Title: Strong Gravity

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Collection: Strong Gravity (2021/2022)

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Abstract: This course will introduce some advanced topics in general relativity related to describing gravity in the strong field and dynamical regime. Topics covered include properties of spinning black holes, black hole thermodynamics and energy extraction, how to define horizons in a dynamical setting, formulations of the Einstein equations as constraint and evolution equations, and gravitational waves and how they are sourced.

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Sans 18

Ergospheres

$$t^a = \left(\frac{\partial}{\partial t}\right)^a \quad \swarrow$$

$$r = r_+ \qquad t^a t_a = g_{tt} = \frac{a^2 \sin^2\theta}{\Sigma} \geq 0$$

spacelike

null $\qquad t^a t_a = g_{tt} = 0$

$$\Rightarrow \quad \Delta = a^2 \sin^2\theta$$

r

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Ergospheres

$$\xi^a = \left(\frac{\partial}{\partial t}\right)^a$$

$$r = r_+ \quad \xi^a \xi_a = g_{tt} = \frac{a^2 \sin^2\theta}{\Sigma} \geq 0$$

spacelike

null $\quad \xi^a \xi_a = g_{tt} = 0$

$$\Rightarrow \Delta = a^2 \sin^2\theta$$

$$r^2 + a^2 \cos^2\theta - 2Mr = 0$$

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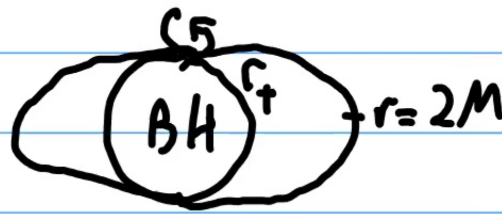
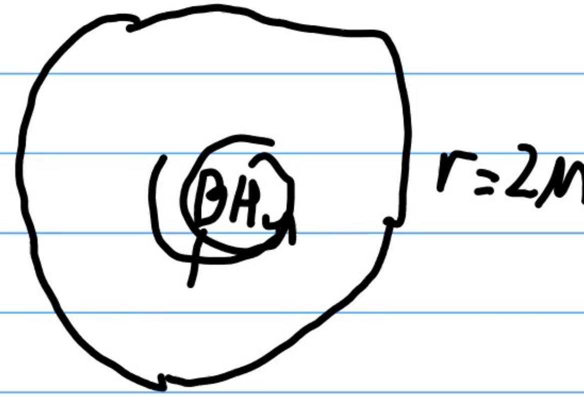
$$r^2 + a^2 \cos^2\theta - 2Mr = 0$$





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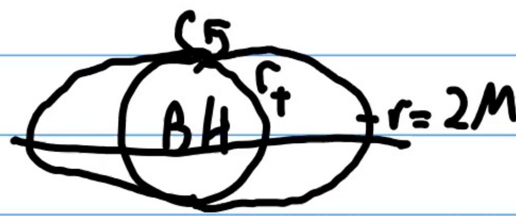
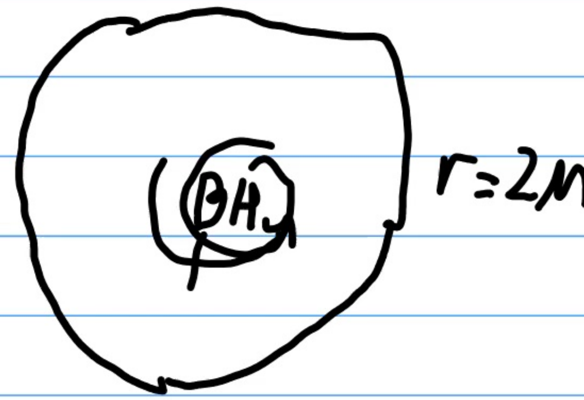
$$r^2 + a^2\cos^2\theta - 2Mr = 0$$





$$-1 = g_{ab}u^a u^b = g_{tt}(u^t)^2 + g_{rr}(u^r)^2 + g_{\theta\theta}(u^\theta)^2 + g_{\phi\phi}(u^\phi)^2 + 2g_{t\phi}u^t u^\phi$$

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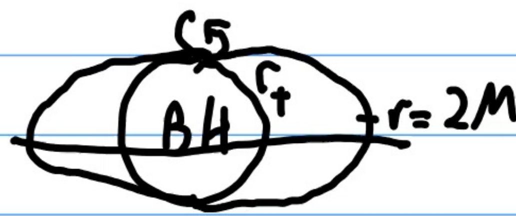
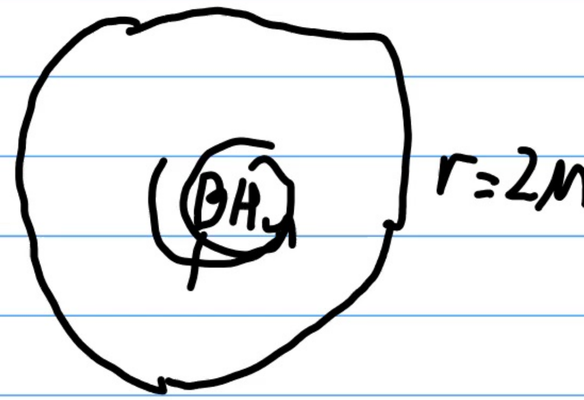
$$r^2 + a^2\cos^2\theta - 2Mr = 0$$





$$-1 = g_{ab}u^a u^b = \underbrace{g_{tt}}_{>0}(u^t)^2 + g_{rr}(u^r)^2 + g_{\theta\theta}(u^\theta)^2 + g_{\phi\phi}(u^\phi)^2 + \underbrace{2\, g_{t\phi}\, u^t u^\phi}_{<0}$$

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$$+2\underbrace{g^{t\phi}}_{<0} u^t u^\phi$$

$$g^{t\phi} < 0 \quad \text{and} \quad u^t = \frac{dt}{d\tau} > 0$$

$$u^\phi = \frac{d\phi}{d\tau} > 0$$

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$g^{t\phi} < 0$ and $u^t = \frac{dt}{d\tau} > 0$

$u^\phi = \frac{d\phi}{d\tau} > 0$

$\chi^a = \alpha\left(\hat{t}^a + \Omega\,\hat{\phi}^a\right)$

ZAMO $\hat{\phi}_a \chi^a = 0 \Rightarrow g_{\phi t} +$

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$$u^{\phi} = \frac{d\phi}{d\tau} > 0$$

$$\chi^{a} = \alpha\left(\hat{t}^{a} + \Omega\, \hat{\phi}^{a}\right)$$

ZAMO $\quad \hat{\phi}_{a}\chi^{a} = 0 \Rightarrow g_{\phi t} + g_{\phi\phi}\Omega = 0$

$$\frac{d\phi}{dt} = \Omega = \frac{-g_{t\phi}}{g_{\phi\phi}}$$

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$$\chi^a = \alpha\left(\hat{t}^a + \Omega\,\hat{\phi}^a\right)$$

ZAMO $\quad \hat{\phi}_a \chi^a = 0 \Rightarrow g_{\phi t} + g_{\phi\phi}\,\Omega = 0$

$$\frac{d\phi}{dt} = \Omega = \frac{-g_{t\phi}}{g_{\phi\phi}}$$

$$r \to r_+ \qquad \Omega = \frac{a}{2r_+} := \Omega_H$$

$$\bar{\chi}^a = \hat{t}_a + \Omega_H\,\hat{\phi}^a$$

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at $\quad \int \Phi\psi$

$r \to r_+ \qquad \Omega = \frac{\bar{a}}{2r_+} := \Omega_H$

$\bar{\chi}^a = \hat{t}_a + \Omega_H \hat{\phi}^a$

Ω_H

For $\bar{a} \ll 1$ $\qquad r_+ = (2 - \frac{1}{2}\bar{a}^2)M$

$\Omega_H \approx \bar{a}/4M$

For $\bar{a} = 1 - \varepsilon$ $\qquad r_+ = (1 + \sqrt{2\varepsilon})M$

$\Omega_H = \frac{1 - \sqrt{2\varepsilon}}{2M}$

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$$\chi^a = \alpha\left(\hat{t}^a + \Omega\,\hat{\phi}^a\right)$$

$\chi^a \chi_a = -1$ ZAMO $\qquad \hat{\phi}_a \chi^a = 0 \Rightarrow g_{\phi t} + g_{\phi\phi}\,\Omega = 0$

$$\frac{d\phi}{dt} = \Omega = \frac{-g_{t\phi}}{g_{\phi\phi}}$$

$$r \to r_+ \qquad \Omega = \frac{a}{2r_+} := \Omega_H$$

$$\to \bar{\chi}^a = \hat{t}_a + \underline{\Omega_H}\,\hat{\phi}^a$$

Ω_H

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Ergospheres

$$\xi^a = \left(\frac{\partial}{\partial t}\right)^a$$

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$$\Omega_H \approx \bar{a}/4M$$

For $\bar{a} = 1-\epsilon$

$$r_+ = (1+\sqrt{2\epsilon})M$$

$$\Omega_H = \frac{1-\sqrt{2\epsilon}}{2M}$$

Geodesics on Kerr

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Geodesics on Kerr

$$U^a = \frac{dx^a}{d\tau}$$

$\uparrow$ proper time

Geodesic Eqn. $U^a \nabla_a U^b = 0$

$$\frac{d^2 x^a}{d\tau^2} + \Gamma^a{}_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0$$

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Geodesic Eqn. $U^b \nabla_b U^a = 0$

τ proper time

$$\frac{d^2 x^a}{d\tau^2} + \Gamma^a{}_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0$$

Constants: $\tilde{E} = -t_a U^a$

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$$\frac{d^2 x^a}{d\tau^2} + \Gamma^a{}_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0$$

Constants: $\tilde{E} = -\hat{t}_a u^a = -u_t$

$$= -\left(g_{tt} u^t + g_{t\phi} u^\phi\right)$$

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Constants: $\tilde{E} = -\hat{t}_a U^a = -U_t$

$= -(g_{tt} U^t + g_{t\phi} U^\phi)$

$= \left(1 - \frac{2Mr}{\Sigma}\right)\frac{dt}{d\tau} + \frac{2Mar\sin^2\theta}{\Sigma}\frac{d\phi}{d\tau}$

$\tilde{J} = \hat{\phi}^a U_a = U_\phi = g_{\phi\phi} U^\phi + g_{t\phi} U^t$

$= (r^2+a^2)^2 \cdot$

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$$= \left(1 - \frac{2Mr}{\Sigma}\right)\frac{dt}{d\tau} + \frac{2Mar\sin^2\theta}{\Sigma}\frac{d\phi}{d\tau}$$

$$\tilde{J} = \phi^a U_a = U_\phi = g_{\phi\phi}U^\phi + g_{t\phi}U^t$$

$$= \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta\,\frac{d\phi}{d\tau}$$

$$- \frac{2Mar\sin^2\theta}{\Sigma}\frac{dt}{d\tau}$$

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Σ $a\tau$

$$C = K_{ab} U^a U^b$$

Restrict to $\theta = \frac{\pi}{2}$ $\quad U^\theta = 0 = U_\theta$

Equatorial geo.

$$-1 = g^{ab} U_a U_b = g^{tt} \tilde{E}^2 + g^{rr} (U_r)^2 + g^{\phi\phi} \tilde{J}^2$$

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$$C = K_{ab} U^a U^b$$

Restrict to $\theta = \frac{\pi}{2}$ $\quad U^\theta = 0 = U_\theta$

Equatorial geo.

$$-1 = g^{ab} U_a U_b = g^{tt} \tilde{E}^2 + g^{rr} (U_r)^2 + g^{\phi\phi} \tilde{J}^2 - 2 g^{t\phi} \tilde{E} \tilde{J}$$

$$U_r = g_{rr} U^r = \frac{r^2}{\Delta} \frac{dr}{d\tau}$$

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$$\theta = \frac{\pi}{2}$$

$$g^{tt} = -\frac{\left((r^2+a^2)^2 - a^2\Delta\right)}{\Delta r^2}$$

$$g^{rr} = \frac{\Delta}{r^2}$$

$$g^{\phi\phi} = \Delta - a^2$$

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Equatorial geo.

$$(*)\quad -1 = g^{ab} U_a U_b = g^{tt}\tilde{E}^2 + g^{rr}(U_r)^2 + g^{\phi\phi}\tilde{J}^2 - 2g^{t\phi}\tilde{E}\tilde{J}$$

$$U_r = g_{rr} U^r = \frac{r^2}{\Delta}\frac{dr}{d\tau}$$

$$\theta = \frac{\pi}{2}$$

$$g^{tt} = \frac{-\left((r^2+a^2)^2 - a^2\Delta\right)}{\Delta r^2}$$

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$$g^{rr} = \frac{\Delta}{r^2}$$

$$g^{\phi\phi} = \frac{\Delta - a^2}{\Delta r^2}, \quad g^{t\phi} = \frac{-2Ma}{\Delta r}$$

$$(*) \Rightarrow \; -1 = \frac{\Delta}{r^2} v_r^2 + \frac{a^2\Delta - (r^2+a^2)^2}{\Delta r^2} \tilde{E}$$

$$+ \left(\frac{\Delta - a^2}{\Delta r}\right) \tilde{J}^2 + \frac{4Ma}{/}$$

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$$g^{rr} = \frac{\Delta}{r^2}$$

$$g^{\phi\phi} = \frac{\Delta - a^2}{\Delta r^2}, \quad g^{t\phi} = \frac{-2Ma}{\Delta r}$$

$$(*) \Rightarrow \; -1 = \frac{\Delta}{r^2} v_r^2 + \frac{a^2 \Delta - (r^2 + a^2)^2}{\Delta r^2} \tilde{E}$$

$$+ \left(\frac{\Delta - a^2}{\Delta r}\right) \tilde{J}^2 + \frac{4Ma}{\Delta r}$$

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Sans 18

$$\Delta r^2$$

$$+\left(\frac{\Delta - a^2}{\Delta r}\right)\tilde{J}^2 + \frac{4Ma}{\Delta r}\tilde{E}\tilde{J}$$

$$-\Delta^2 U r^2 = (1-\tilde{E}^2) r^4 - 2M r^3$$
$$+\left[a^2 (1-\tilde{E}^2) + \tilde{J}^2\right] r^2$$
$$-2M (a\tilde{E} - \tilde{J}) r$$

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$$\Delta r^2$$

$$+ \left(\frac{\Delta - a^2}{\Delta r}\right) \tilde{J}^2 + \frac{4Ma}{\Delta r} \tilde{E} \tilde{J}$$

$$-\Delta^2 U r^2 = (1-\tilde{E}^2) r^4 - 2M r^3$$

$$+ \left[a^2 (1-\tilde{E}^2) + \tilde{J}^2\right] r^2$$

$$- 2M (a\tilde{E} - \tilde{J}) r$$

$$:= V(\tilde{E}, \tilde{J}, r)$$

$\uparrow$

$\sim r^4$

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$$\Delta r^2$$

$$+ \frac{(\Delta - a^2)}{\Delta r} \tilde{J}^2 + \frac{4Ma}{\Delta r} \tilde{E} \tilde{J}$$

$$-\Delta^2 U r^2 = (1-\tilde{E}^2) r^4 - 2M r^3$$
$$+ \left[a^2 (1-\tilde{E}^2) + \tilde{J}^2 \right] r^2$$
$$- 2M (a\tilde{E} - \tilde{J}) r$$

$\sim r^4$

$$:= V(\tilde{E}, \tilde{J}, r)$$

Turning points $(\dot{r}=0)$ when $V=0$

(**)

$$-\Delta^2 U r^2 = (1-\tilde{E}^2)\, r^4 - 2M r^3 + \left[a^2(1-\tilde{E}^2) + \tilde{J}^2\right] r^2 - 2M(a\tilde{E} - \tilde{J})\, r$$

$\sim r^4$

$$:= V(\tilde{E}, \tilde{J}, r)$$

Turning points ($\dot{r}=0$) when $V=0$

$\tilde{E} < 1$ for $r \to \infty$

$$V(r) \to (1-\tilde{E}^2)\, r^4 \to \infty$$

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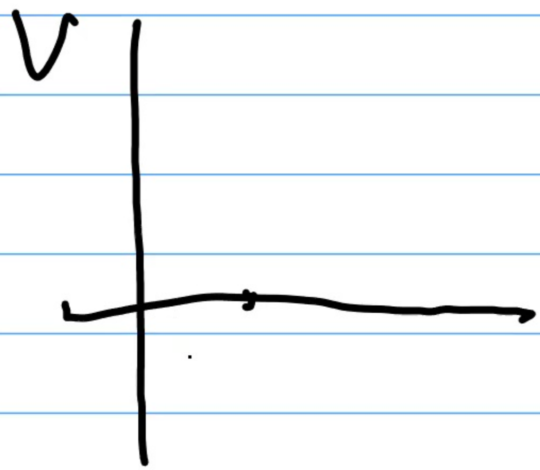
$$V(r_+) = -\left(2Mr_+\tilde{E} - a\tilde{J}\right)^2 \leq 0$$



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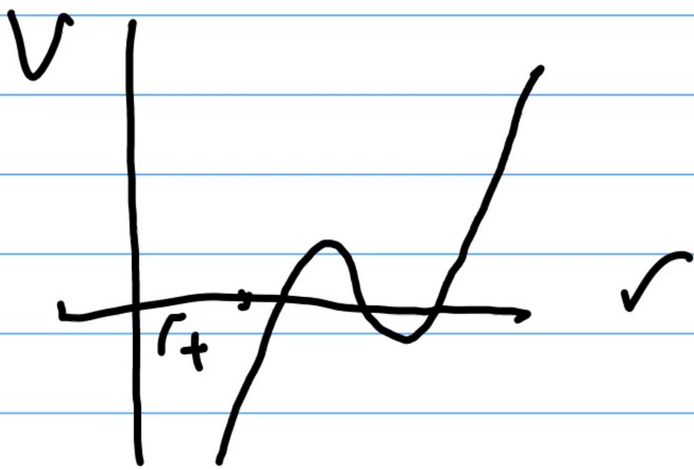
$$V(r_+) = -\left(2Mr_+\tilde{E} - a\tilde{J}\right)^2 \leq 0$$



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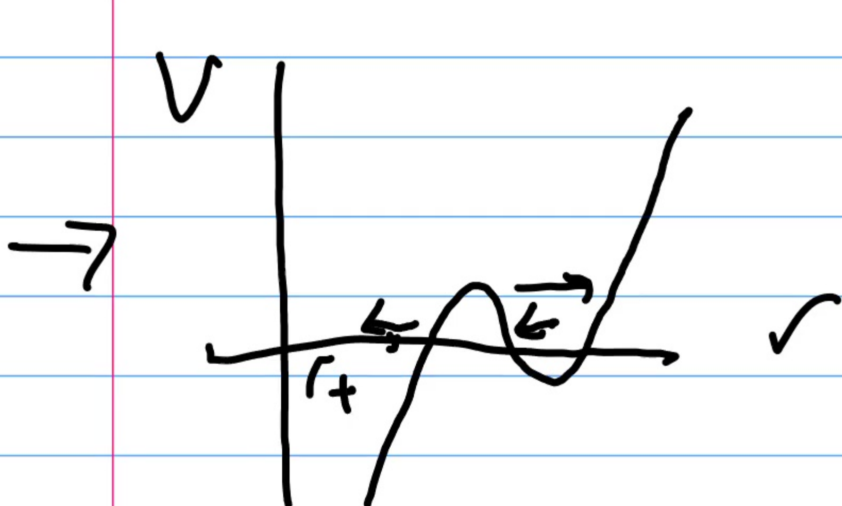
$$V(r_+) = -\left(2Mr_+\tilde{E} - a\tilde{J}\right)^2 \leq 0$$



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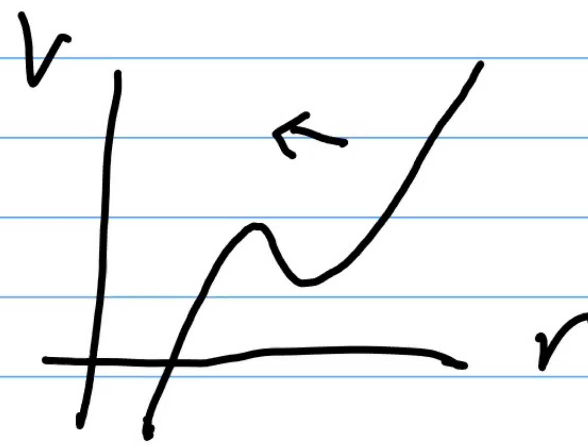

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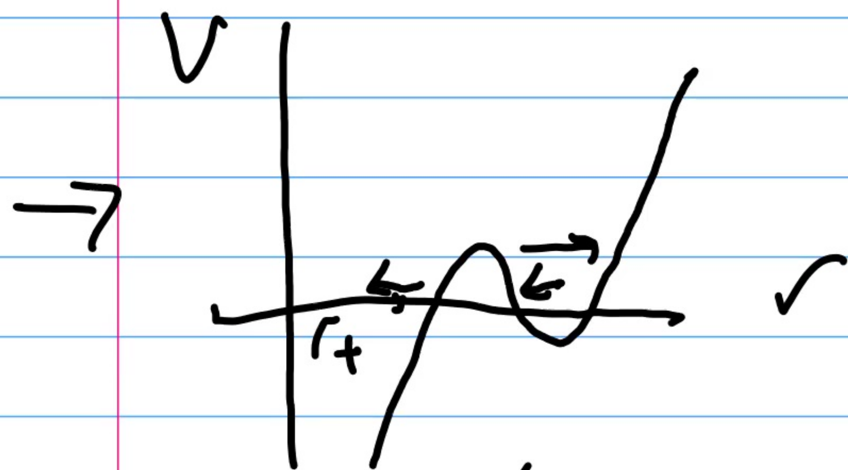

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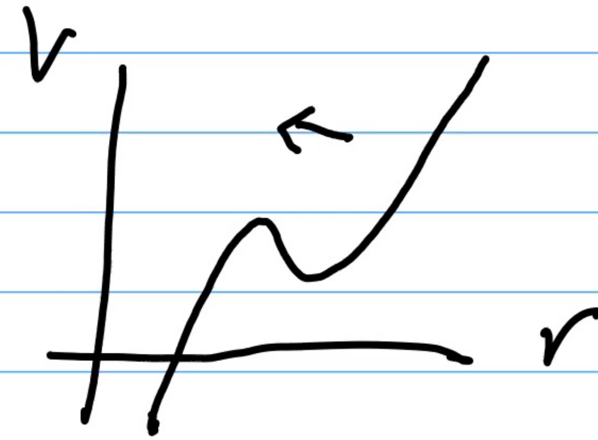

(2)

Marginal case : $V(r_c) = \frac{dV}{dr}(r_c) = 0$

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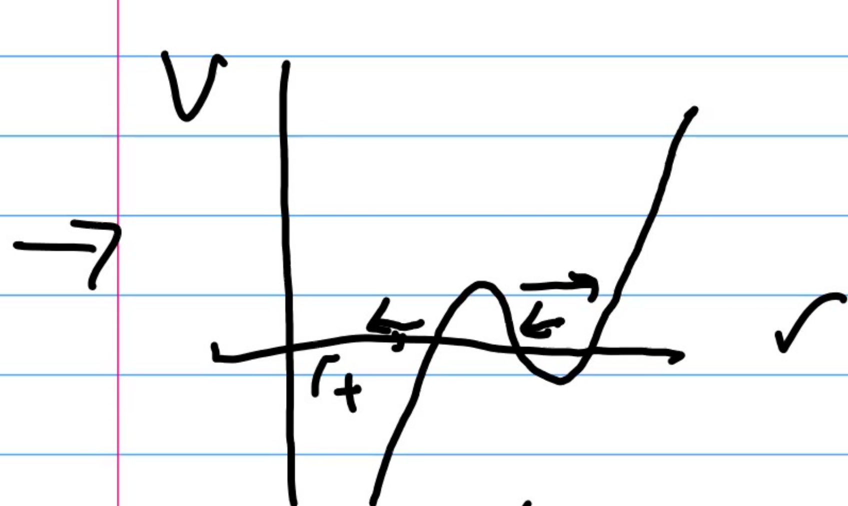

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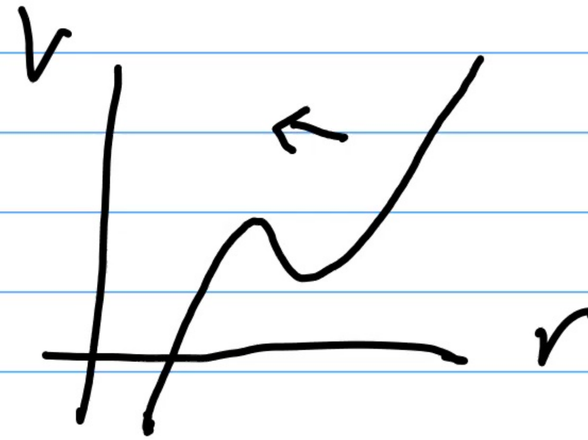

(2)

Marginal case : $V(r_c) = \frac{dV}{dr}(r_c) = 0$

$r_c > r_+$

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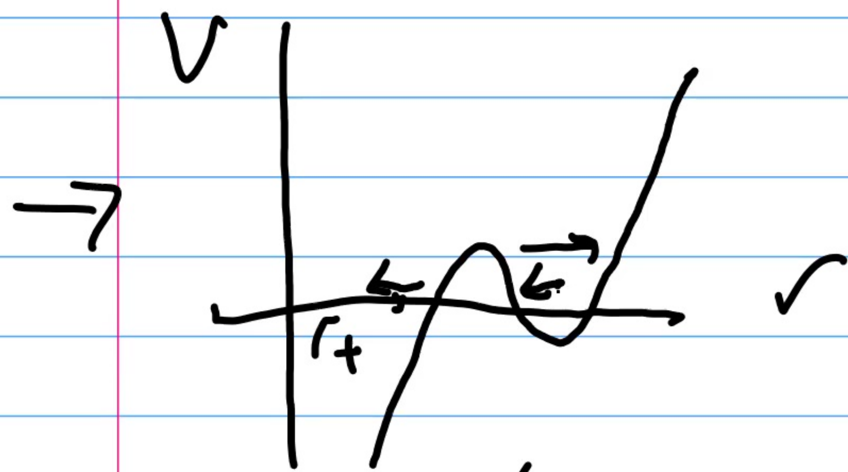

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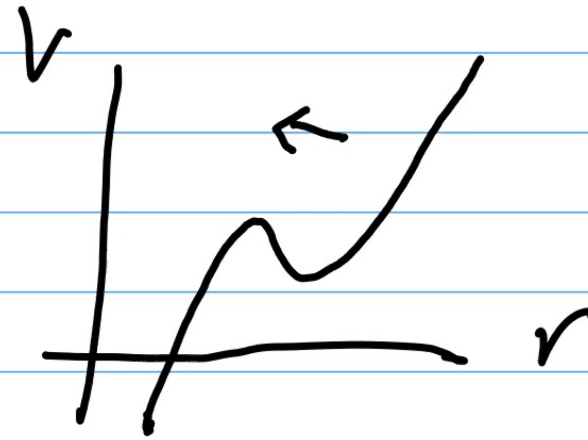

(2)

Marginal case : $V(r_c) = \frac{dV}{dr}(r_c) = 0$

$r_c > r_+$

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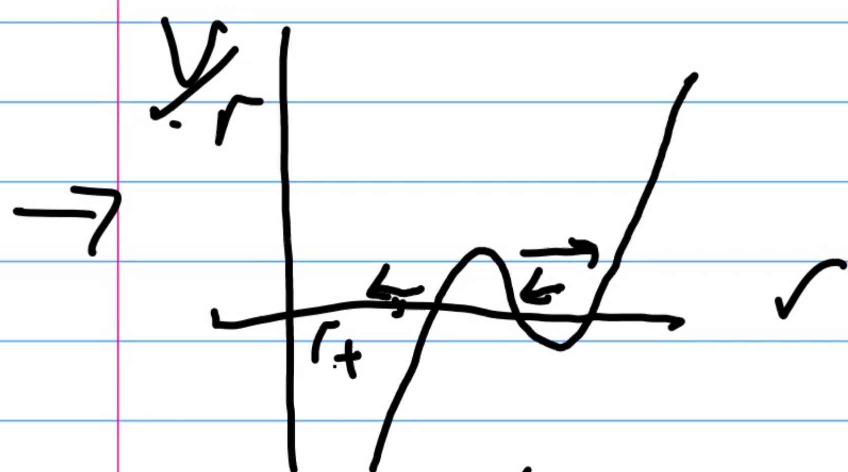
$$V(r_+) = -\left(2Mr_+\tilde{E} - a\tilde{J}\right)^2 \le 0$$



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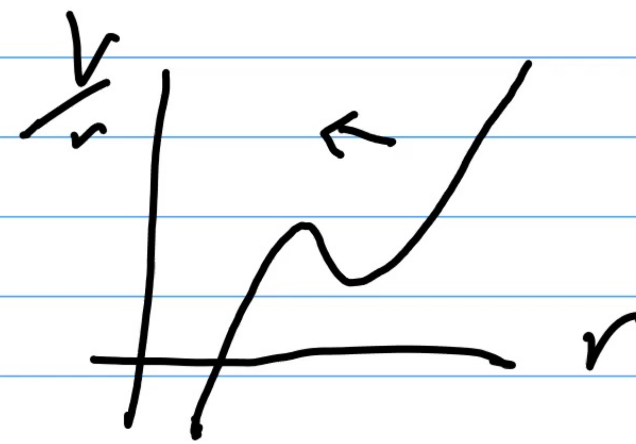

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r_c / r_+

$$\tilde{E}_c = \frac{1 - \frac{2M}{r_c} + \frac{a}{r_c}\sqrt{\frac{M}{r_c}}}{\sqrt{1 - \frac{3M}{r_c} + \frac{2a}{r_c}\sqrt{\frac{M}{r_c}}}}$$

$$\tilde{J}_c = \sqrt{M r_c} - 2a\, M/r_c + a^2/r_c \sqrt{\;/}$$

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$$\tilde{E}_c = \frac{1 - \frac{2M}{r_c} + \frac{a}{r_c}\sqrt{\frac{M}{r_c}}}{\sqrt{1 - \frac{3M}{r_c} + \frac{2a}{r_c}\sqrt{\frac{M}{r_c}}}}$$

$$\tilde{J}_c = \frac{\sqrt{M r_c} - 2a M/r_c + a^2/r_c \sqrt{\frac{M}{r_c}}}{\sqrt{1 - \frac{3M}{r_c} + 2a/r_c \sqrt{\frac{M}{r_c}}}}$$

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$$\sqrt{1-\frac{3M}{r_c}+\frac{2a}{r_c}\sqrt{\frac{M}{r_c}}}$$

$$\tilde{J}_c = \frac{\sqrt{M r_c} - 2a\,M/r_c + a^2/r_c\sqrt{\frac{M}{r_c}}}{\sqrt{1-\frac{3M}{r_c}+2a/r_c\sqrt{\frac{M}{r_c}}}}$$

For $r_c \gg 1$

$$\tilde{E}_c \simeq 1-\frac{M}{2r_c}$$

$$\tilde{J}_c \simeq \sqrt{M r_c}$$

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For $r_c \gg 1$

$$\hat{E}_c \approx 1 - \frac{M}{2r_c}$$

$$\hat{J}_c \approx \sqrt{M r_c}$$

$\hat{J}_c, \hat{E}_c \to \infty$ when

$$r_c^{3/2} - 3M r_c^{1/2} + 2a\sqrt{M} = 0$$

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$$\hat{\tilde{J}}_c \approx \sqrt{M r_c}$$

$\hat{\tilde{J}}_c, \hat{E}_c \to \infty$ when

$$r_c^{3/2} - 3M r_c^{1/2} + 2a\sqrt{M} = 0$$

For $a = 0$ $\quad r_c > 3M$

$\bar{a} = 1 \quad r_c > M$

$\bar{a} = -1 \quad r_c > 4$

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$\hat{\tilde{J}}_c, \hat{E}_c \to \infty$ when

$$r_c^{3/2} - 3M r_c^{1/2} + 2a\sqrt{M} = 0$$

For $a=0$ $\quad r_c > 3M$

$\bar{a} = 1 \quad r_c > M$

$\bar{a} = -1 \quad r_c > 4M$

$$r_c^{min} = 2M\left[1 + \cos\left(\tfrac{2}{3}\cos^{-1}(\bar{a})\right)\right]$$

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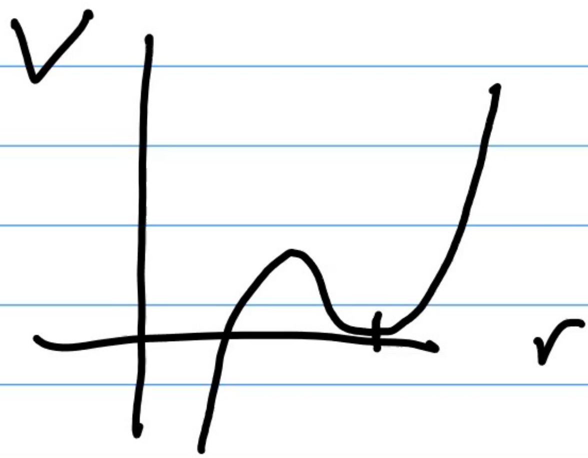
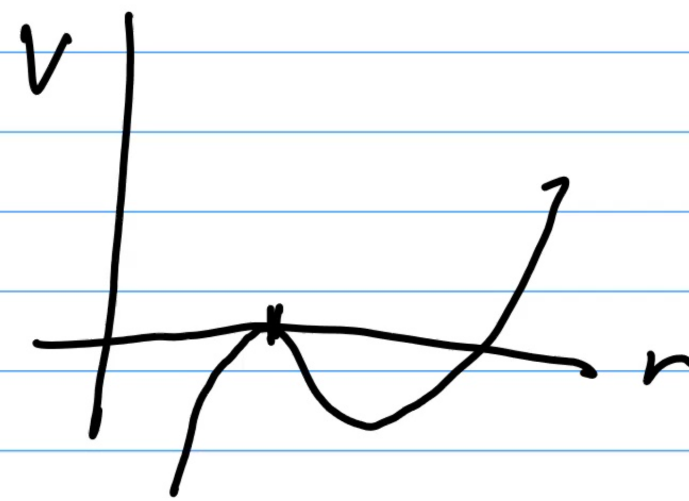
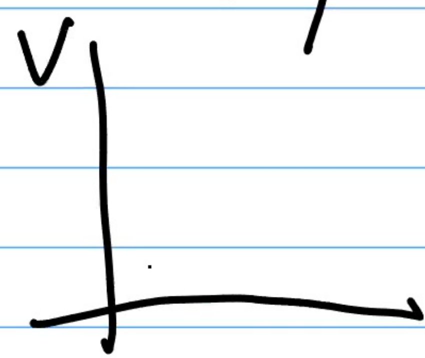
$$r_c^{min} = 2M\left[1+\cos\left(\frac{2}{3}\cos^{-1}(\bar{a})\right)\right]$$





Marginal case



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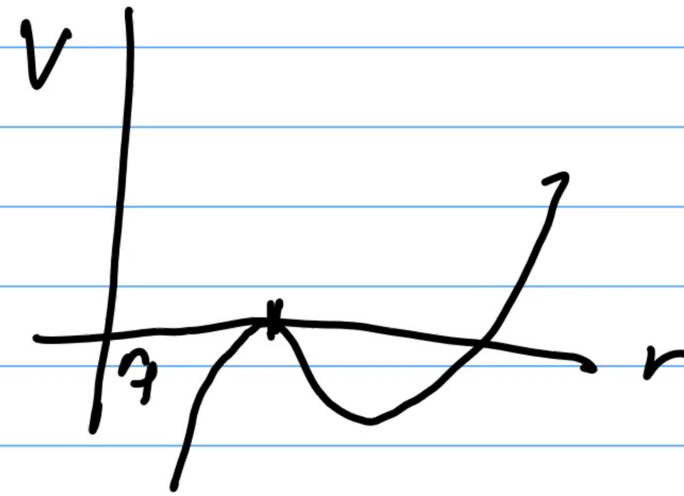
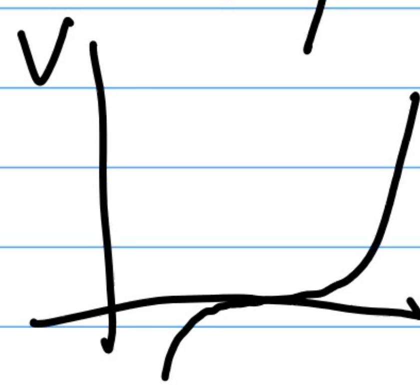
$$r_c^{min} = 2M\left[1+\cos\left(\frac{2}{3}\cos^{-1}(\bar{a})\right)\right]$$



Marginal case



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$$\tilde{E}_c = \frac{1 - \frac{\cdots}{r_c} + \frac{\cdots}{r_c}\sqrt{\frac{M}{r_c}}}{\sqrt{1-\frac{3M}{r_c}+\frac{2a}{r_c}\sqrt{\frac{M}{r_c}}}}$$

$$\tilde{J}_c = \frac{\sqrt{Mr_c} - 2aM/r_c + \frac{a^2}{r_c}\sqrt{\frac{M}{r_c}}}{\sqrt{1-\frac{3M}{r_c}+\frac{2a}{r_c}\sqrt{\frac{M}{r_c}}}}$$

For $r_c \gg 1$ $\quad \tilde{E}_c \simeq 1 - \frac{M}{2r_c}$

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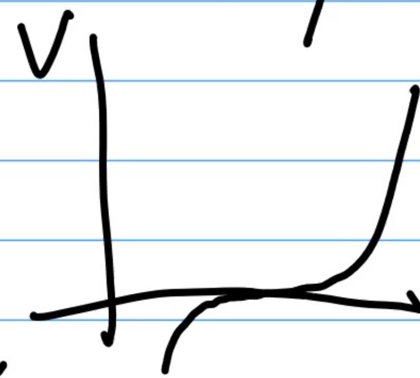
Marginal case



ISCO $\qquad \dfrac{d\tilde{E}_L}{dr_c}\left(r_{ISCO}\right) = 0$

r_{ISCO}

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Sans 18

$$(*) \Rightarrow \; -1 = \frac{\Delta}{r^2} u_r^2 + \frac{a^2\Delta - (r^2+a^2)^2}{\Delta r^2} \tilde{E}^2$$

$$+ \left(\frac{\Delta - a^2}{\Delta r}\right) \tilde{J}^2 + \frac{4Ma}{\Delta r} \tilde{E}\tilde{J}$$

$(**)$

$$-\Delta u_r^2 = (1-\tilde{E}^2) r^4 - 2Mr^3 + \left[a^2(1-\tilde{E}^2) + \tilde{J}^2\right] r^2 - 2M(a\tilde{E} - \tilde{J}) r$$

$\uparrow$

$\sim r^4$

$$:= V(\tilde{E}, \tilde{J}, r)$$

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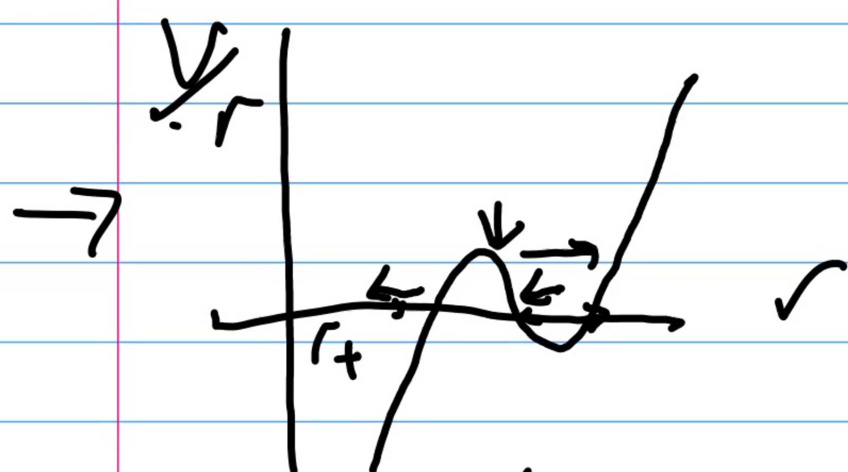
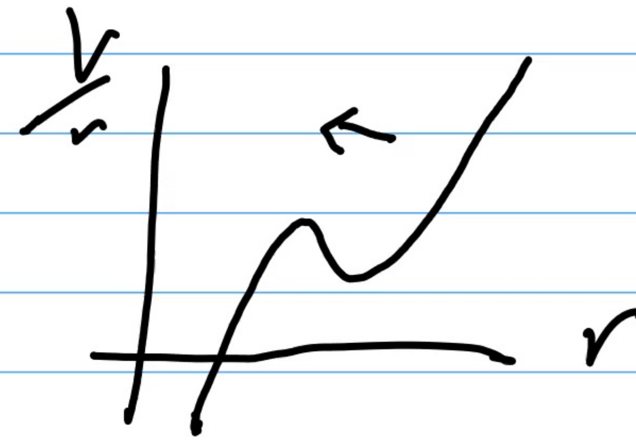
$$V(r_+) = -(2Mr_+\tilde{E} - a\tilde{J})^2 \leq 0$$



(1)

or



(2)

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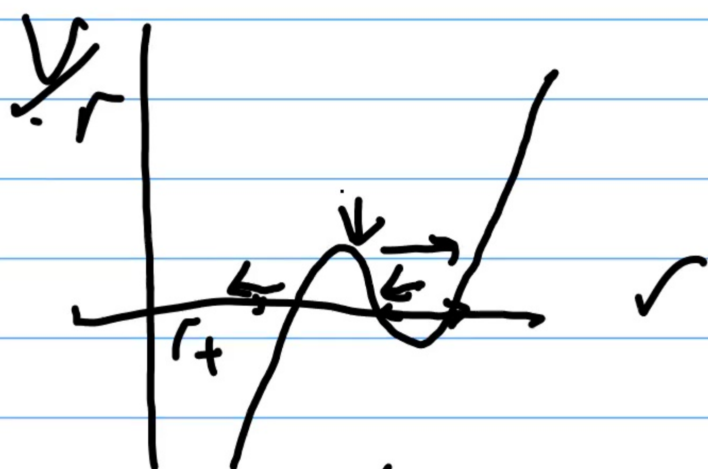
$$V(r_+) = -\left(2Mr_+\tilde{E} - a\tilde{J}\right)^2 \leq 0$$



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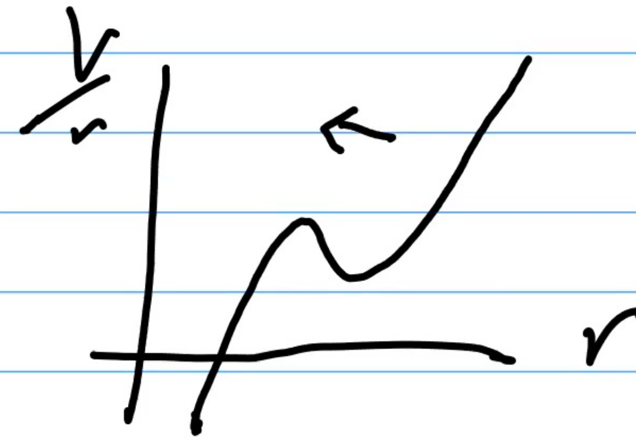

(2)

Marginal case : $V(r_c) = \frac{dV}{dr}(r_c) = 0$

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$$r_c > r_+$$

$$\tilde{E}_c = \frac{1 - \frac{2M}{r_c} + \frac{a}{r_c}\sqrt{\frac{M}{r_c}}}{\sqrt{1 - \frac{3M}{r_c} + \frac{2a}{r_c}\sqrt{\frac{M}{r_c}}}}$$

$$\tilde{J}_c = \frac{\sqrt{M r_c} - 2aM/r_c + a^2/r_c\sqrt{\frac{M}{r_c}}}{\sqrt{1 - \frac{3M}{r_c} + 2a/r_c\sqrt{\frac{M}{r_c}}}}$$

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dr_c

$$r_{ISCO}^2 - 6M\, r_{ISCO} + 8a\sqrt{M}\, r_{ISCO}^{1/2} - 3a^2 = 0$$

For $a = 0$ $\quad r_{ISC}$

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For $a=0$ $\quad r_{ISCO}=6M$

$a=M$ $\quad r_{ISCO}=M$

$a=-M$ $\quad r_{ISCO}=9M$

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For $a = 0$ $\quad r_{ISCO} = 6M$

$a = M$ $\quad r_{ISCO} = M$

$a = -M$ $\quad r_{ISCO} = 9M$

For $r_c^{min} < r < r_{ISCO}$

unstable circular

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For $a=0 \qquad r_{ISCO} = 6M$

$a=M \qquad r_{ISCO} = M$

$a=-M \qquad r_{ISCO} = 9M$

For $r_c^{min} < r < r_{ISCO}$

unstable circular orbits

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For $r_c^{min} < r < r_{ISCO}$ unstable circular orbits

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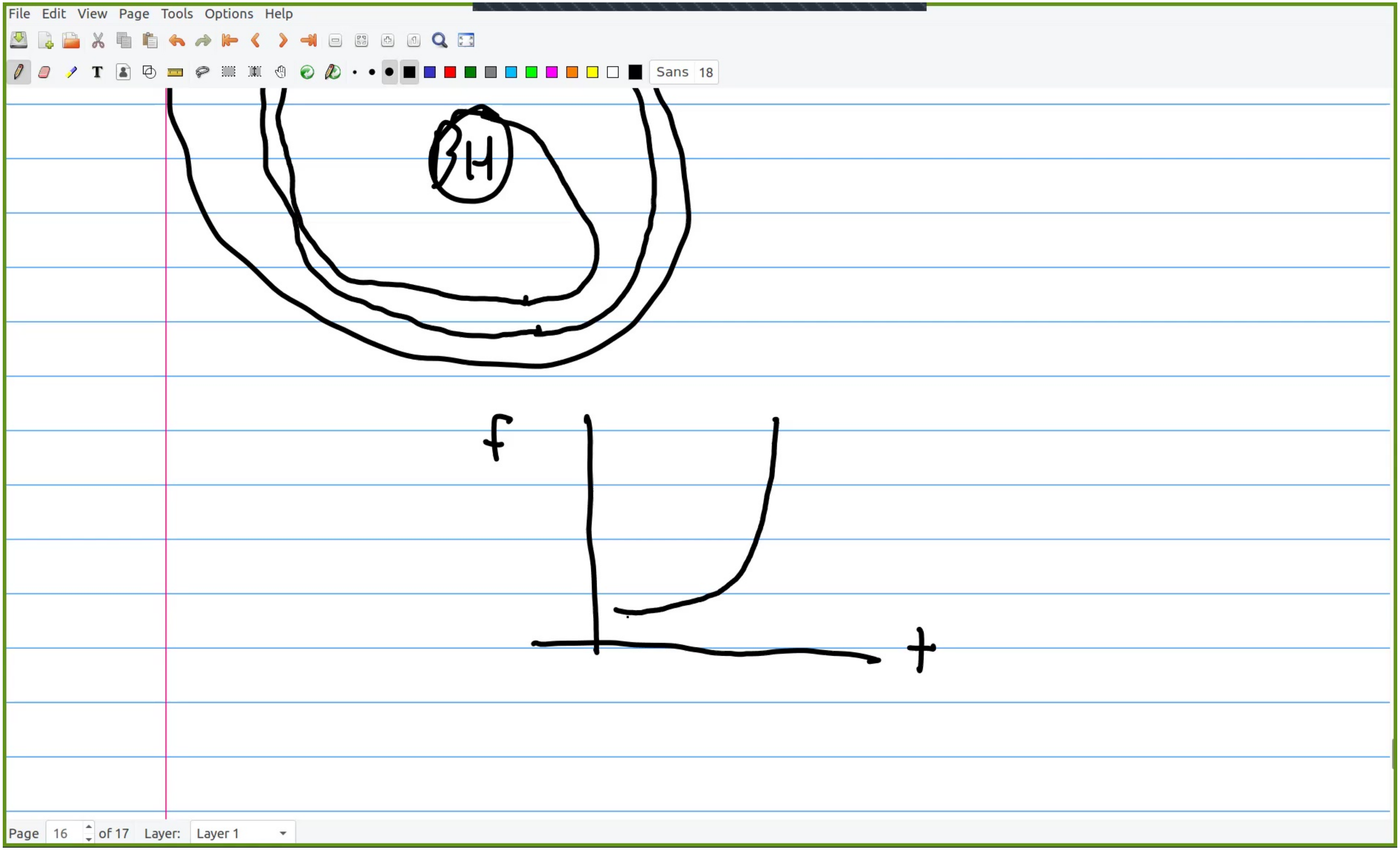
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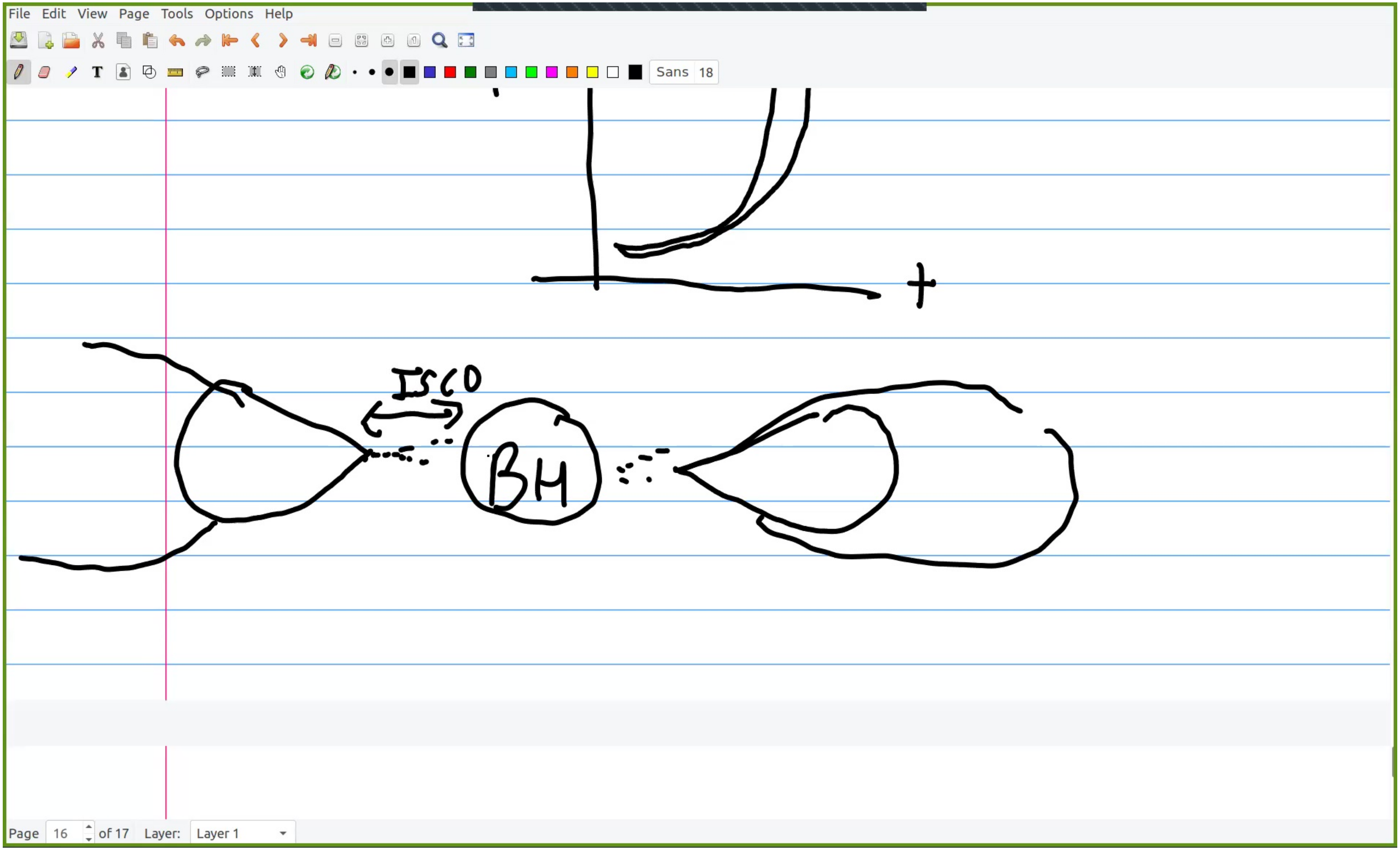
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