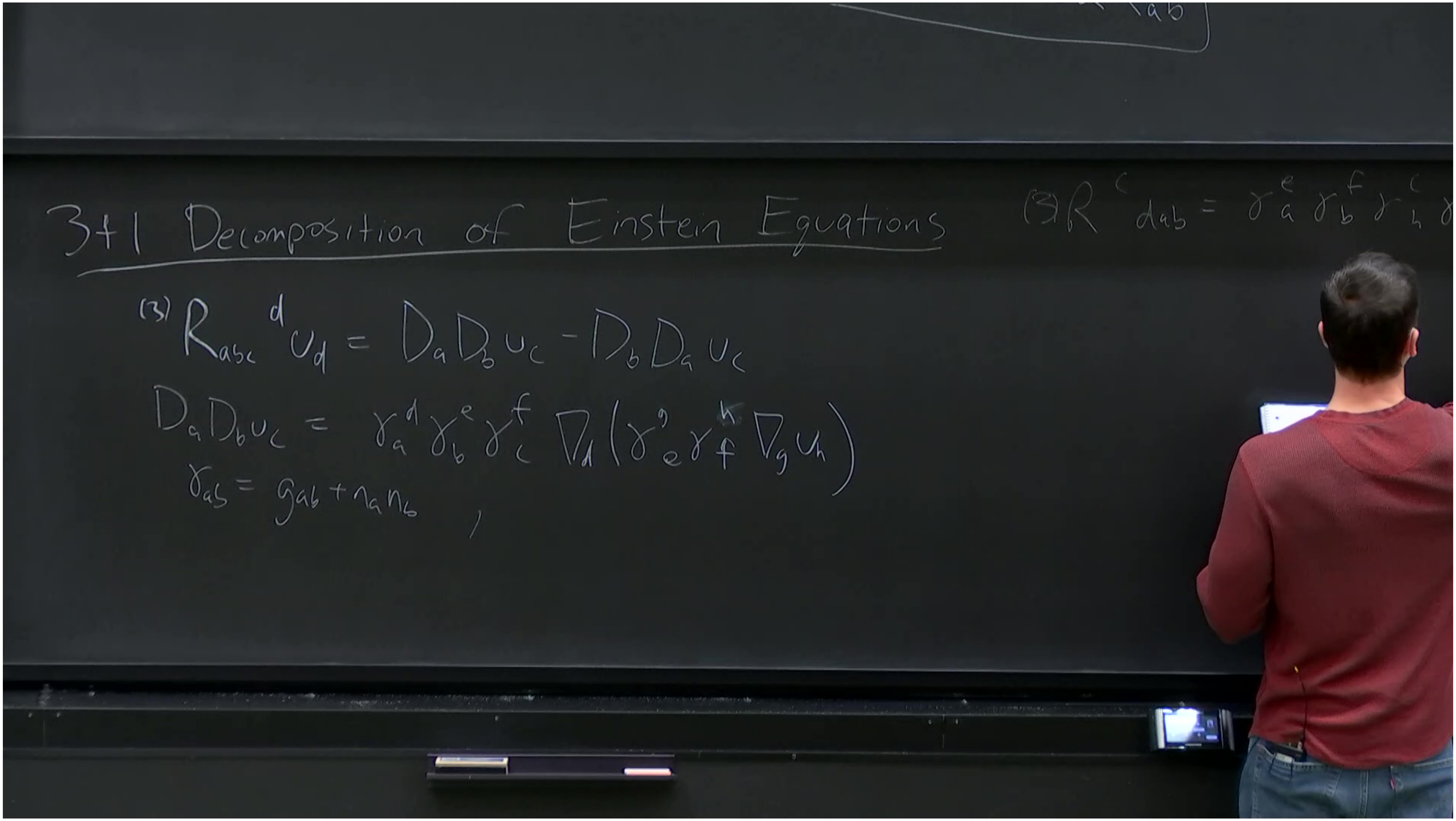
$\rho) + \delta t$

$$\mathcal{L}_{\vec{n}}\gamma_{ab} = n^c\nabla_c\gamma_{ab} + \gamma_{ab}\nabla_c n^c + \gamma_{ac}\nabla_b n^c$$

Use $\nabla_c\gamma_{ab} = \nabla_c(n_a n_b)$; $\nabla_a n^c = -K_a^{\ c} - \underbrace{(n^d\nabla_d n^c)}_{a^c} n_a$ tangent to Σ_t

$$\rightarrow \mathcal{L}_n \gamma_{ab} = -2K_{ab}$$

$$\boxed{\mathcal{L}_{\vec{m}}\gamma_{ab} = -2\alpha K_{ab}}$$


3+1 Decomposition of Einstein Equations
$^{(3)}R^{c}{}_{dab} = \gamma_a^e \gamma_b^f \gamma_h^c \gamma$
$^{(3)}R_{abc}{}^{d} u_d = D_a D_b u_c - D_b D_a u_c$
$D_a D_b u_c = \gamma_a^d \gamma_b^e \gamma_c^f \nabla_d (\gamma_e^g \gamma_f^h \nabla_g u_h)$
$\gamma_{ab} = g_{ab} + n_a n_b$

$$ {}^{(3)}R^{c}{}_{dab} = \gamma^{e}_{a}\gamma^{f}_{b}\gamma^{c}_{h}\gamma^{g}_{d} R^{h}{}_{gef} - K^{c}_{a} K_{db} + K^{c}_{b} K_{ad} $$

$$ {}^{(3)}R_{db} + K K_{db} - K^{a}_{b} K_{ad} = \gamma^{f}_{b}\gamma^{e}_{h}\gamma^{g}_{d} R^{h}{}_{gef} \qquad (\diamond) $$

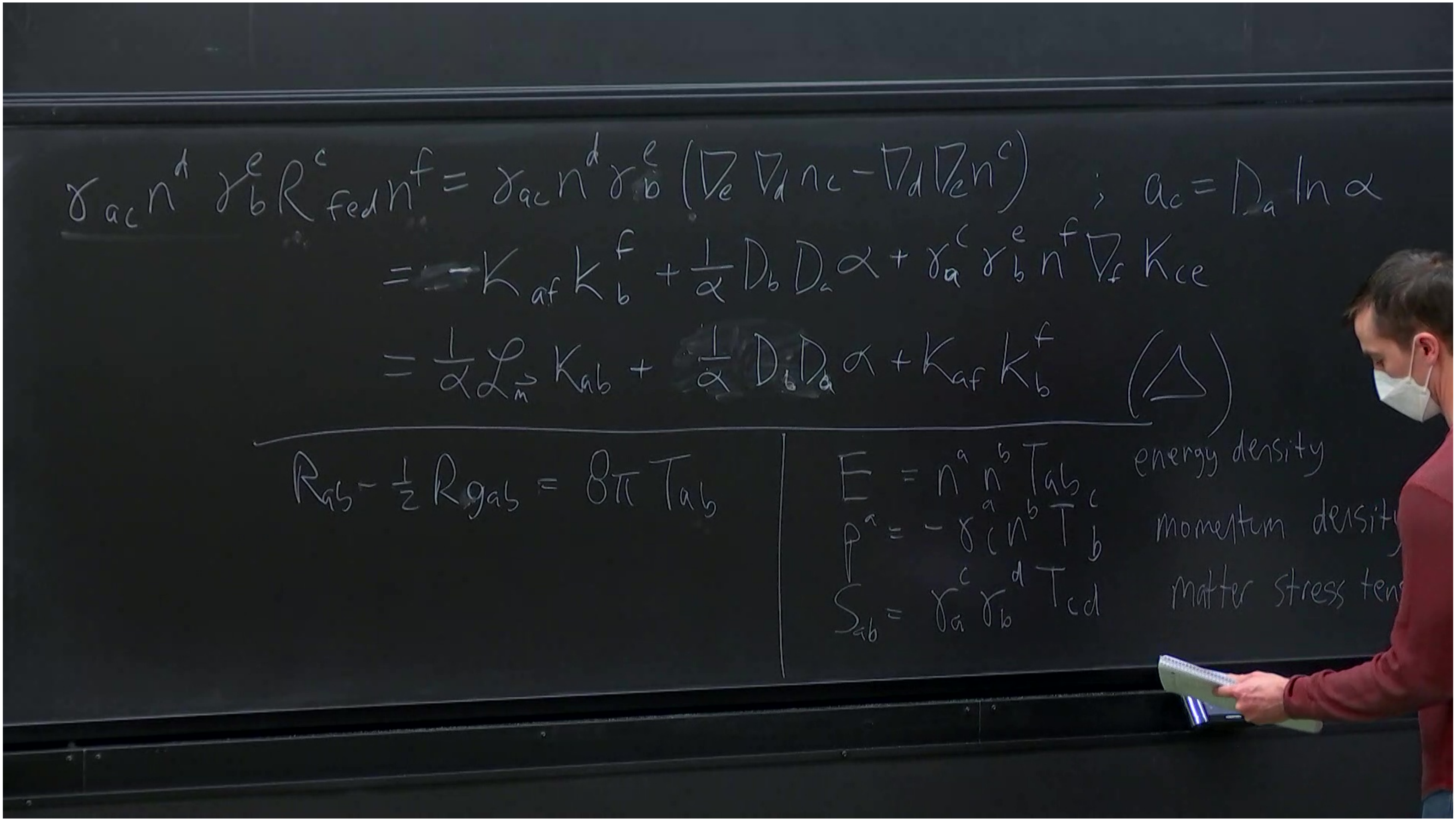
$$ {}^{(3)}R + K^{2} - K_{ab}K^{ab} = R + 2R_{ab}n^{a}n^{b} \qquad (*) $$

$$ R^{c}{}_{dab}n^{d} = (\nabla_a\nabla_b - \nabla_b\nabla_a)n^{c} $$

$$ \gamma^{c}_{d} n^{e} \gamma^{f}_{a}\gamma^{g}_{b} R^{d}{}_{efg} = D_b K^{c}_{a} - D_a K^{c}_{b} $$

$$ n^{e}\gamma^{g}_{b} R_{eg} = D_b K - D_a K^{a}_{b} \qquad (☺) $$

Gauss-Codacci Eqns.



$\gamma_{ac} n^d \gamma^e_b R^c{}_{fed} n^f = \gamma_{ac} n^d \gamma^e_b (\nabla_e \nabla_d n^c - \nabla_d \nabla_e n^c)$; $a_c = D_a \ln \alpha$
$= -K_{af} K^f{}_b + \frac{1}{\alpha} D_b D_a \alpha + \gamma^c_a \gamma^e_b n^f \nabla_f K_{ce}$
$= \frac{1}{\alpha} \mathcal{L}_{\vec{m}} K_{ab} + \frac{1}{\alpha} D_b D_a \alpha + K_{af} K^f{}_b \quad (\triangle)$
$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}$
$E = n^a n^b T_{ab}$ energy density
$p^a = -\gamma^a_c n^b T^c{}_b$ momentum density
$S_{ab} = \gamma^c_a \gamma^d_b T_{cd}$ matter stress ten

$$= \frac{1}{\alpha}\mathcal{L}_{\vec{m}} K_{ab} + \frac{1}{\alpha} D_b D_a \alpha + K_{af} K^f{}_b \qquad (\triangle)$$

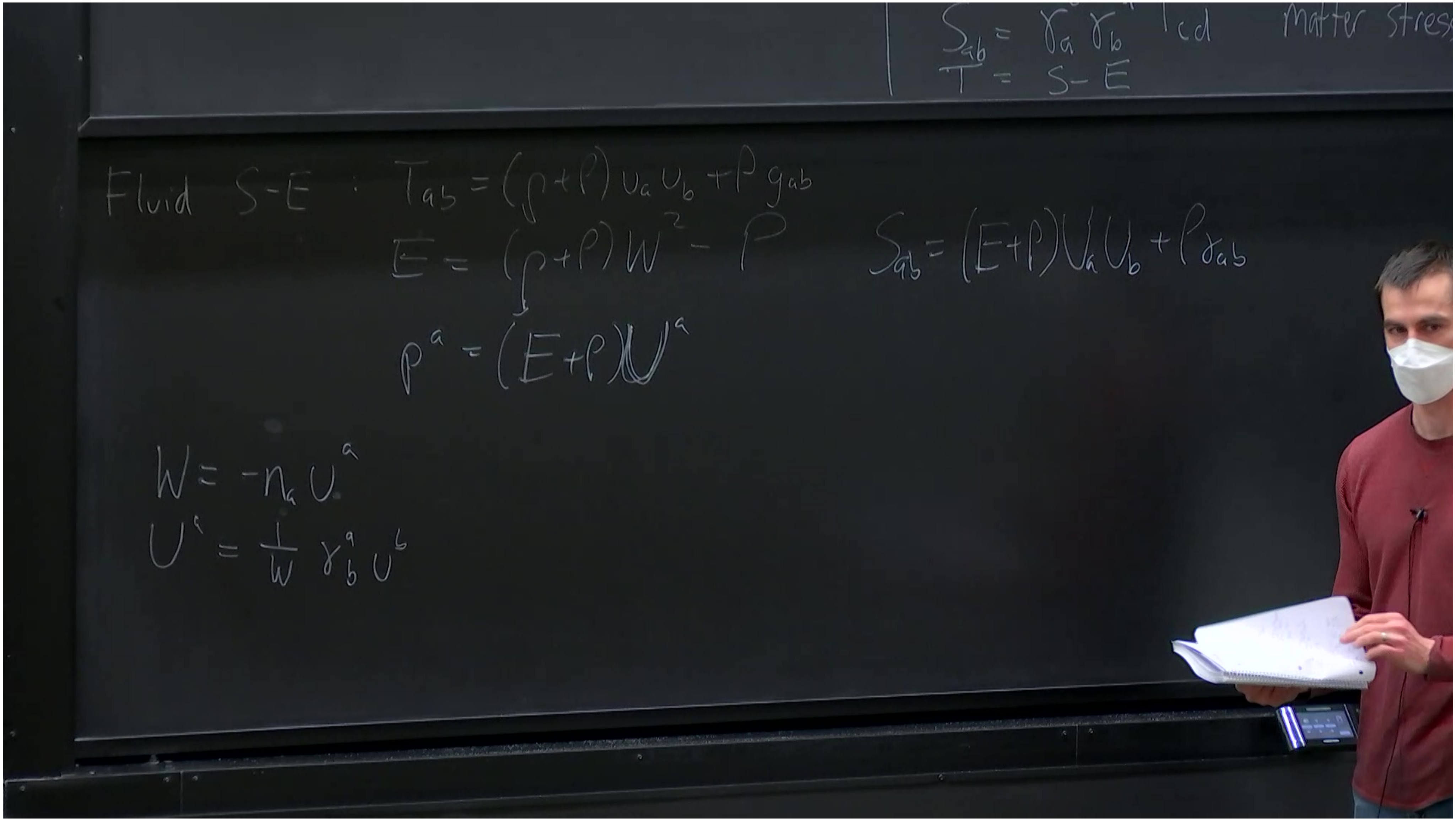
$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}$$

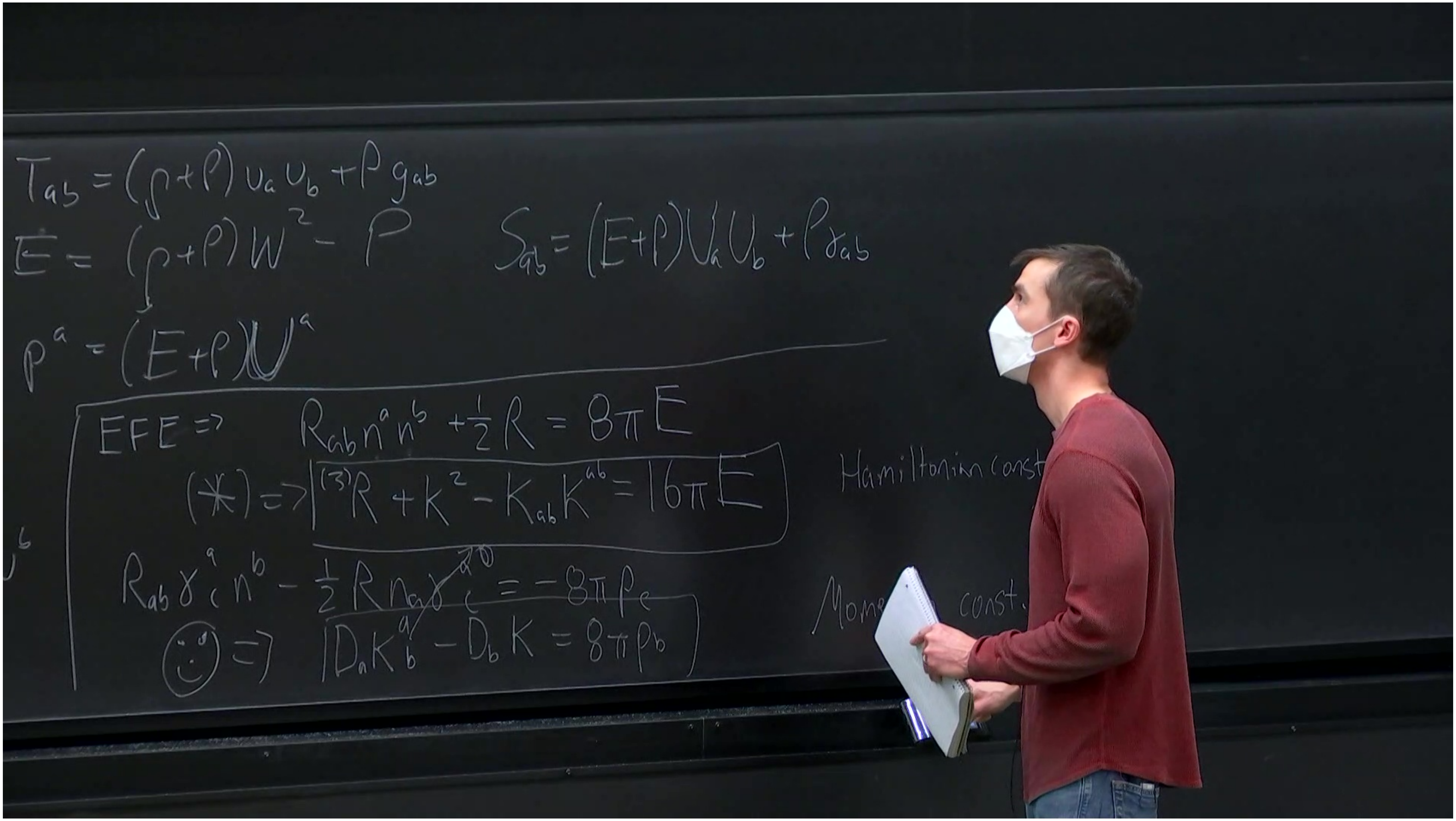
$E = n^a n^b T_{ab}$ energy density

$p^a = -\gamma^a{}_c n^b T^c{}_b$ momentum density

$S_{ab} = \gamma^c{}_a \gamma^d{}_b T_{cd}$ matter stress tensor

$T = S - E$


$S_{ab} = \gamma_a^{\ c} \gamma_b^{\ d} T_{cd}$ matter stress
$T = S - E$
Fluid S-E : $T_{ab} = (\rho + P) u_a u_b + P g_{ab}$
$E = (\rho + P) W^2 - P$
$S_{ab} = (E + P) U_a U_b + P \gamma_{ab}$
$p^a = (E + P) U^a$
$W = -n_a u^a$
$U^a = \frac{1}{W} \gamma^a_{\ b} u^b$


$T_{ab} = (\rho + P) u_a u_b + P g_{ab}$
$E = (\rho + P) W^2 - P$
$S_{ab} = (E + P) U_a U_b + P \gamma_{ab}$
$p^a = (E + P) U^a$
EFE ⇒ $R_{ab} n^a n^b + \frac{1}{2} R = 8\pi E$
$(*) \Rightarrow {}^{(3)}R + K^2 - K_{ab} K^{ab} = 16\pi E$
Hamiltonian const
$R_{ab} \gamma^a{}_c n^b - \frac{1}{2} R n_a \gamma^a{}_c = -8\pi p_c$
$\Rightarrow D_a K^a{}_b - D_b K = 8\pi p_b$
Mome const.

$$\Rightarrow \boxed{D_a K^a_{\ b} - D_b K = 8\pi p_b}$$

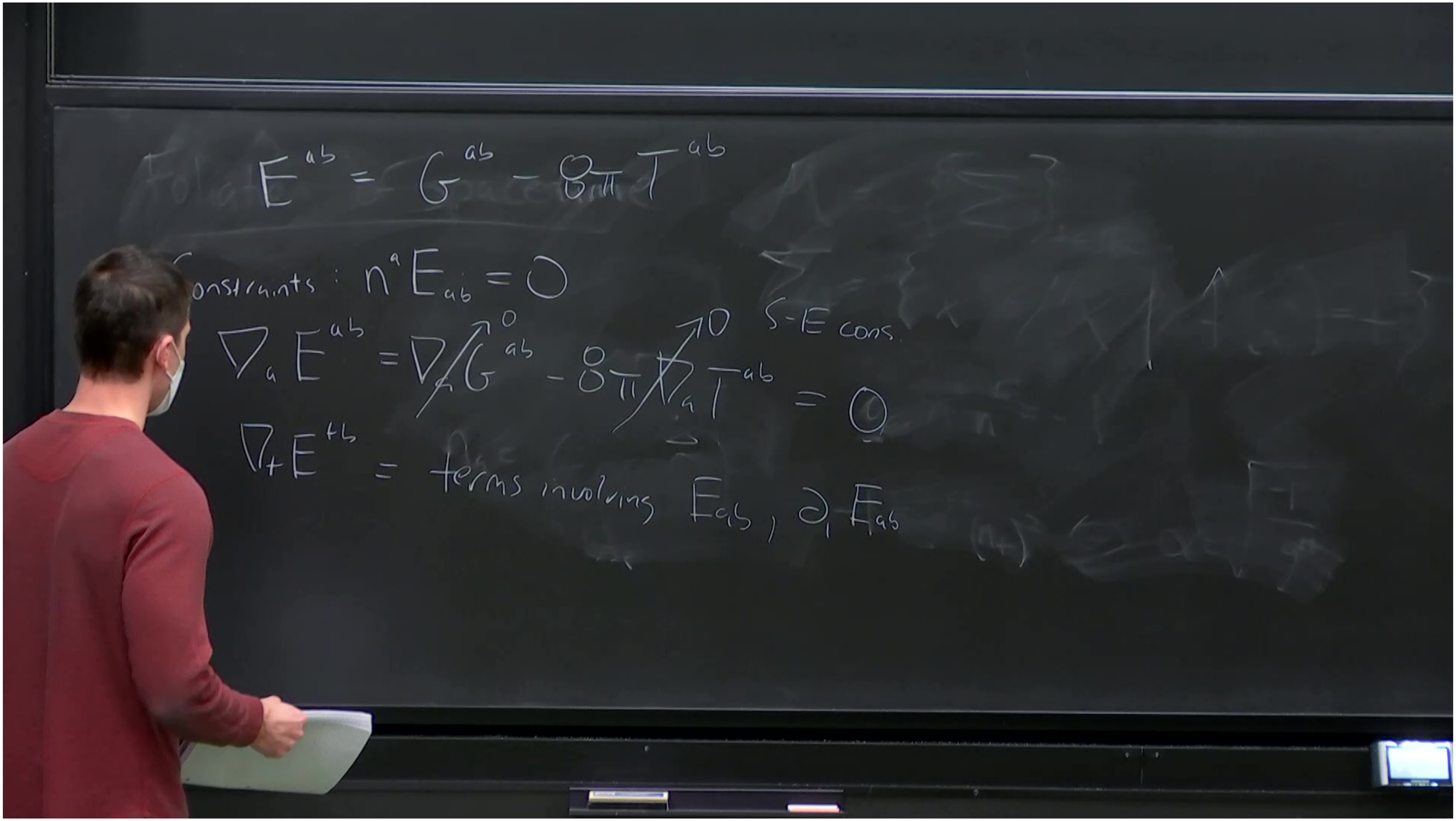
Momentum const.

$$R_{ab} = 8\pi\left(T_{ab} - \tfrac{1}{2} T g_{ab}\right)$$

$$\gamma^a_{\ c}\gamma^b_{\ d} R_{ab} = 8\pi\left[S_{cd} - \tfrac{1}{2}(S-E)\gamma_{cd}\right]$$

$$(\diamond, \triangle) \Rightarrow -\tfrac{1}{2}\mathcal{L}_{\vec{m}} K_{cd} - \tfrac{1}{\alpha} D_c D_d \alpha + {}^{(3)}R_{cd} + K K_{cd} - 2 K_{ca} K^a_{\ d} = 8\pi[\ldots]$$

Recall $\mathcal{L}_{\vec{m}} \gamma_{ab} = -2\alpha K_{ab}$



$E^{ab} = G^{ab} - 8\pi T^{ab}$
Constraints : $n^a E_{ab} = 0$
$\nabla_a E^{ab} = \cancel{\nabla_a} G^{ab} - 8\pi \cancel{\nabla_a} T^{ab} = 0$
0
0 S-E cons.
$\nabla_t E^{tb} =$ terms involving E_{ab}, $\partial_i E_{ab}$

Maxwell's Eqns

Constraints: $\nabla \cdot \vec{E} = \rho$, $\nabla \cdot \vec{B} = 0$

Evolution Eqns: $\partial_t \vec{E} = \nabla \times \vec{B} - \vec{J}$, $\partial_t \vec{B} = -\nabla \times \vec{E}$

$$\partial_t (\nabla \cdot \vec{B}) = 0$$

$$\partial_t (\nabla \cdot \vec{E}) = -\nabla \cdot \vec{J} \quad , \quad \partial_t \rho + \nabla \cdot \vec{J} = 0$$

$$\partial_t (\nabla \cdot \vec{E} - \rho) = 0$$