Title: Strong Gravity

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Collection: Strong Gravity (2021/2022)

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Abstract: This course will introduce some advanced topics in general relativity related to describing gravity in the strong field and dynamical regime. Topics covered include properties of spinning black holes, black hole thermodynamics and energy extraction, how to define horizons in a dynamical setting, formulations of the Einstein equations as constraint and evolution equations, and gravitational waves and how they are sourced.

3+1 Decomposition of Spacetime

Ref: GR, Wald Chap 10.2

3+1 Formalism, Gourgoulhon arXiv: gr-qc/07033035

Hypersurfaces of spacetime

4-dim spacetime manifold $\mathcal{M}$, 3-dim manifold Σ

Embedding of $\hat{\Sigma}$ in $\mathcal{M}$

$\Phi : \hat{\Sigma} \to \mathcal{M}$, 1:1, Φ, Φ^{-1} are C^0

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Then $\Sigma = \Phi(\hat{\Sigma})$ is a hypersurface

Level set $\Sigma = \{x^a \in \mathcal{M} \mid F(x^a) = 0\}$

anifold $\hat{\Sigma}$

$\partial_a F$ is normal to Σ

$\Leftrightarrow \partial_a F \, v^a = 0$ for any vector tangent to Σ

$\partial_a F$ timelike : Σ spacelike ← restrict to this case

$\partial_a F$ spacelike : Σ timelike

$\partial_a F$ null : Σ null

Choose coordinates $x^a = (t, x^i)$ (x^i coordinates in $\hat{\Sigma}$)

x^i ← 3-index

with $\Sigma = \{x^a \in M \mid t = 0\}$

embedding $\Phi : \hat{\Sigma} \to M$

$x^i \to (0, x^i)$

Unit normal to Σ:

$$n_a = \frac{-\nabla_a t}{\sqrt{-\nabla_a t \nabla^a t}}, \quad n_a n^a = -1$$

Choose coordinates $x^a = (t, x^i)$ (x^i coordinates in $\hat{\Sigma}$)

$\uparrow$ 3-index

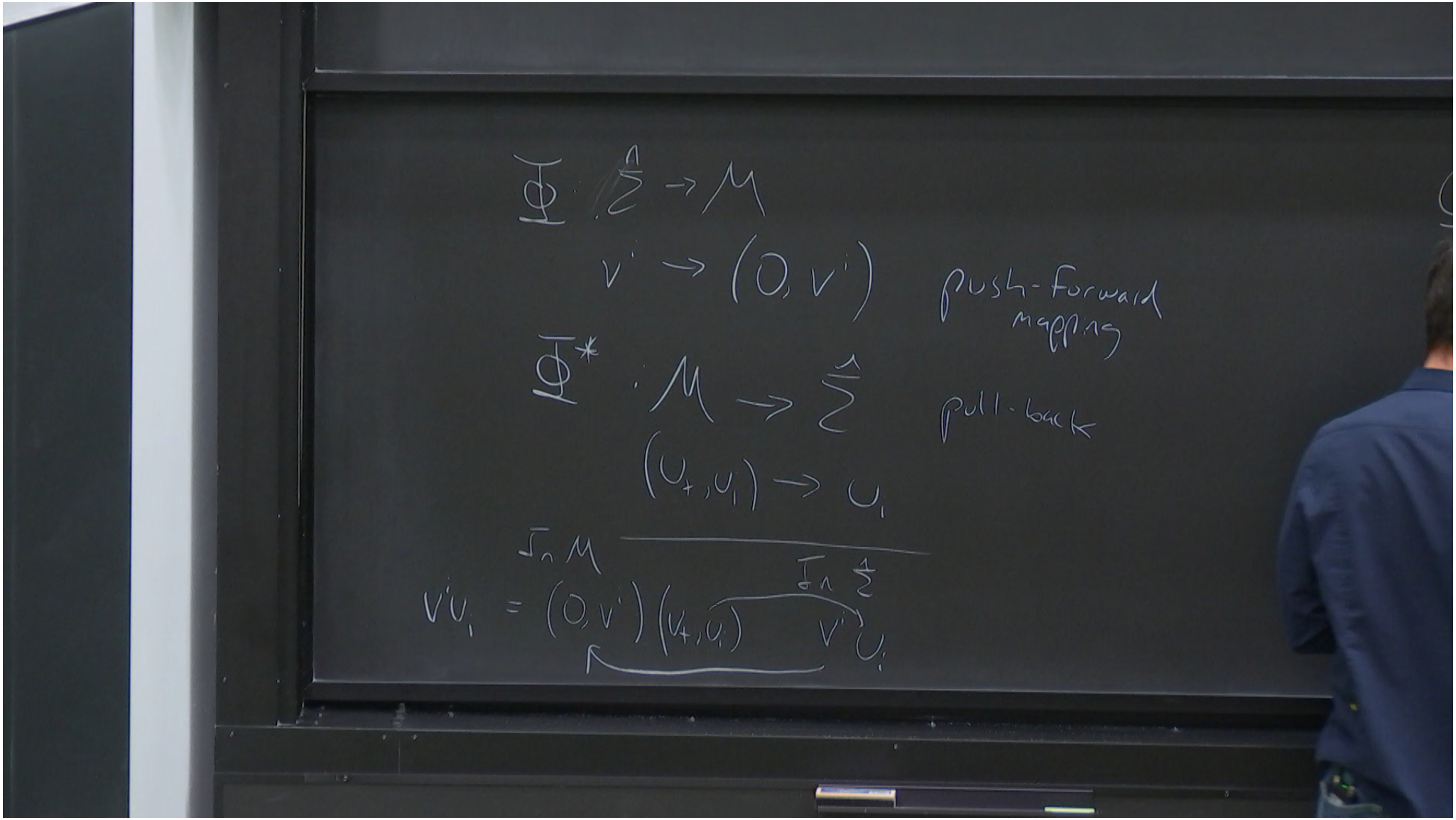
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Unit normal to Σ:

$$n_a = \frac{-\nabla_a t}{\sqrt{-\nabla_a t \nabla^a t}}, \quad n_a n^a = -1$$


$\Phi : \hat{\Sigma} \to M$
$v^i \to (0, v^i)$
push-forward mapping
$\Phi^* : M \to \hat{\Sigma}$
pull-back
$(U_+, U_-) \to U_-$
$T_n M$
$T_n \hat{\Sigma}$
$V^i U_- = (0, V^i)(U_+, U_-) \quad V^i U_-$

$\sqrt{-\nabla_a t \nabla^a t}$

Caution

M

$(U^t, U^i) \not\to \Sigma$ U^i

Doff hats

$\hat{\Sigma} \cong \Sigma$

For gas metric M, induced metric γ_{ij} on Σ is $\gamma_{ij} = g_{ij}$ in our adapted coördinates

$$v^a = \underbrace{-v^b n_b n^a}_{\text{orthogonal to } \Sigma} + \underbrace{(v^a + v^b n_b n^a)}_{\text{tangent} := \gamma^a_b v^b}$$

$$\gamma^a_b = \delta^a_b + n_b n^a \quad (*)$$

$$n_a(\gamma^a_b v^b) = 0$$

$$n_a(v^a + v_b n^b n^a) = 0$$

$M \to \Sigma$

vectors $V^a \to \gamma^i_a V^a$

$\Sigma \to M$

$U_i \to \gamma^i_a U_i$

$\gamma_{ab} = \gamma^i_a \gamma^j_b \gamma_{ij} = g_{ab} + n_a n_b$

$\gamma^b_a n^a$)

Check $\quad u^a n_a = 0 = v^a n_a = 0$

$$\gamma_{ab} v^a u^b = g_{ab} v^a u^b$$

$$w^a = \lambda n^a \qquad \gamma_{ab} w^a = 0$$

$$(g_{ab} + n_a n_b)\lambda n^a = \lambda n_b + n_a (\lambda n^a) n_b = 0$$

Check: $U^a n_a = 0 = V^a n_a = 0$

$\gamma_{ab} V^a U^b = g_{ab} V^a U^b$

$W^a = \lambda n^a \qquad \gamma_{ab} W^a = 0$

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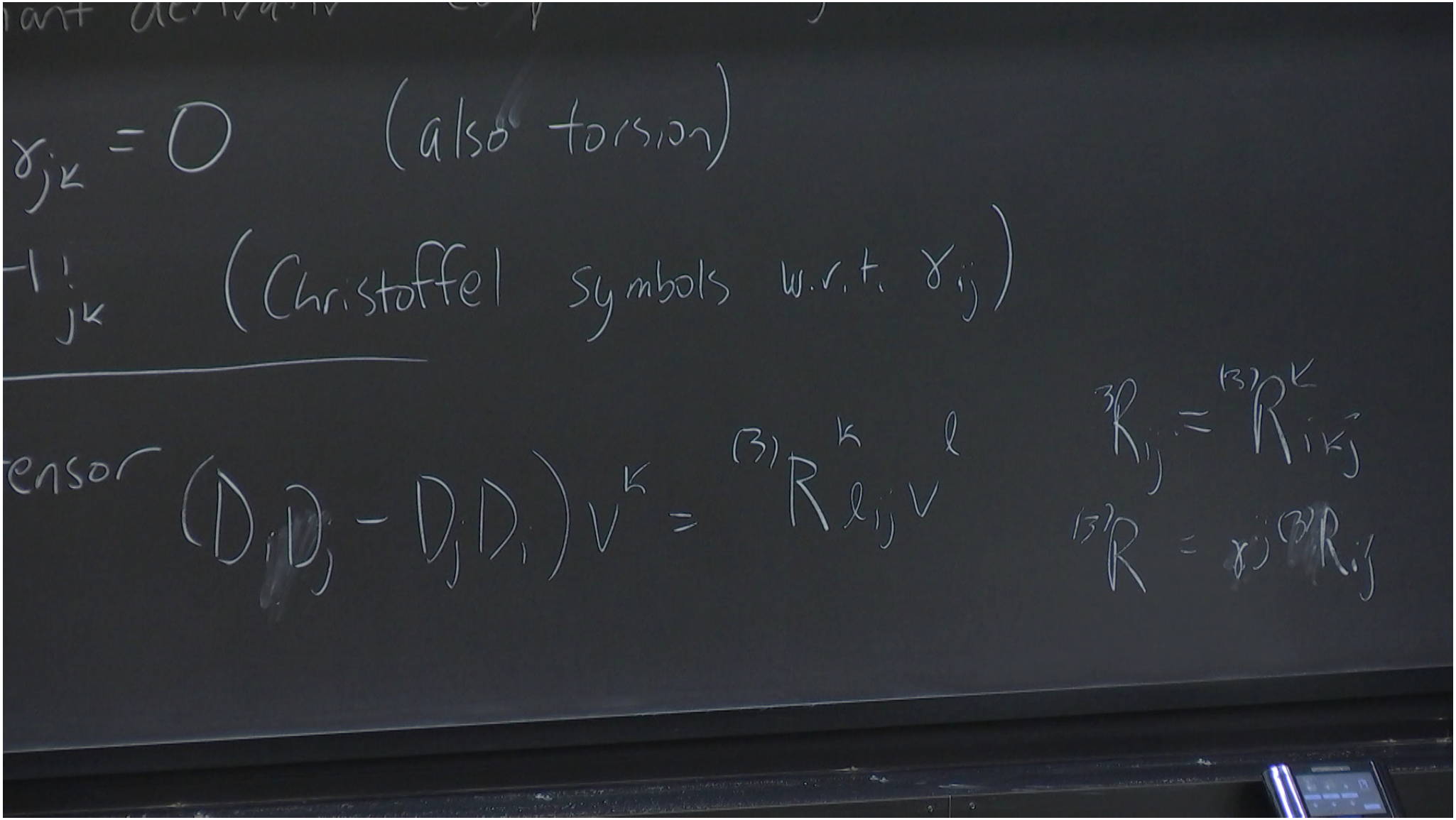
Intrinsic curvature: same but $(g_{ab}, M) \to (\gamma_{ij}, \Sigma)$

Introduce covariant derivative compatible γ_{ij}

$$D_i \gamma_{jk} = 0 \quad \text{(also torsion)}$$

defined w/ ${}^{(3)}\Gamma^{i}_{jk}$ (Christoffel symbols w.r.t. γ_{ij})

3D Riemann tensor $(D_i D_j - D_j D_i) V^k = {}^{(3)}R^{k}_{\ell ij} V^{\ell}$ ${}^{3}R_{ij}$ ${}^{(3)}R$

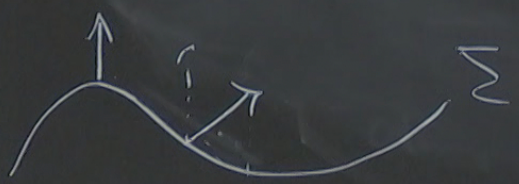

$\gamma_{jk} = 0$ (also torsion)
$^{(3)}\Gamma^{i}_{jk}$ (Christoffel symbols w.r.t. γ_{ij})
$(D_i D_j - D_j D_i) V^k = {}^{(3)}R^{k}{}_{\ell ij} V^{\ell}$
${}^{3}R_{ij} = {}^{(3)}R^{k}{}_{ikj}$
${}^{(3)}R = \gamma^{ij} {}^{(3)}R_{ij}$

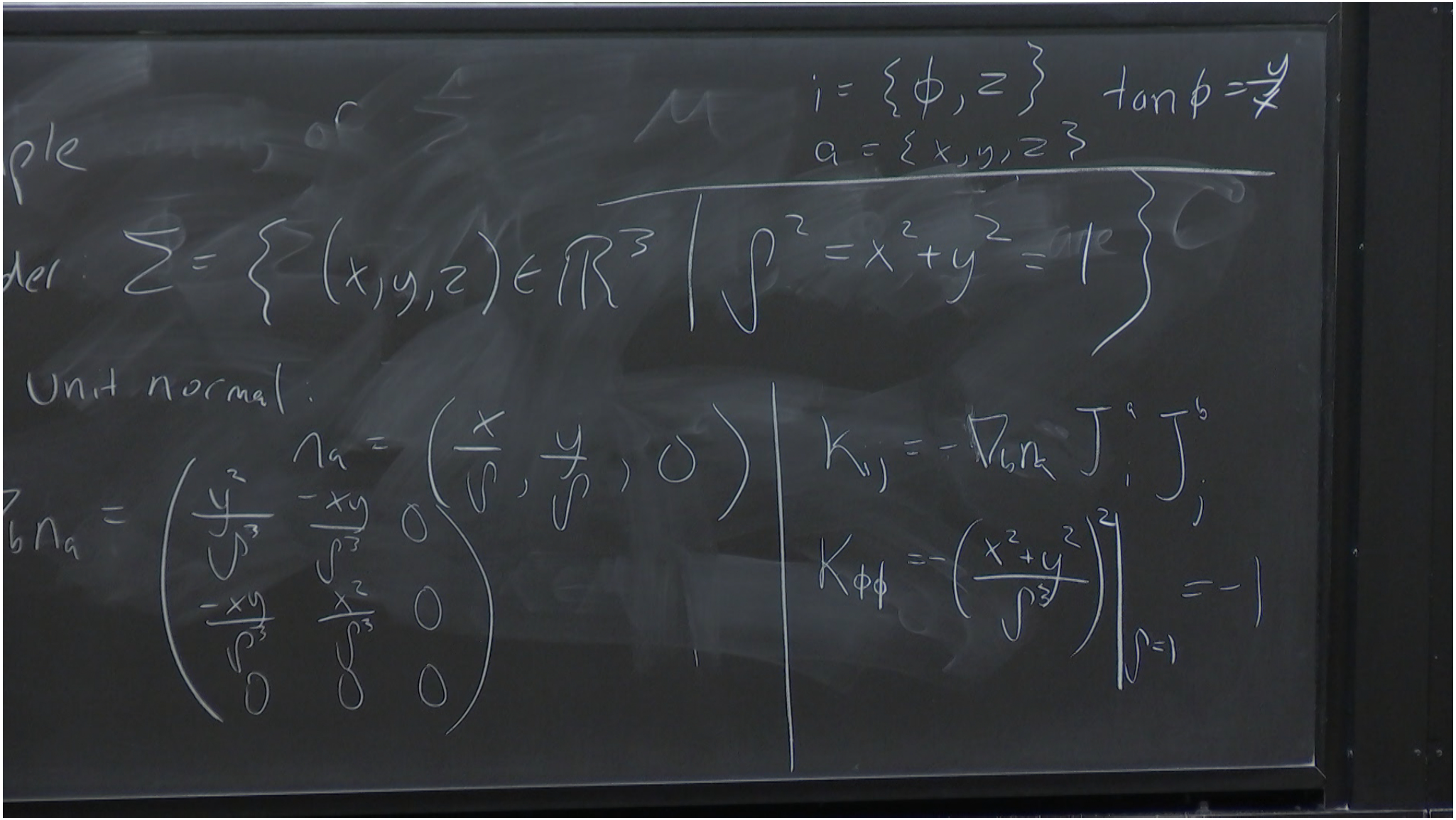
$$D_a T^{b_1 b_2 \ldots}{}_{c_1 c_2 \ldots} = \gamma^{b_1}_{d_1} \ldots \gamma^{e_1}_{c_1} \ldots \gamma^{f}_{a} \nabla_a T^{d_1 \ldots}$$

$$D_a \gamma_{bc} = \gamma^{d}_{b} \gamma^{e}_{c} \gamma^{f}_{a} \nabla_f \left(g_{de} + n_d n_e \right) = 0$$

Extrinsic Curvature

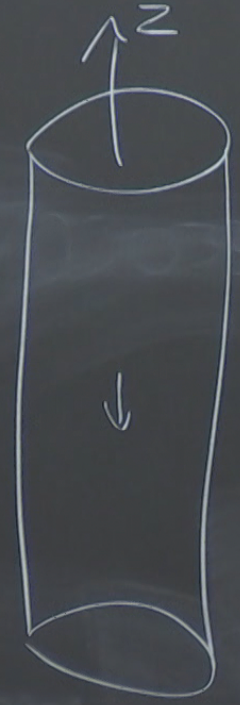
$$K_{ij} = -\nabla_j n_i$$




ple
der
Unit normal:
$i = \{\phi, z\}$
$a = \{x, y, z\}$
$\tan\phi = \frac{y}{x}$
$\Sigma = \{(x,y,z) \in \mathbb{R}^3 \mid \rho^2 = x^2 + y^2 = 1\}$
$n_a = \left(\frac{x}{\rho}, \frac{y}{\rho}, 0\right)$
$\nabla_b n_a = \begin{pmatrix} \frac{y^2}{\rho^3} & -\frac{xy}{\rho^3} & 0 \\ -\frac{xy}{\rho^3} & \frac{x^2}{\rho^3} & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$K_{ij} = -\nabla_b n_a J^a_i J^b_j$
$K_{\phi\phi} = -\left(\frac{x^2+y^2}{\rho^3}\right)\Big|_{\rho=1} = -1$

$e_1 \dots$

Example



Cylinder $\Sigma = \{(x,y,z) \in \mathbb{R}^3 \mid \rho$

Unit normal:

$$n_a = \left(\frac{x}{\rho}, \frac{y}{\rho}, 0\right)$$

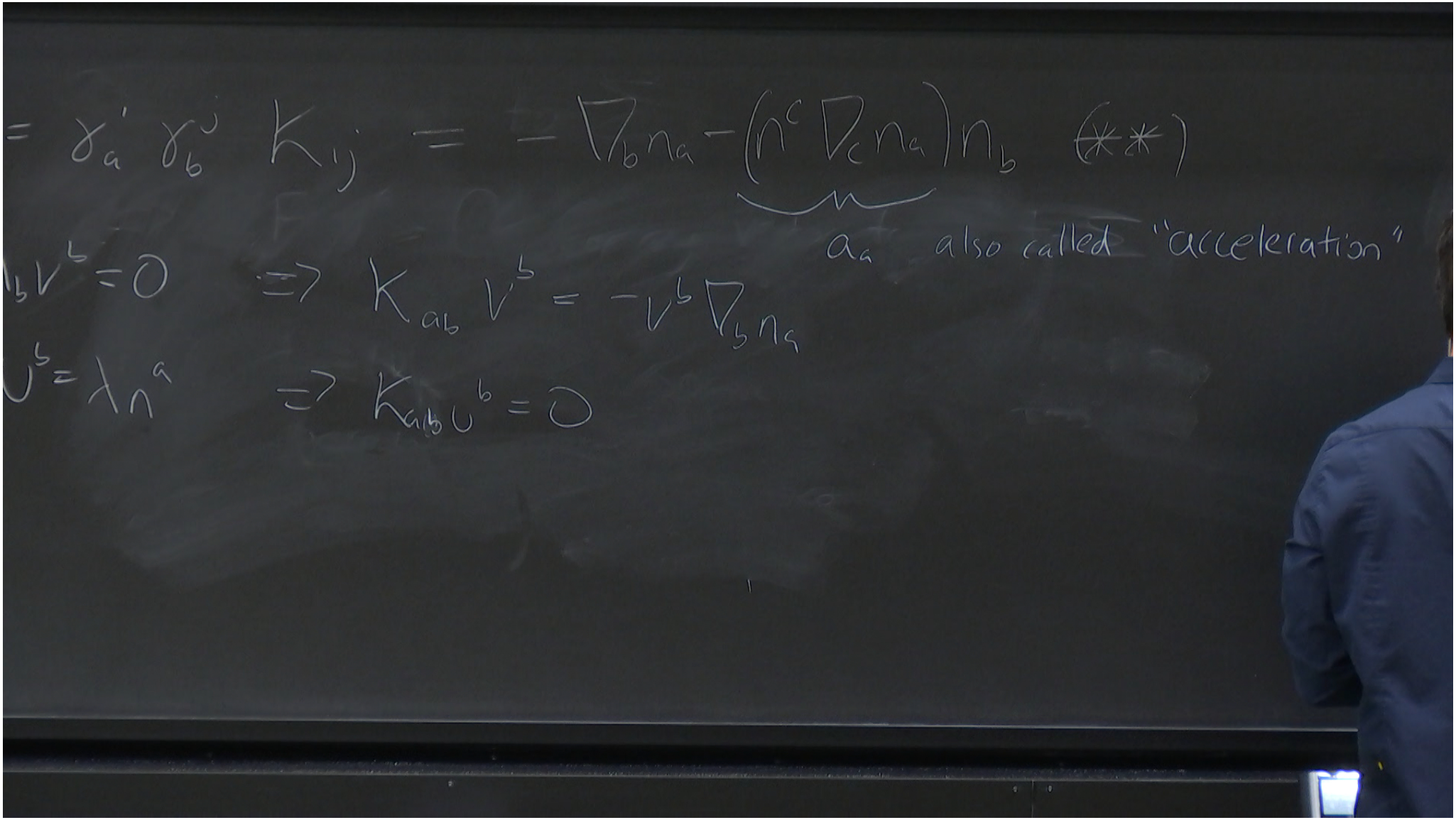
$$\nabla_b n_a = \begin{pmatrix} \frac{y^2}{\rho^3} & -\frac{xy}{\rho^3} & 0 \\ -\frac{xy}{\rho^3} & \frac{x^2}{\rho^3} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$K_{ab} = \gamma_a^i \gamma_b^j K_{ij} = -\nabla_b n_a - \underbrace{(n^c \nabla_c n_a)}_{a_a} n_b \quad (*$$

also called

$$n_b v^b = 0 \Rightarrow K_{ab} v^b = -v^b \nabla_b n_a$$

$$v^b = \lambda n^a \Rightarrow K_{ab} v^b = 0$$


$= \gamma_a^i \gamma_b^j K_{ij} = -\nabla_b n_a - (n^c \nabla_c n_a) n_b$ (**)
a_a also called "acceleration"
$v_b v^b = 0 \Rightarrow K_{ab} v^b = -v^b \nabla_b n_a$
$U^b = \lambda n^a \Rightarrow K_{ab} U^b = 0$

Foliation

$$\mathcal{M} = \{\Sigma_t\}_{t\in\mathbb{R}}$$

$$\Sigma_t := \{x^a \in \mathcal{M} \mid \hat{t}(x^a) = t\}$$

Σ_t: slice or leaf

$\hat{t}$: smooth, $\nabla_a \hat{t} \neq 0$

$\mathcal{M}$ is called globally hyperbolic

Foliation

$$\mathcal{M} = \{\Sigma_t\}_{t\in\mathbb{R}}$$

$$\Sigma_t := \{x^a \in \mathcal{M} \mid \hat{f}(x^a) = t\}$$

Σ_0

Σ_t: slice or leaf

$\hat{f}$: smooth, $\nabla_a \hat{f} \neq 0$

$\mathcal{M}$ is called globally hyperbolic