Title: Cosmology (2021/2022)

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Collection: Cosmology (2021/2022)

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Abstract: This class is an introduction to cosmology. We'll cover expansion history of the universe, thermal history, dark matter models, and as much cosmological perturbation theory as time permits.

$$S = S_{EH} + S_m \qquad S_m = \int d^4x \sqrt{-g}\left[-\frac{1}{2}(\nabla^\mu \varphi)(\nabla_\mu \varphi) - V(\varphi)\right]$$

$$\delta S_m = \int d^4x \sqrt{-g}\left(-\frac{1}{2}T_{\mu\nu}\right)\delta g^{\mu\nu} \qquad \text{DEFINES } T_{\mu\nu} \qquad \delta g_{\mu\nu} = -g_{\mu\rho}\, g_{\nu\sigma}\,(\delta g^{\rho\sigma})$$

$$= \int d^4x \sqrt{-g}\left(\frac{1}{2}T^{\mu\nu}\right)\delta g_{\mu\nu}$$

$\delta(\sin x) = (\cos x)\,\delta x$

$\delta(\det A) = (\det A)\,\mathrm{Tr}\left\{A^{-1}\delta A\right\}$ "JACOBI'S FORMULA"

$(\delta g^{\rho\sigma})$

$$\delta(\sin x) = (\cos x)\,\delta x$$

$$\delta(\mathrm{Det}\ A) = (\mathrm{Det}\ A)\,\mathrm{Tr}\left[A^{-1}\delta A\right] \qquad \text{"JACOBI'S FORMULA"}$$

$$\delta\sqrt{-g} = \delta\left[(-\mathrm{Det}\ G)^{-1/2}\right] \qquad G = \{g_{\mu\nu}\}_{0\le\mu,\nu\le 3}$$

$$= -\frac{1}{2}(-\mathrm{Det}\ G)^{-3/2}(-\mathrm{Det}\ G)\,\mathrm{Tr}\left[G^{-1}\delta G\right]$$

$$= -\frac{1}{2}\sqrt{-g}\; g_{\mu\nu}\,\delta g^{\mu\nu}$$

$$\delta S_m = \delta \int d^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V \right)$$

$$= \int d^4x \sqrt{-g} \left[-\frac{1}{2} (\delta g^{\mu\nu}) (\partial_\mu \phi)(\partial_\nu \phi) \right]$$

$$+ \int d^4x \left(-\frac{1}{2} \sqrt{-g} \, g_{\mu\nu} \delta g^{\mu\nu} \right) \left(-\frac{1}{2} (\nabla^\rho \phi)(\nabla_\rho \phi) - V(\phi) \right)$$

$$= \int d^4x \sqrt{-g} \, \delta g^{\mu\nu} \Bigg[\underbrace{-\frac{1}{2} (\nabla_\mu \phi)(\nabla_\nu \phi) + \frac{1}{4} (\nabla^\rho \phi)(\nabla_\rho \phi) + \frac{1}{2} V(\phi) g_{\mu\nu}}_{-\frac{1}{2} T_{\mu\nu}} \Bigg]$$

$$\Rightarrow T_{\mu\nu} = (\nabla_\mu \phi)(\nabla_\nu \phi) - \frac{1}{2}(\nabla^\rho \phi)(\nabla_\rho \phi) g_{\mu\nu} - V(\phi) g_{\mu\nu}$$

EOMs

$$\nabla^2 \phi = V'(\phi)$$

$$G_{\mu\nu} = \frac{1}{M_{Pl}^2} T_{\mu\nu}$$

$g_{\mu\nu}$]

BACKGROUND SOLUTION

- INF

FRW SYMMETRIES:

$$\phi = \phi(t) \qquad g_{\mu\nu} = \begin{pmatrix} -1 & \\ & a(t)^2 \delta_{ij} \end{pmatrix}$$

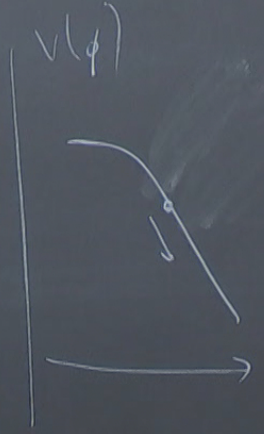
PARAMETERIZED BY $(\phi(t), a(t))$

NEXT: FIND EOMs

- INFLATON: $\nabla^\mu \nabla_\mu \phi = V'(\phi)$

FRW: $\nabla^\mu \nabla_\mu \phi = -\frac{\partial^2 \phi}{\partial t^2} - 3H\frac{\partial \phi}{\partial t} + \frac{1}{a^2}\partial_i \partial_i \phi$

$$\Rightarrow \boxed{\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0}$$



- METRIC: $H^2 = \frac{\rho}{3M_{Pl}^2}$ $\qquad \frac{\ddot{a}}{a} = -\frac{1}{6M_{Pl}^2}(\rho + 3p)$ $\qquad 8\pi G = \frac{1}{M_{Pl}^2}$

WHERE ρ, p ARE DEFINED BY $T_{\mu\nu} = \begin{pmatrix} \rho & & \\ & a^2 p & \\ & & a^2 p \end{pmatrix}$

$$T_{\mu\nu} = (\nabla_\mu \phi)(\nabla_\nu \phi) - \frac{1}{2}(\nabla^\rho \phi)(\nabla_\rho \phi) g_{\mu\nu} - V(\phi) g_{\mu\nu}$$

$$T_{00} = \dot{\phi}^2 - \frac{1}{2}(-\dot{\phi}^2)(-1) - V(-1)$$

$$\Rightarrow \boxed{\rho = \frac{1}{2}\dot{\phi}^2 + V}$$

$$T_{ij} = 0 - \frac{1}{2}(-\dot{\phi}^2)(a^2 \delta_{ij}) - V(a^2 \delta_{ij}) \Rightarrow \boxed{p = \frac{1}{2}\dot{\phi}^2 - V}$$

$$= \frac{1}{2}\dot{\phi}^2 a^2 \delta_{ij} - V a^2 \delta_{ij}$$

A POTENTIAL IS SLOW-ROLL IF $\epsilon, \eta \ll 1$

$$\epsilon = \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2 \qquad \eta = M_{pl}^2\left(\frac{V''}{V}\right) \qquad [\text{FEW} \times 10^{-2}]$$

EXACT EOMs:

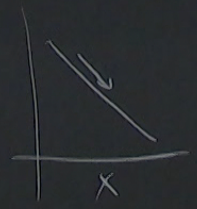
$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \qquad H^2 = \frac{1}{3M_{pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V\right)$$

SLOW-ROLL APPROX:

$$3H\dot{\phi} + V'(\phi) \approx 0 \qquad H^2 \approx \frac{1}{3M_{pl}^2} V$$

IF $\varepsilon, \eta \ll 1$

$\eta = M_{Pl}^2 \left(\frac{V''}{V}\right)$ [FEW $\times 10^{-2}$]



$\ddot{x} + \mu\dot{x} = -V'(x)$

$\mu\dot{x} \simeq -V'$

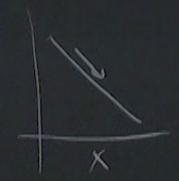
$H^2 = \frac{1}{3M_{Pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V\right)$

$H^2 \simeq \frac{1}{3M_{Pl}^2} V$

A POTENTIAL IS SLOW-ROLL IF $\varepsilon, \eta \ll 1$

$$\varepsilon = \frac{M_{Pl}^2}{2}\left(\frac{V'}{V}\right)^2 \qquad \eta = M_{Pl}^2\left(\frac{V''}{V}\right)$$

[FEW $\times 10^{-2}$]



EXACT EOMs:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \qquad \left| \; H^2 = \frac{1}{3M_{Pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V\right)\right.$$

SLOW-ROLL APPROX:

$$3H\dot{\phi} + V'(\phi) \approx 0 \qquad H^2 \approx \frac{1}{3M_{Pl}^2} V \qquad (\Phi)$$

DEFINE $(\phi_{SR}(t), a_{SR}(t))$ = EXACT SOLUTION TO $(\star)$

$$\dot{\phi}_{SR} = -\frac{V'}{3H}$$

$$= -M_{Pl}\frac{V'}{(3V)^{1/2}}$$

$$\ddot{\phi}_{SR} = \frac{d}{dt}\left[-M_{Pl}\frac{V'}{(3V)^{1/2}}\right]$$

$$= -M_{Pl}\underbrace{\left[\frac{V''}{(3V)^{1/2}} - \frac{1}{2}\frac{V'^2}{(3V)^{1/2}V}\right]}_{\frac{d}{d\phi}[\cdots]}$$

$$= M_{Pl}^2\left(\frac{V'V''}{3V} - \frac{V'^3}{6V^2}\right)$$

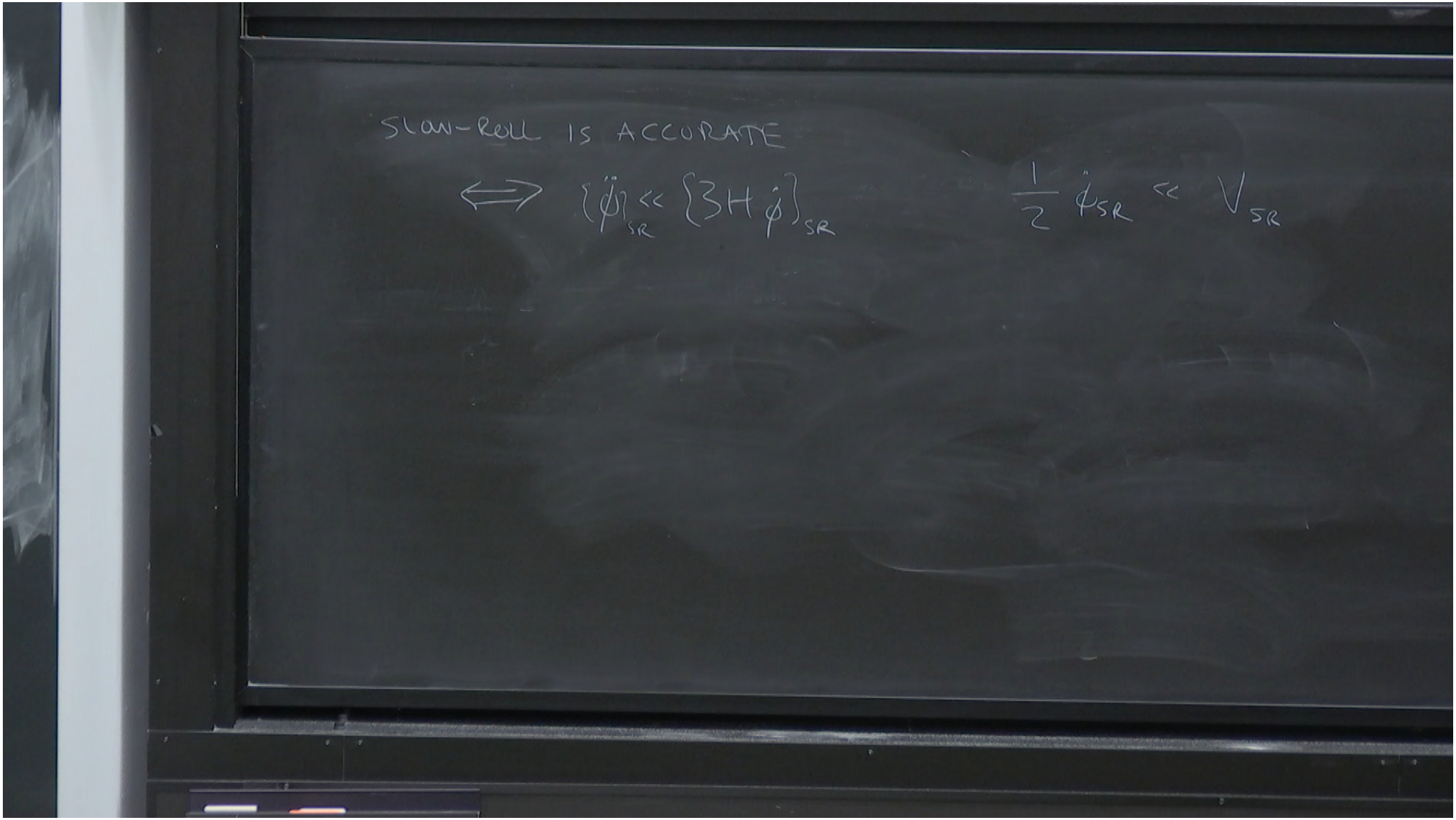
$$H_{SR} = M_{Pl}^{-1}\left(\frac{V}{3}\right)^{1/2}$$

EXACT SOLUTION TO (8)

$$\ddot{\phi}_{SR} = \frac{d}{dt}\left[-M_{pl}\frac{V'}{(3V)^{1/2}}\right]$$

$$= -M_{pl}\underbrace{\left[\frac{V''}{(3V)^{1/2}} - \frac{1}{2}\frac{V'^2}{(3V)^{1/2}V}\right]}_{\frac{d}{d\phi}[\dots]}\underbrace{\left[-M_{pl}\frac{V'}{(3V)^{1/2}}\right]}_{\frac{d\phi}{dt}}$$

$$= \boxed{M_{pl}^2\left(\frac{V'V''}{3V} - \frac{V'^3}{6V^2}\right)}$$


SLOW-ROLL IS ACCURATE
$\Longleftrightarrow \{\ddot{\phi}\}_{SR} \ll \{3H\dot{\phi}\}_{SR}$
$\frac{1}{2}\dot{\phi}_{SR} \ll V_{SR}$

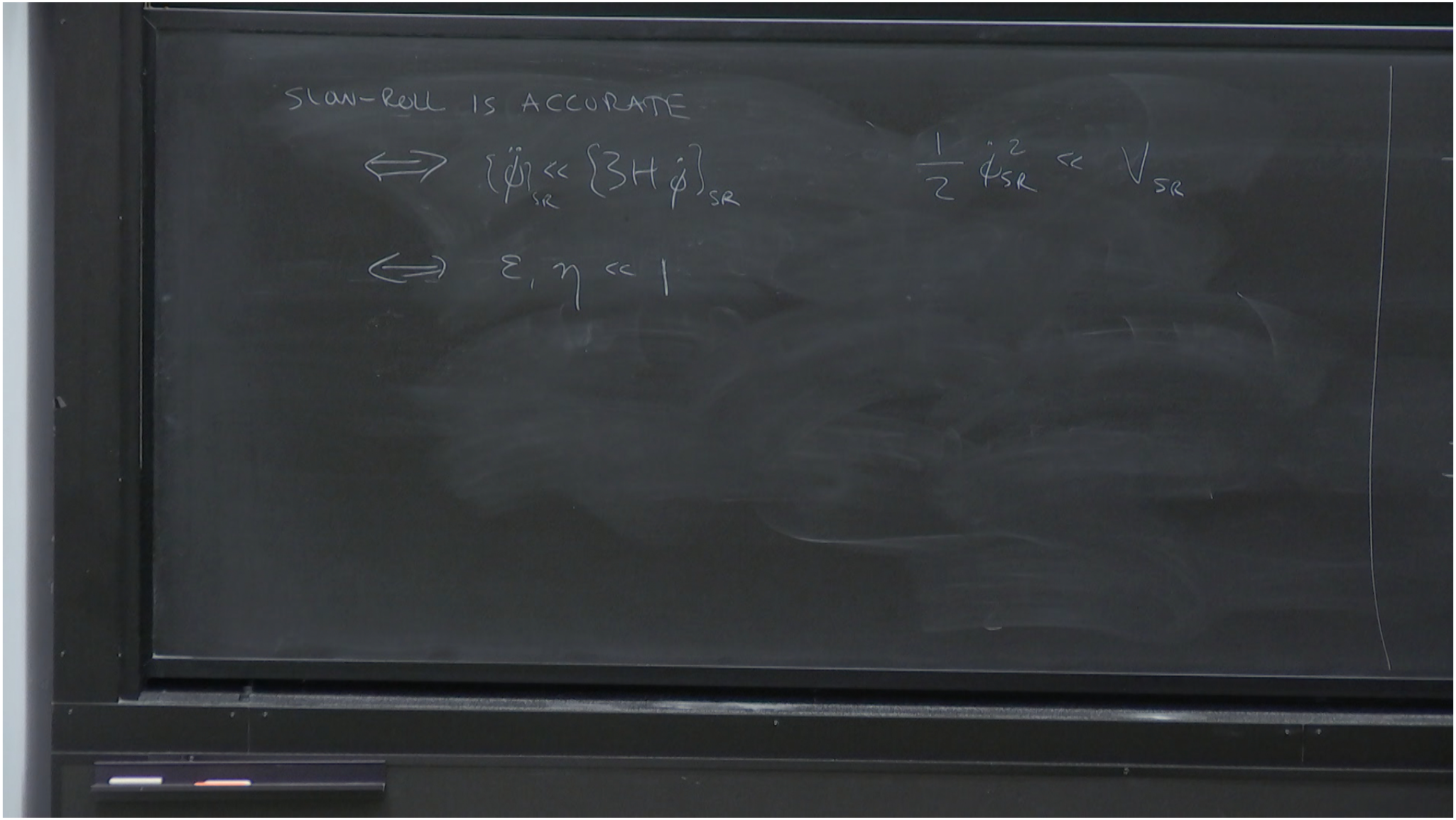
$$\frac{\ddot{\phi}_{SR}}{3H\dot{\phi}_{SR}} = \frac{M_{Pl}^2\left(V'V''/3V - V'^3/6V^2\right)}{3\left[M_{Pl}^{-1}\left(\frac{V}{3}\right)\right]\left[-M_{Pl}V'(3V)^{1/2}\right]}$$

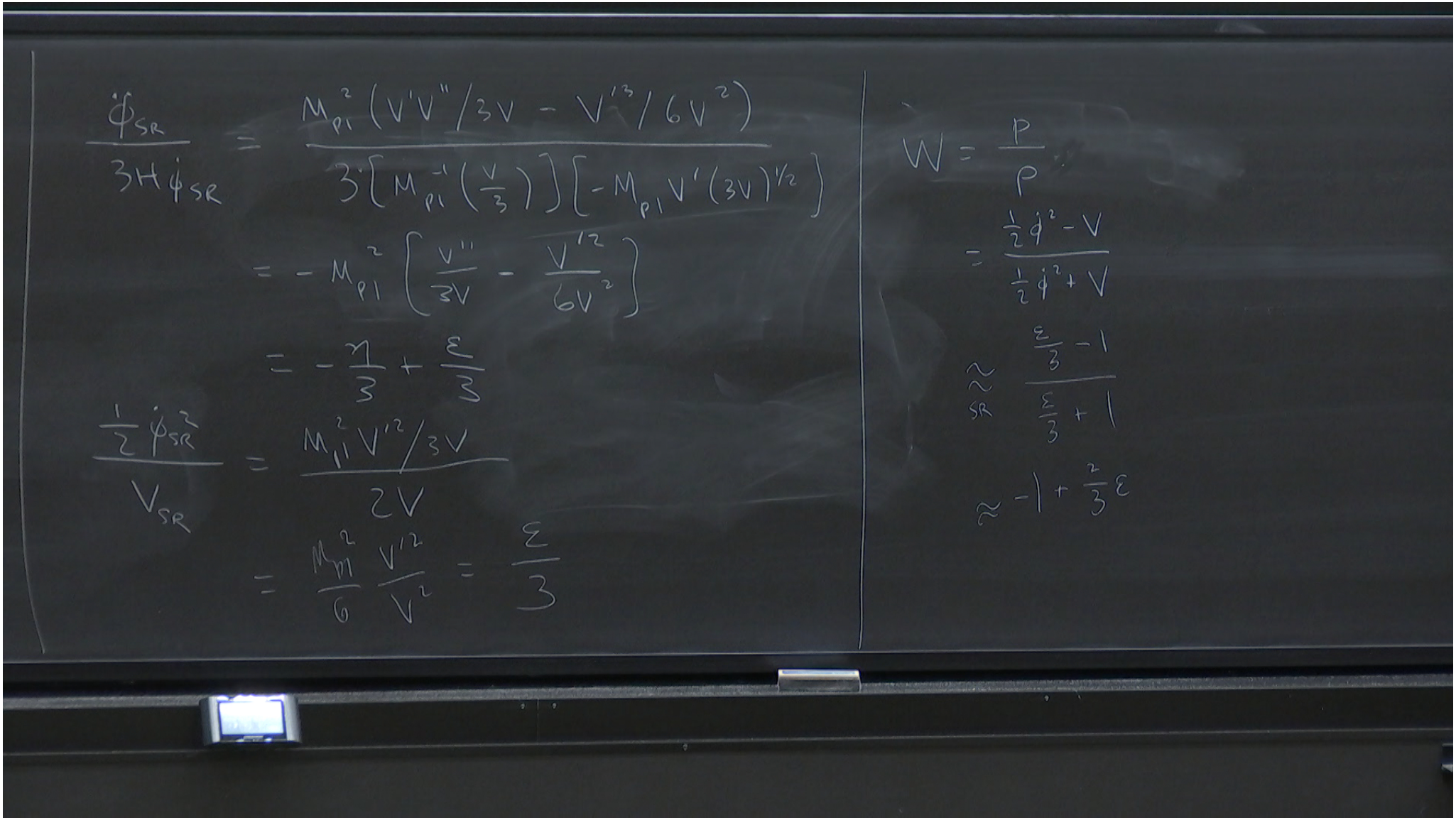
$$= -M_{Pl}^2\left[\frac{V''}{3V} - \frac{V'^2}{6V^2}\right]$$

$$= -\frac{\eta}{3} + \frac{\varepsilon}{3}$$

$$\frac{\frac{1}{2}\dot{\phi}_{SR}^2}{V_{SR}} = \frac{M_{Pl}^2 V'^2/3V}{2V}$$

$$= \frac{M_{Pl}^2}{6}\frac{V'^2}{V^2} = \frac{\varepsilon}{3}$$


SLOW-ROLL IS ACCURATE
$\Leftrightarrow \{\ddot{\phi}\}_{SR} \ll \{3H\dot{\phi}\}_{SR}$
$\frac{1}{2}\dot{\phi}^2_{SR} \ll V_{SR}$
$\Leftrightarrow \varepsilon, \eta \ll 1$


$\frac{\dot\phi_{SR}}{3H\dot\phi_{SR}} = \frac{M_{Pl}^2\,(V'V''/3V - V'^3/6V^2)}{3\left[M_{Pl}^{-1}\left(\frac{V}{3}\right)\right]\left[-M_{Pl}V'(3V)^{1/2}\right]}$
$= -M_{Pl}^2\left[\frac{V''}{3V} - \frac{V'^2}{6V^2}\right]$
$= -\frac{\eta}{3} + \frac{\varepsilon}{3}$
$\frac{\frac{1}{2}\dot\phi_{SR}^2}{V_{SR}} = \frac{M_{Pl}^2 V'^2/3V}{2V}$
$= \frac{M_{Pl}^2}{6}\frac{V'^2}{V^2} = \frac{\varepsilon}{3}$
$w = \frac{p}{\rho}$
$= \frac{\frac{1}{2}\dot\phi^2 - V}{\frac{1}{2}\dot\phi^2 + V}$
$\underset{SR}{\approx} \frac{\frac{\varepsilon}{3} - 1}{\frac{\varepsilon}{3} + 1}$
$\approx -1 + \frac{2}{3}\varepsilon$

$V^2)$

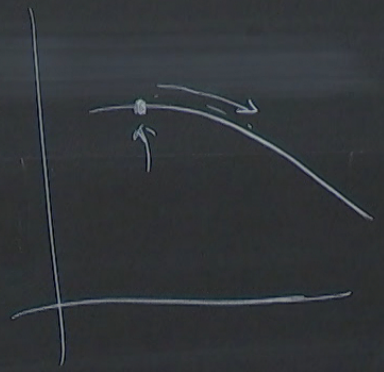
$\dot{\phi}'(3V)^{1/2}$ }

$$W = \frac{P}{\rho}$$

$$= \frac{\frac{1}{2}\dot{\phi}^2 - V}{\frac{1}{2}\dot{\phi}^2 + V}$$

$$\overset{SR}{\approx} \frac{\frac{\varepsilon}{3} - 1}{\frac{\varepsilon}{3} + 1}$$

$$\approx -1 + \frac{2}{3}\varepsilon$$

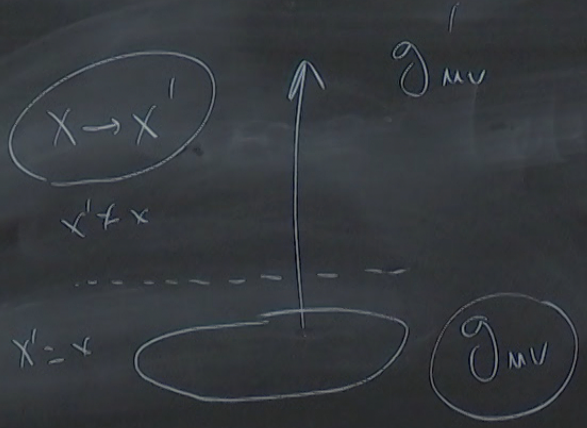


GENERAL COMMENTS ON GR

$$G_{\mu\nu} = \frac{T_{\mu\nu}}{M_{Pl}^2}$$

10 DOFs $\{g_{\mu\nu}\}$

2 PHYSICAL DOFs ?



Fs $\{g_{\mu\nu}\}$

SICAL DOFs ?

TOY ODE EXAMPLE: $x(t)$, $y(t)$, $z(t)$

$$\dot{x} + \dot{y} = y + z$$

$$\dot{y} + \dot{z} = x - z$$

$$\dot{x} - \dot{z} = -x + y + 2z$$

DEFINE $u = x + y \quad v = y + z$

$$\dot{u} = v$$

$$\dot{v} = u - v$$

$$\dot{u} - \dot{v} = -u + 2v$$

TOY ODE EXAMPLE: $x(t)$, $y(t)$, $z(t)$

$$\dot{x} + \dot{y} = y + z$$

$$\dot{y} + \dot{z} = x - z$$

$$\dot{x} - \dot{z} = -x + y + 2z$$

DEFINE $u = x + y \quad v = y + z$

$$\dot{u} = v$$

$$\dot{v} = u - v$$

$$\dot{u} - \dot{v} = -u + 2v$$

ONE "GAUGE

$x(t) \to x(t$

$y(t) \to y($

$z(t) \to z($

$x(t),\ y(t),\ z(t)$

ONE "GAUGE MODE"

$x(t) \to x(t) + \varepsilon(t)$

$y(t) \to y(t) - \varepsilon(t)$

$z(t) \to z(t) + \varepsilon(t)$

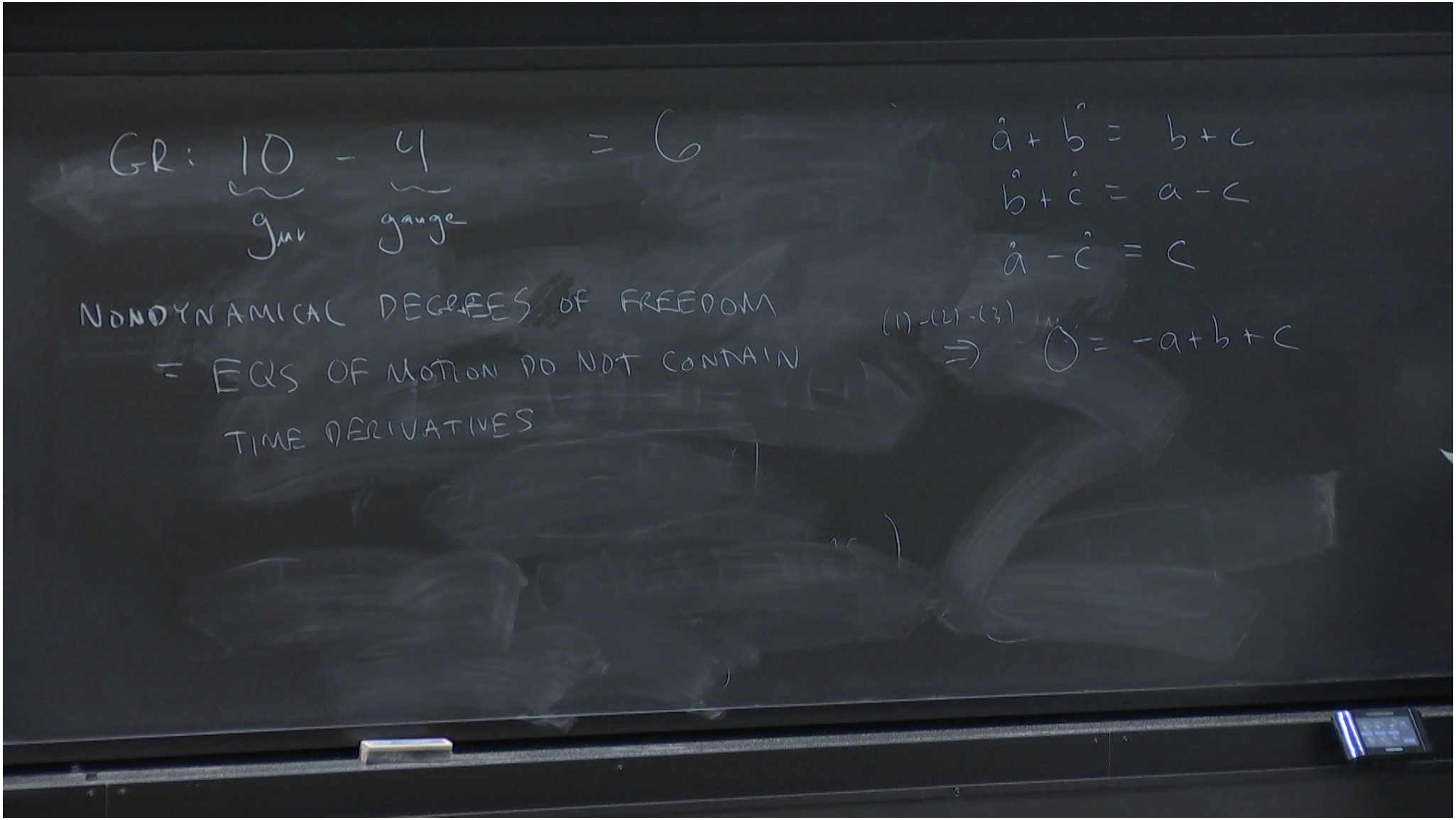
FIX BY "CHOOSING A GAUGE"

$z = 0$

$\dot{x} + \dot{y} = y$

$\dot{y} = x$

$\dot{x} = -x + y$


GR: 10 - 4 = 6
$g_{\mu\nu}$ gauge
NONDYNAMICAL DEGREES OF FREEDOM
= EQS OF MOTION DO NOT CONTAIN TIME DERIVATIVES
$\dot{a} + \dot{b} = b + c$
$\dot{b} + \dot{c} = a - c$
$\dot{a} - \dot{c} = c$
(1)-(2)-(3)
$\Rightarrow \ddot{0} = -a + b + c$

GR: 10 - 4 - 4 = 6

$g_{\mu\nu}$ gauge nondynamical

NONDYNAMICAL DEGREES OF FREEDOM

= EQS OF MOTION DO NOT CONTAIN TIME DERIVATIVES

$\dot{a}+$

$\dot{b}+$

$\dot{a}-$

(1)-(2)-(3)

$\Rightarrow$ C

REPLACE C=

$\dot{a}+\dot{b}$

$\dot{a}$

GR: 10 - 4 - 4 = 2

$g_{\mu\nu}$ gauge nondynamical

NONDYNAMICAL DEGREES OF FREEDOM

= EQS OF MOTION DO NOT CONTAIN TIME DERIVATIVES

$\dot{a}$ +

$\dot{b}$ +

$\dot{a}$ -

(1)-(2)-(3)

$\Rightarrow$

REPLACE C=

$\dot{a}$ + $\dot{b}$

$\dot{a}$

INFLATION

$$1 + 10 - 4 - 4 = 3$$

ϕ $g_{\mu\nu}$ gauge nondynamical

NONDYNAMICAL DEGREES OF FREEDOM

= EQS OF MOTION DO NOT CONTAIN TIME DERIVATIVES

(1)-(2)-(3) ⇒

REPLACE

PERTURBATIONS

$$\phi(x,t) = \bar{\phi}(t) + \delta\phi(x,t)$$

$$g_{\mu\nu}(x,t) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(x,t)$$

ALL CALCULATIONS TO LINEAR ORDER IN $(\delta\phi)$, $(\delta g_{\mu\nu})$

STEP 1. CHOOSING A GAUGE

- SMALL GAUGE TRANSFORMATION $X^{M\prime} = X^M - \varepsilon^M(x)$ IS PARAMETERIZED BY 4-VECTOR FIELD ε^M

START WITH $(\delta\phi, \delta g_{\mu\nu}) \xrightarrow[x'^\mu = x^\mu - \xi^\mu(x)]{\text{GAUGE}} (\delta\phi', \delta g'_{\mu\nu})$

$$\delta\phi' = \delta\phi + \dot{\bar{\phi}}\,\xi^0$$
$$\delta g'_{00} = \delta g_{00} - 2\dot{\xi}^0$$
$$\delta g'_{0i} = \delta g_{0i} + a^2\dot{\xi}^i - \partial_i\xi^0$$
$$\delta g'_{ij} = \delta g_{ij} + 2a^2H\xi^0\delta_{ij} + a^2\left(\partial_i\xi_j + \partial_j\xi_i\right)$$

$$\delta g'_{\mu\nu} = \delta g_{\mu\nu} + \mathcal{L}_\xi \bar{g}_{\mu\nu}$$

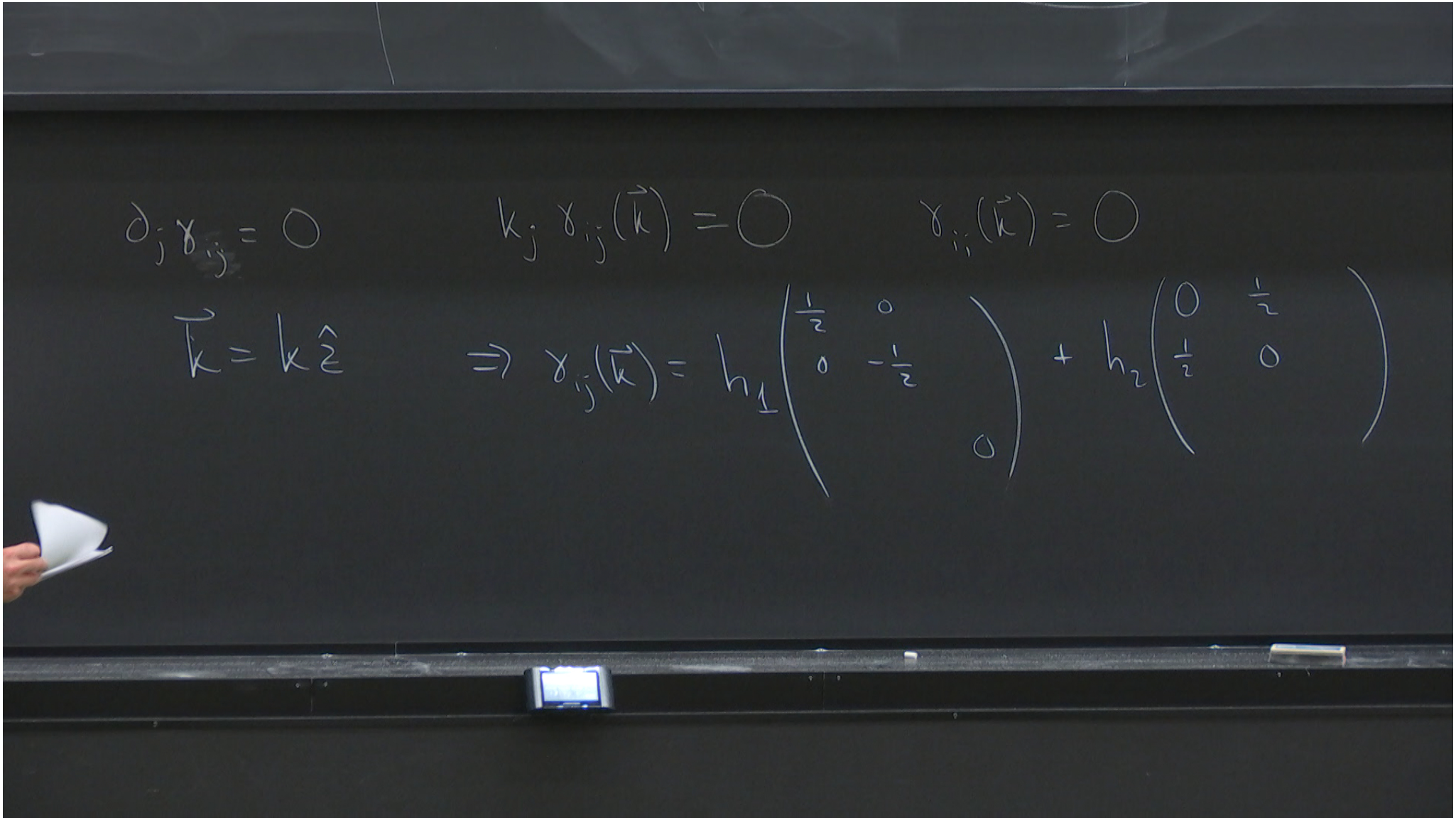
NOTATION:

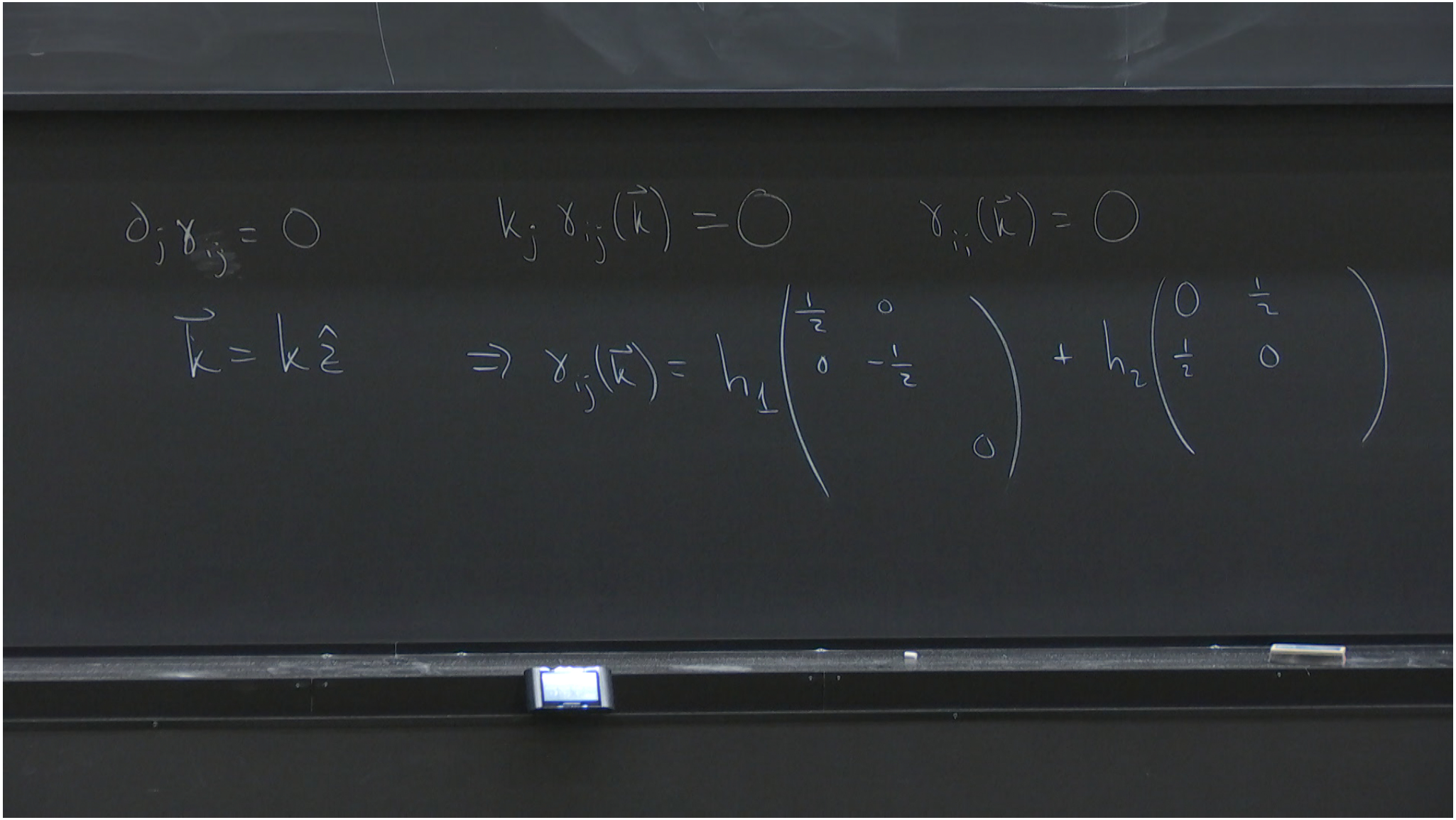
$$g_{ij} = a^2\left[(1+2\zeta)\delta_{ij} + \gamma_{ij}\right] \qquad \text{WHERE } \gamma_{ii} = 0$$

- 1: BACKGROUND
- ζ: "TRACE" PERTURBATION
- γ_{ij}: "TRACELESS" PERTURBATION

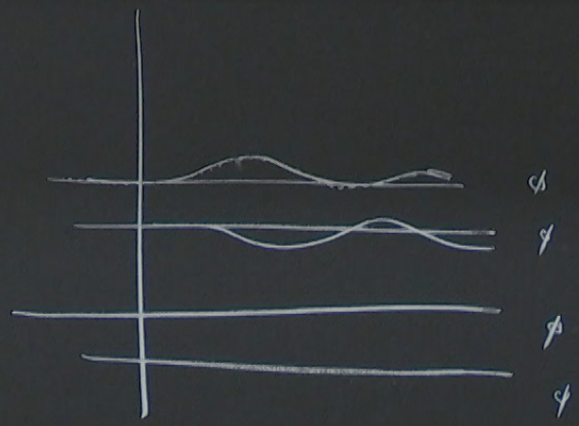
CHANGE OF VARIABLES $(\delta g_{ij}) \leftrightarrow (\zeta, \gamma_{ij})$, NOT A GAUGE CHOICE

- UNITARY GAUGE: $\boxed{\delta\phi = 0}$ $\boxed{\partial_j \gamma_{ij} = 0}$


$\partial_j \gamma_{ij} = 0$
$k_j \gamma_{ij}(\vec{k}) = 0$
$\gamma_{ii}(\vec{k}) = 0$
$\vec{k} = k\hat{z}$
$\Rightarrow \gamma_{ij}(\vec{k}) = h_1 \begin{pmatrix} \frac{1}{2} & 0 & \\ 0 & -\frac{1}{2} & \\ & & 0 \end{pmatrix} + h_2 \begin{pmatrix} 0 & \frac{1}{2} & \\ \frac{1}{2} & 0 & \\ & & \end{pmatrix}$


$\partial_j \gamma_{ij} = 0$
$k_j \gamma_{ij}(\vec{k}) = 0$
$\gamma_{ii}(\vec{k}) = 0$
$\vec{k} = k\hat{z}$
$\Rightarrow \gamma_{ij}(\vec{k}) = h_1 \begin{pmatrix} \frac{1}{2} & 0 & \\ 0 & -\frac{1}{2} & \\ & & 0 \end{pmatrix} + h_2 \begin{pmatrix} 0 & \frac{1}{2} & \\ \frac{1}{2} & 0 & \\ & & \end{pmatrix}$

$\delta\phi = 0$



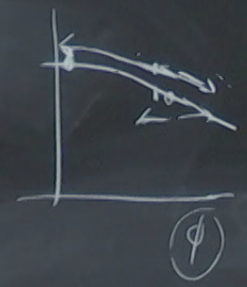
$\partial_j \gamma_{ij} = 0$

$\vec{k} = k\hat{z}$

STEP 1. CHOOSING A GAUGE

- SMALL GAUGE TRANSFORMATION $X^{\mu\prime} = X^{\mu} - \xi^{\mu}(x)$ IS PARAMETERIZED BY 4-VECTOR FIELD ξ^{μ}

$w \approx -1 \Rightarrow a \propto e^{Ht}$ $\qquad \frac{1}{a^2}$



ATION $X^{M\prime} = X^M - \varepsilon^M(x)$

4-VECTOR FIELD ε^M

$a \propto e^{Ht}$

$\frac{1}{a^2}$

