Title: Quantum Gravity

Speakers: Aldo Riello

Collection: Quantum Gravity (2021-2022)

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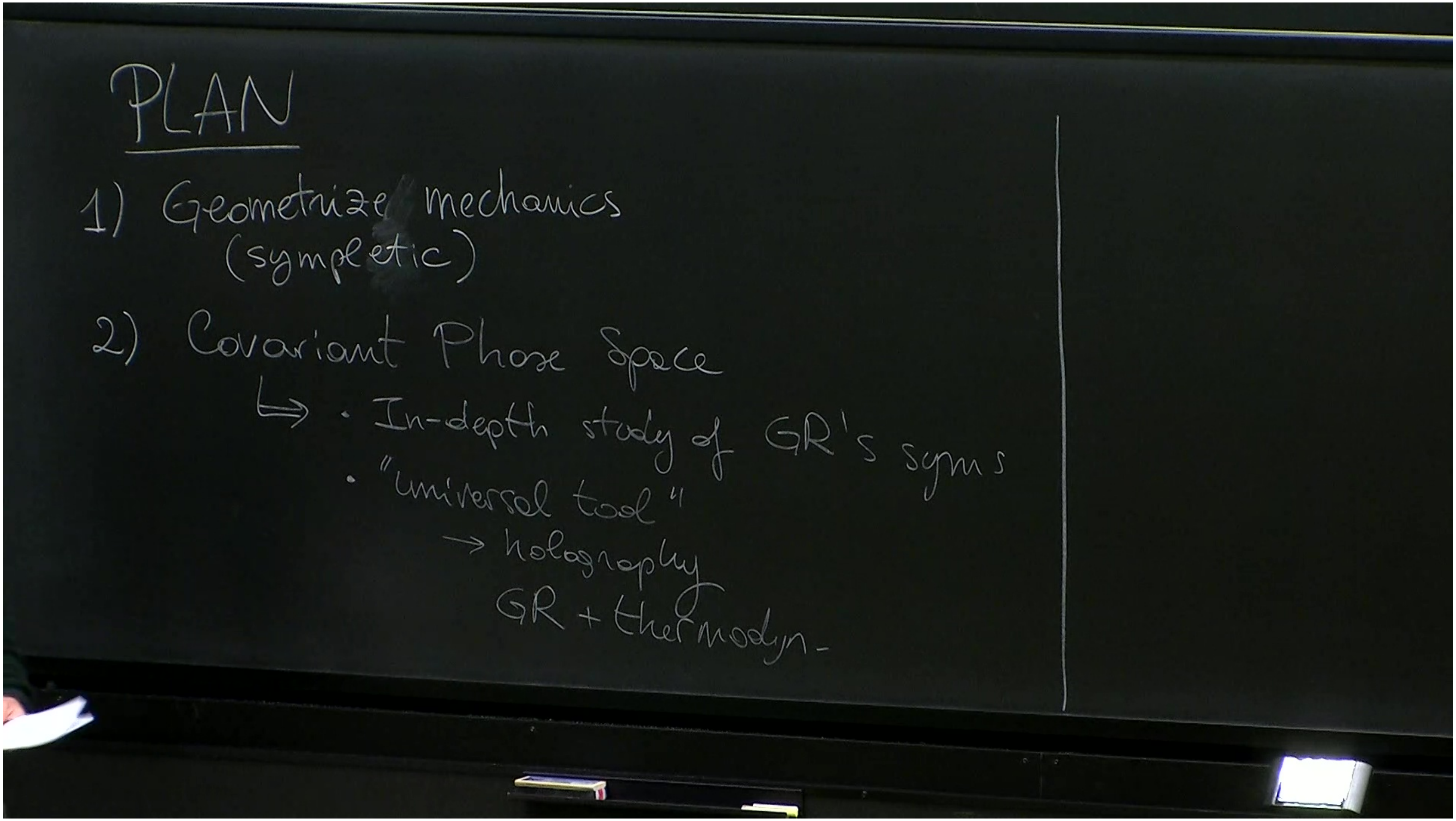
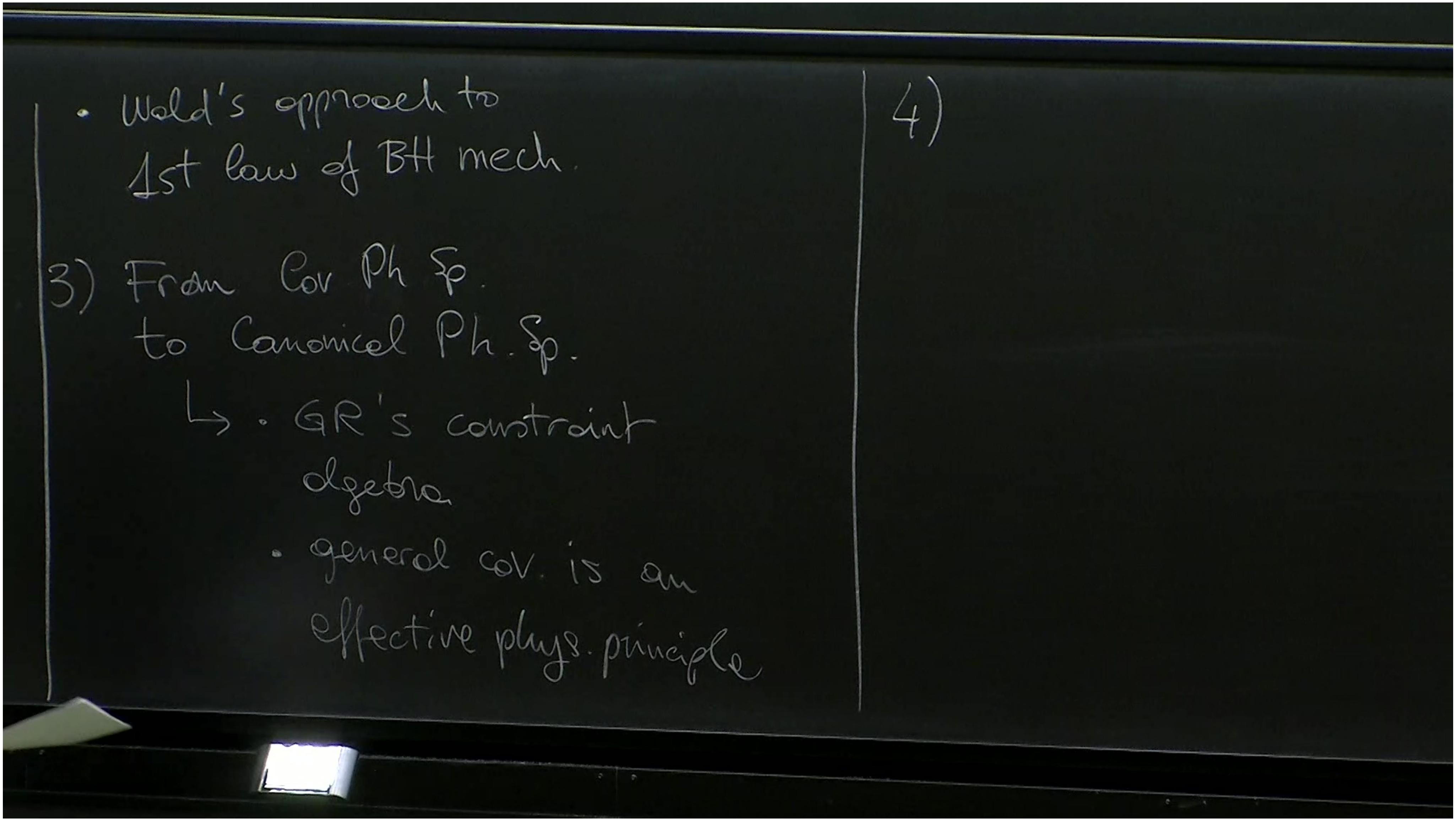
Abstract: Topics will include (but are not limited to): Canonical formulation of constrained systems, The Dirac program, First order formalism of gravity, Loop Quantum Gravity, Spinfoam models, Research at PI and other approaches to quantum gravity.



PLAN

1) Geometrize mechanics
(symplectic)

2) Covariant Phase Space
↳ · In-depth study of GR's syms
· "universal tool"
→ holography
GR + thermodyn.



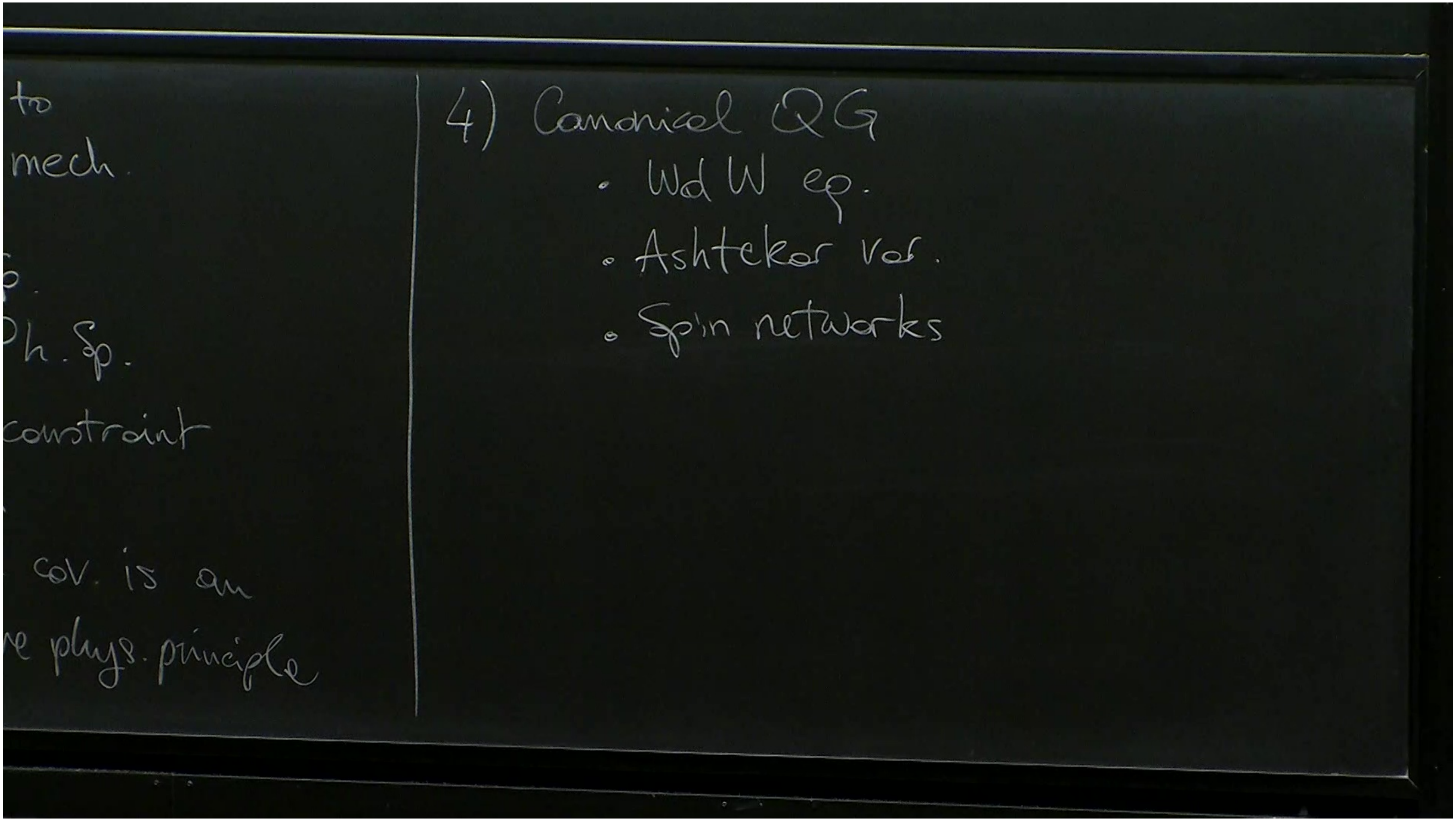
- Wald's approach to 1st law of BH mech.

3) From Cov Ph Sp. to Canonical Ph. Sp.

↳ • GR's constraint algebra

• general cov. is an effective phys. principle

4)



to

mech.

p.

Ph. Sp.

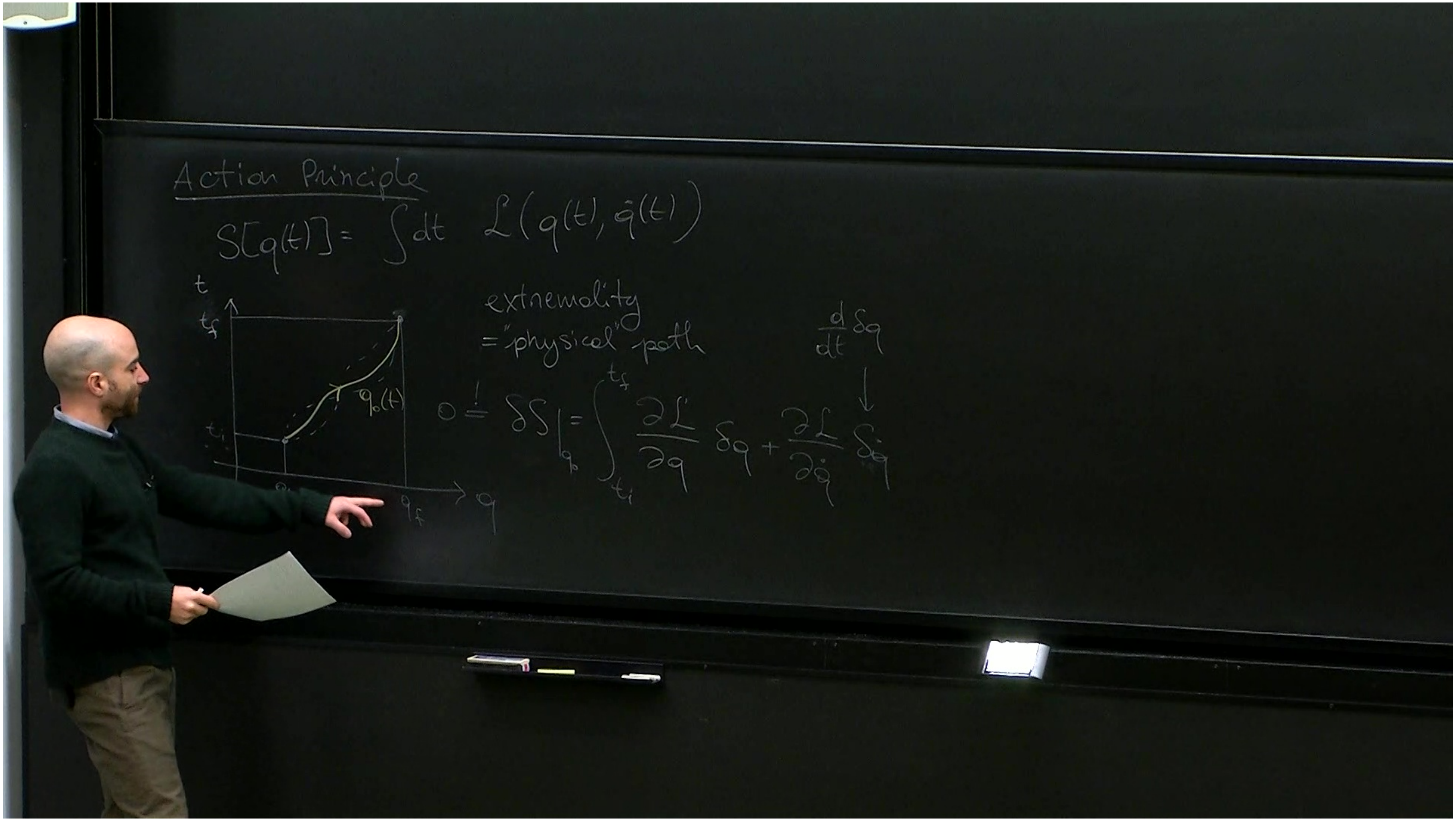
constraint

cov. is an

re phys. principle

4) Canonical QG

- WdW eq.
- Ashtekar var.
- Spin networks



Action Principle

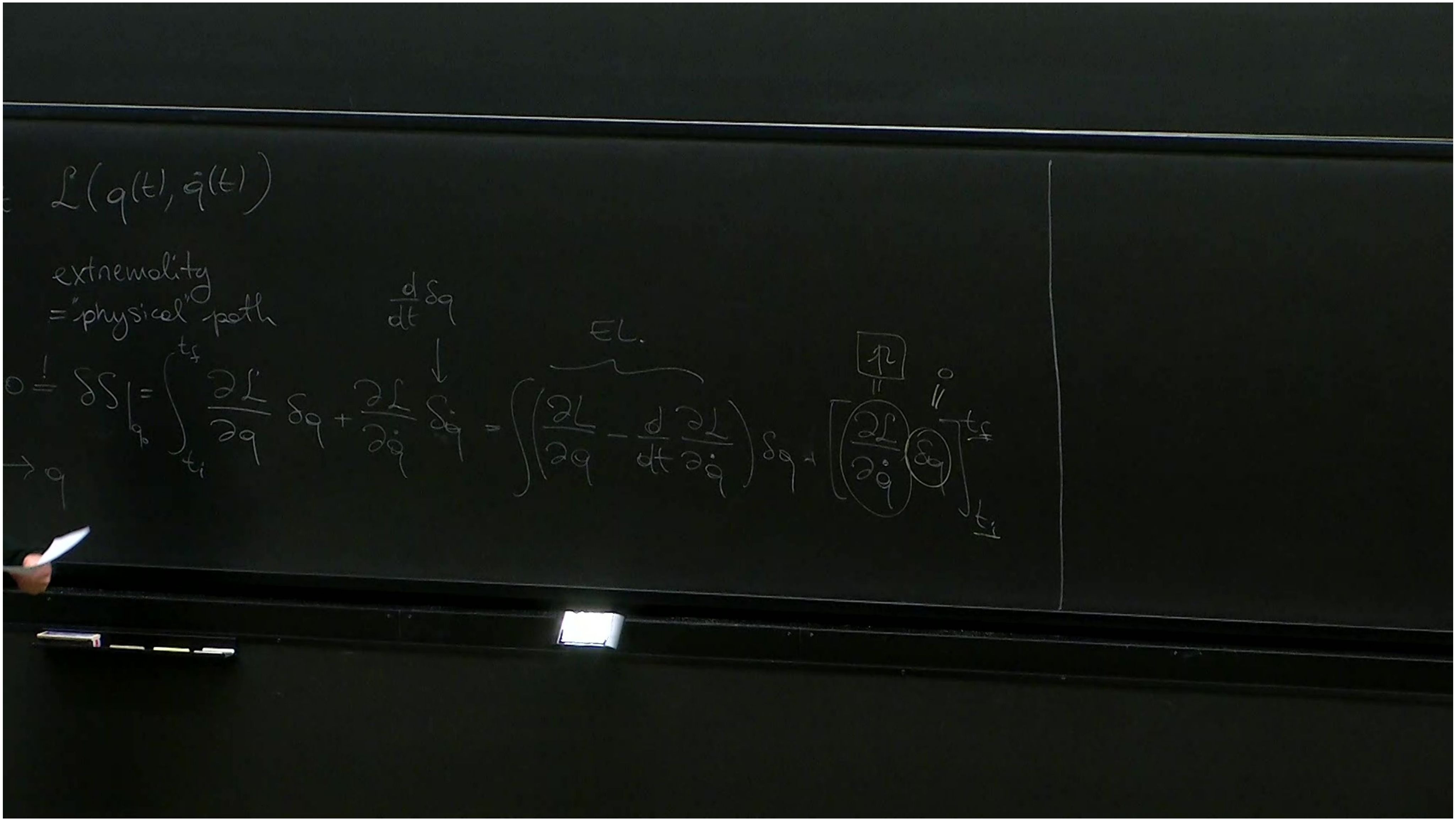
$$S[q(t)] = \int dt \; \mathcal{L}(q(t), \dot{q}(t))$$

extremality
= "physical" path

$$\frac{d}{dt}\delta q$$

$$\downarrow$$

$$0 \overset{!}{=} \delta S\big|_{q_0} = \int_{t_i}^{t_f} \frac{\partial \mathcal{L}}{\partial q}\,\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\,\delta \dot{q}$$



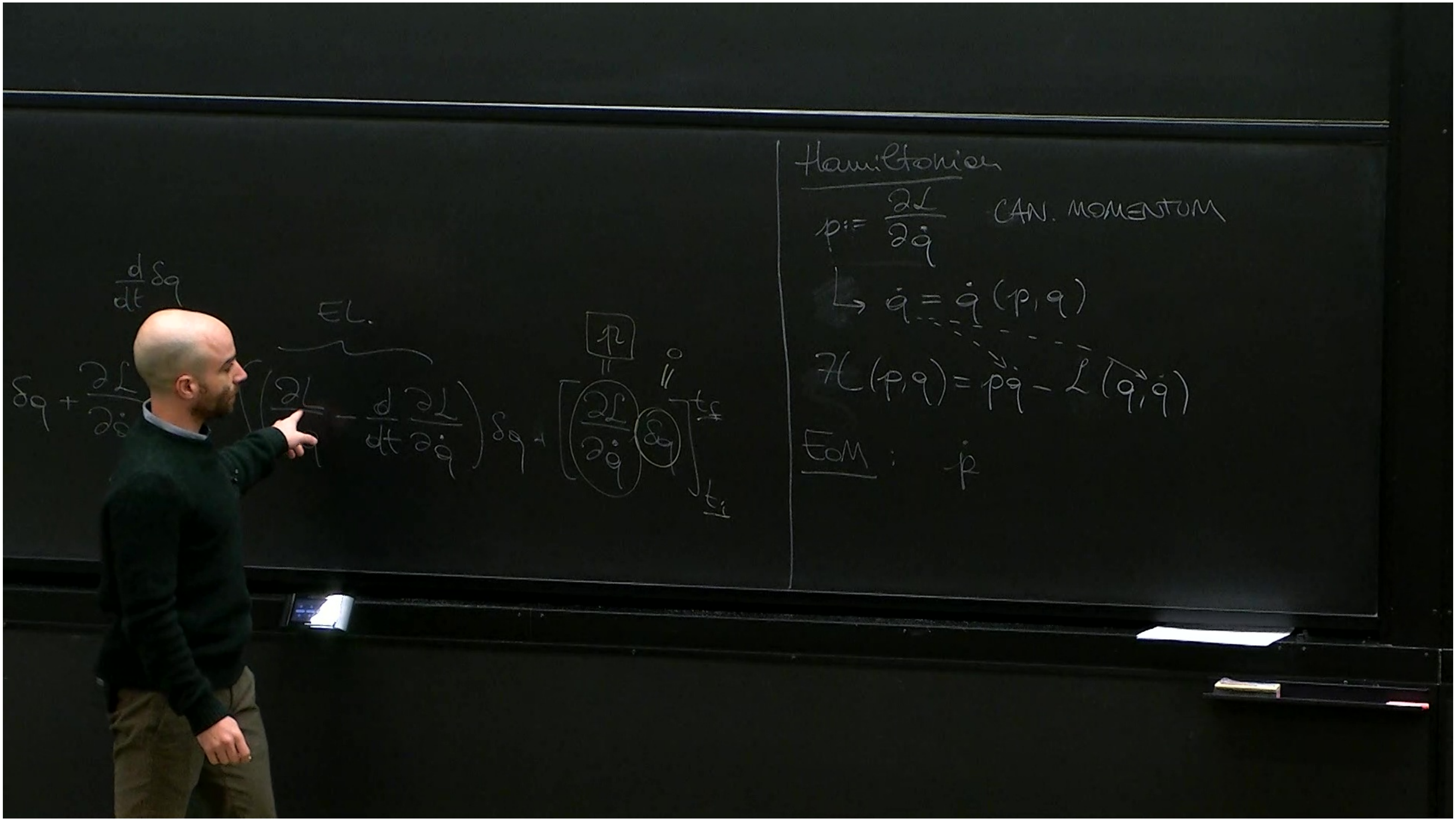
$\mathcal{L}(q(t), \dot{q}(t))$

extremality

= "physical" path

$$0 \overset{!}{=} \delta S\Big|_{q_0} = \int_{t_i}^{t_f} \frac{\partial L}{\partial q}\,\delta q + \frac{\partial L}{\partial \dot q}\,\overset{\overset{\frac{d}{dt}\delta q}{\downarrow}}{\delta \dot q} = \int \overbrace{\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)}^{EL.}\delta q + \left[\overset{\overset{\boxed{p}}{\|}}{\frac{\partial L}{\partial \dot q}}\;\overset{\overset{0}{\|}}{\delta q}\right]_{t_i}^{t_f}$$

$\rightarrow q$



$\frac{d}{dt}\delta q$

$$\delta q + \frac{\partial L}{\partial \dot{q}} \quad \Big(\underbrace{\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}}_{\text{EL.}}\Big)\delta q + \Big[\overset{\boxed{p}}{\frac{\partial L}{\partial \dot{q}}}\ \overset{0}{\delta q}\Big]_{t_i}^{t_f}$$

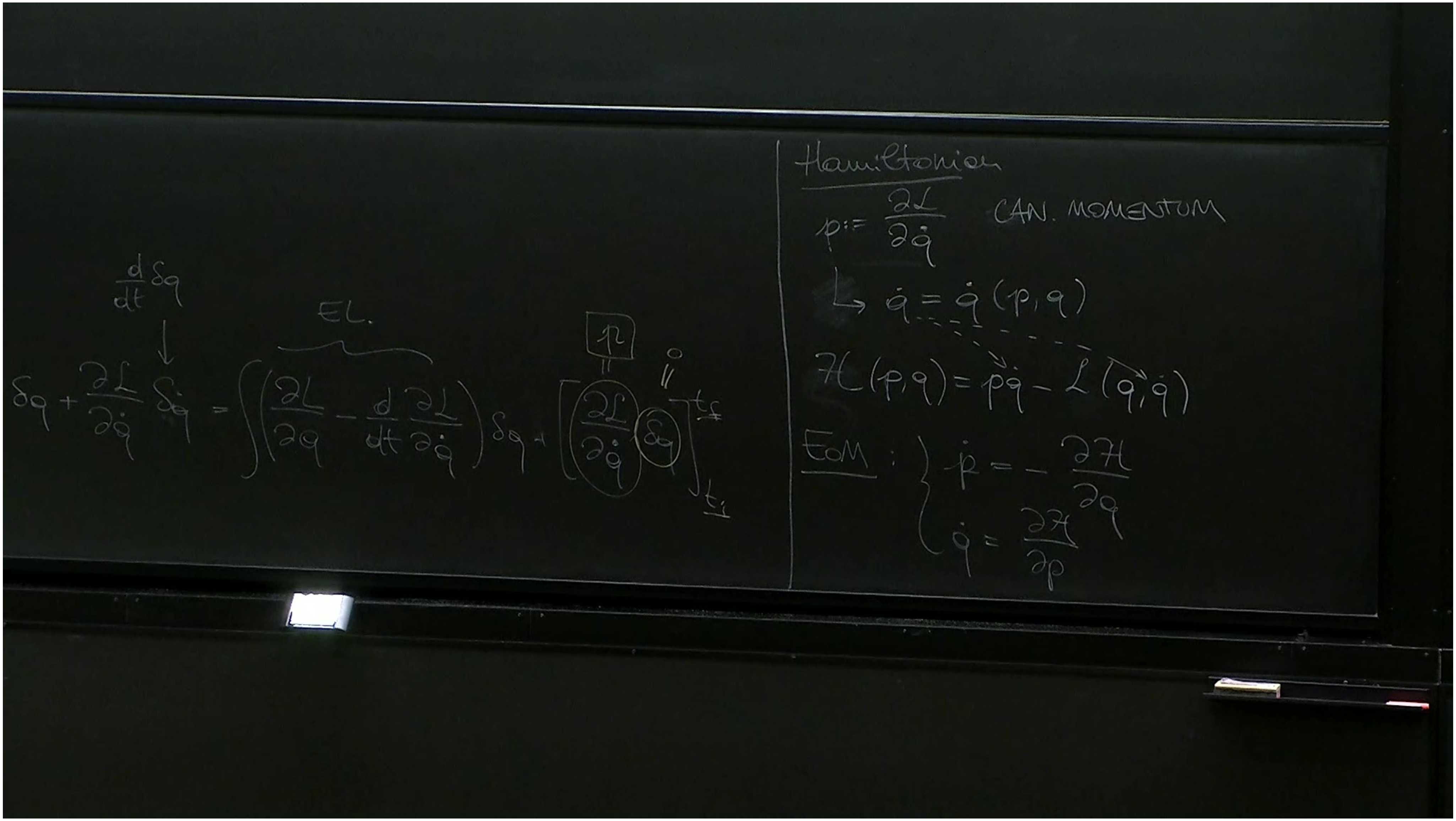
Hamiltonian

$p := \frac{\partial L}{\partial \dot{q}}$ CAN. MOMENTUM

$\hookrightarrow \dot{q} = \dot{q}(p, q)$

$\mathcal{H}(p, q) = p\dot{q} - \mathcal{L}(q, \dot{q})$

EoM : $\dot{p}$



$$\delta q + \frac{\partial L}{\partial \dot q}\,\overset{\overset{\frac{d}{dt}\delta q}{\downarrow}}{\delta \dot q} = \int \overbrace{\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)}^{EL.}\delta q + \left[\overset{\overset{\boxed{p}}{\parallel}}{\frac{\partial L}{\partial \dot q}}\,\overset{\overset{0}{\parallel}}{\delta q}\right]_{t_i}^{t_f}$$

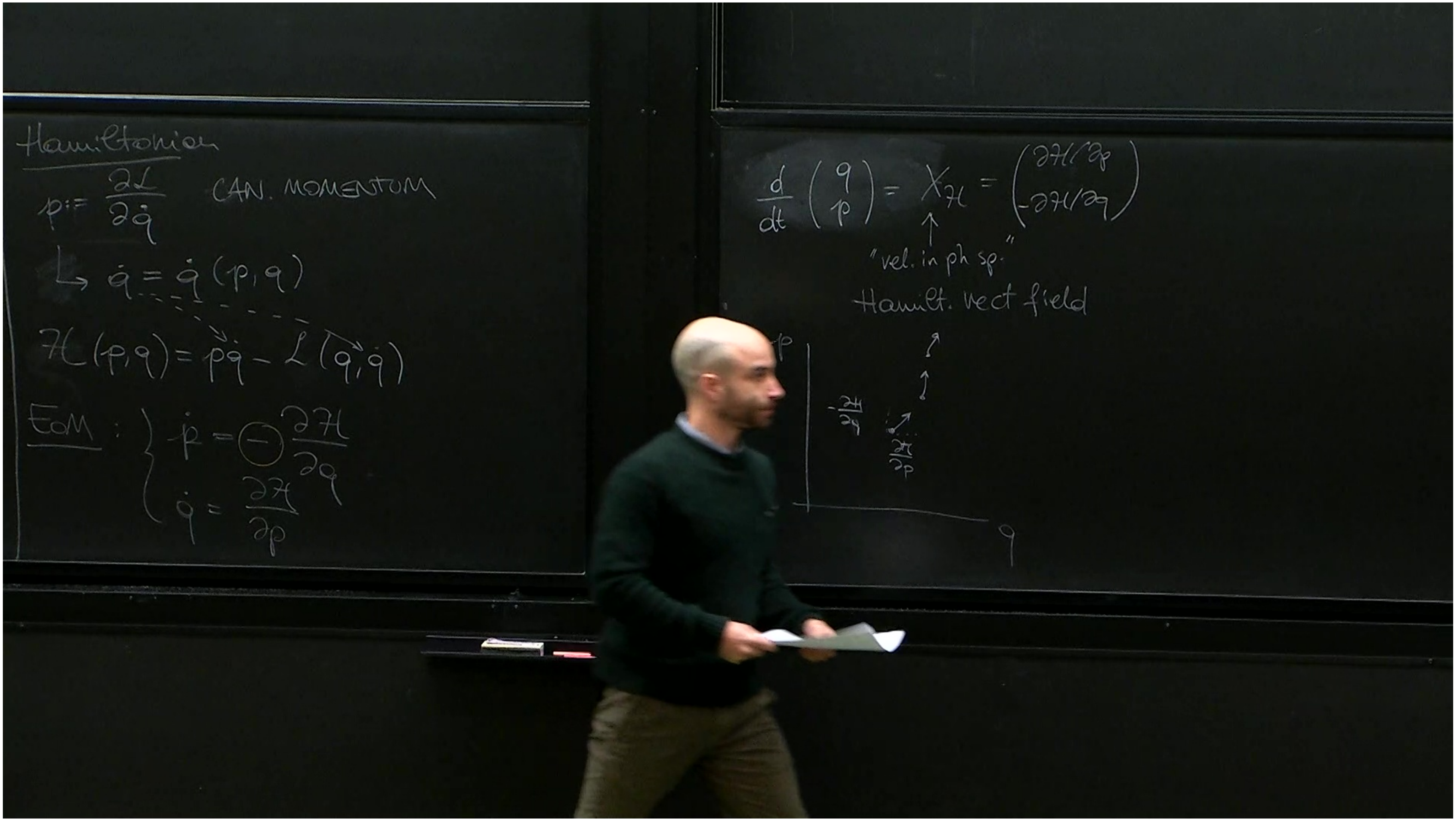
Hamiltonian

$p := \frac{\partial L}{\partial \dot q}$ CAN. MOMENTUM

$\hookrightarrow \dot q = \dot q(p,q)$

$\mathcal{H}(p,q) = p\dot q - \mathcal{L}(q,\dot q)$

EoM :
$$\begin{cases} \dot p = -\frac{\partial \mathcal{H}}{\partial q} \\ \dot q = \frac{\partial \mathcal{H}}{\partial p} \end{cases}$$



Hamiltonian

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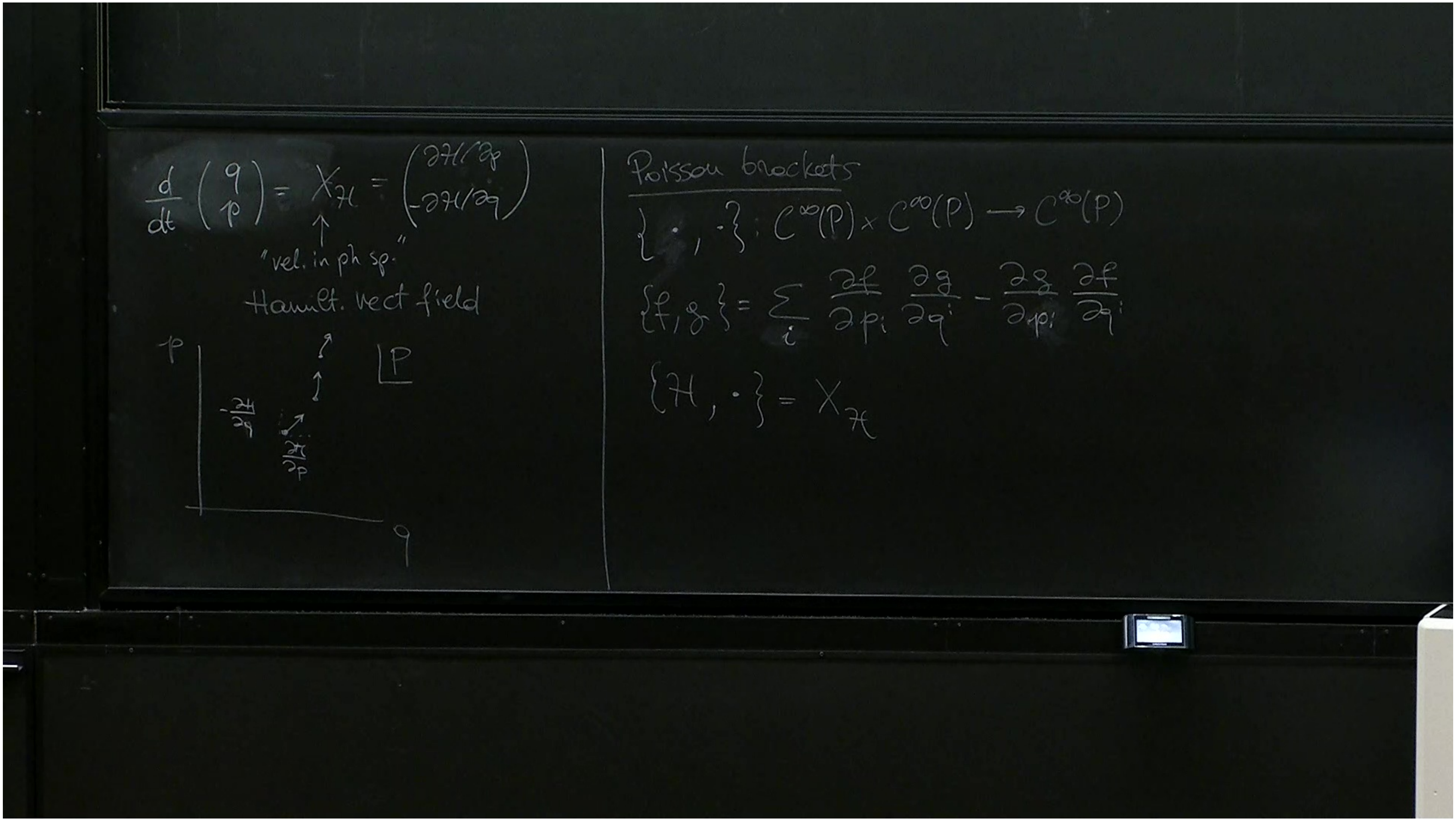
$$\mathcal{H}(p,q) = p\dot q - L(q, \dot q)$$

EoM : $\begin{cases} \dot p = \ominus \dfrac{\partial\mathcal{H}}{\partial q} \\ \dot q = \dfrac{\partial\mathcal{H}}{\partial p} \end{cases}$

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = X_{\mathcal{H}} = \begin{pmatrix} \partial\mathcal{H}/\partial p \\ -\partial\mathcal{H}/\partial q \end{pmatrix}$$

"vel. in ph sp."

Hamilt. vect field



$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \underset{\substack{\uparrow \\ \text{"vel. in ph sp."}}}{X_{\mathcal{H}}} = \begin{pmatrix} \partial\mathcal{H}/\partial p \\ -\partial\mathcal{H}/\partial q \end{pmatrix}$$

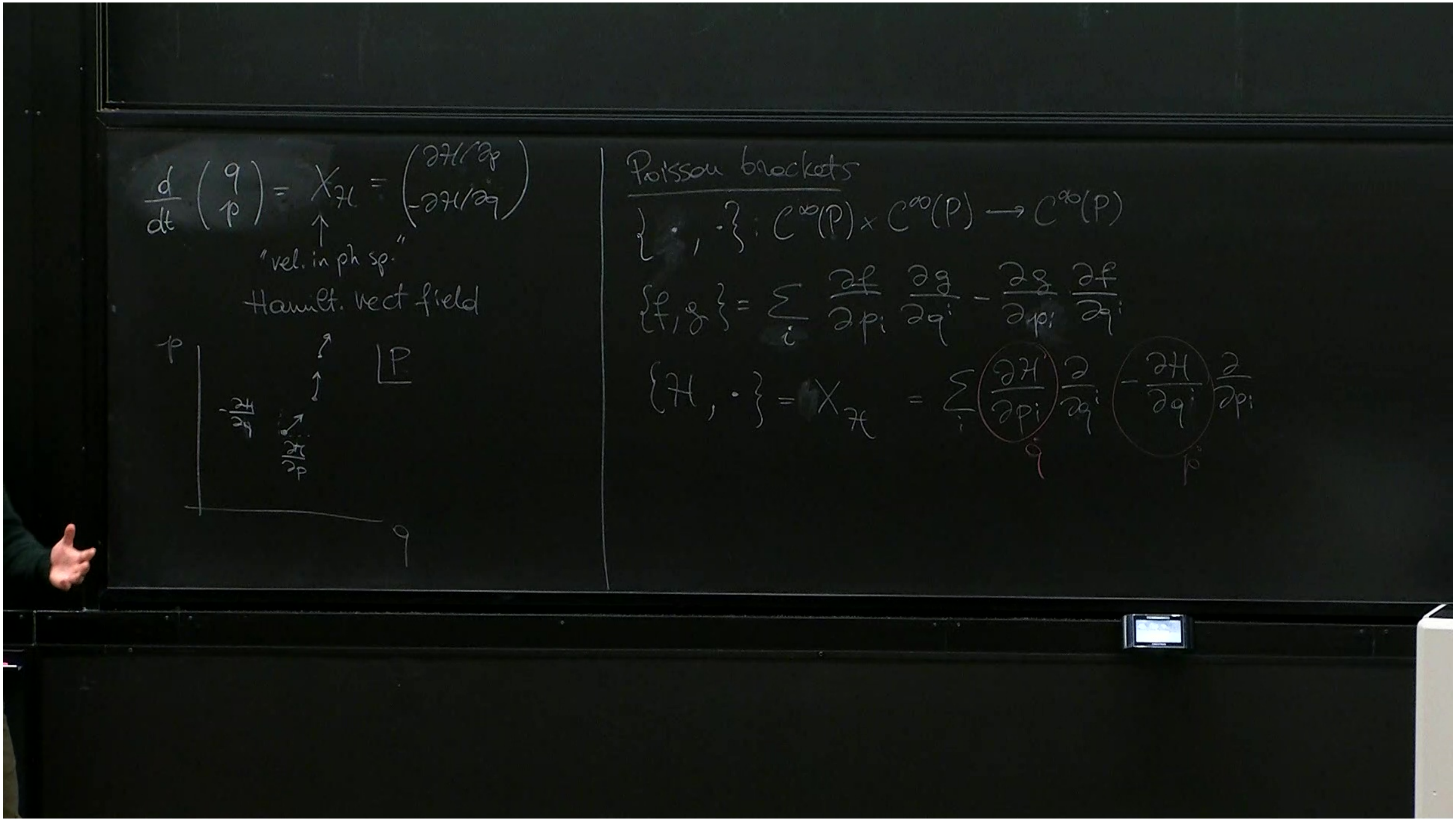
Hamilt. vect field

Poisson brackets

$$\{\cdot\,,\cdot\}: C^\infty(P) \times C^\infty(P) \longrightarrow C^\infty(P)$$

$$\{f,g\} = \sum_i \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q^i}$$

$$\{\mathcal{H}\,,\cdot\} = X_{\mathcal{H}}$$



$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = X_{\mathcal{H}} = \begin{pmatrix} \partial\mathcal{H}/\partial p \\ -\partial\mathcal{H}/\partial q \end{pmatrix}$$

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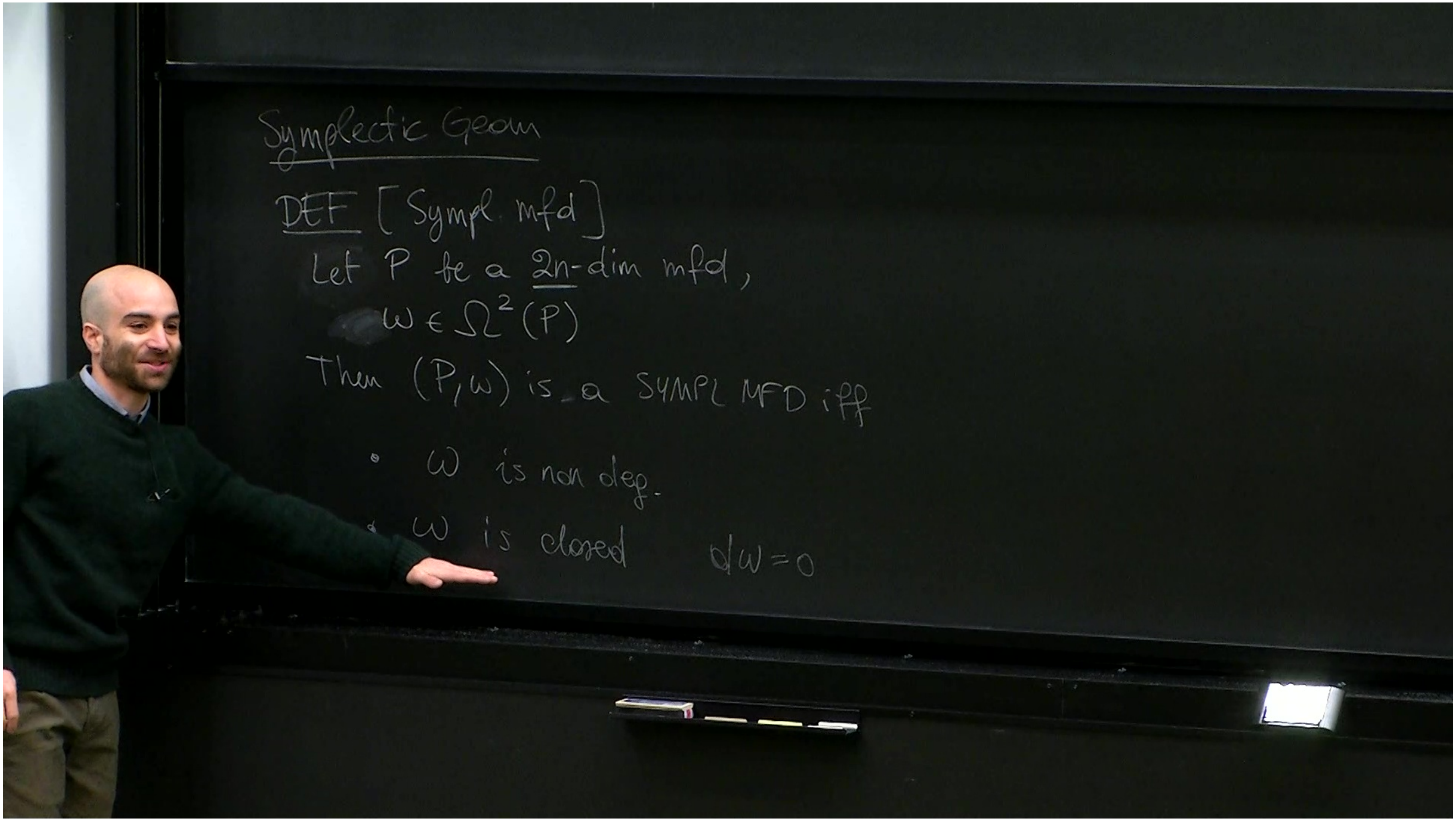
$$\{\cdot\,,\cdot\} : C^\infty(P) \times C^\infty(P) \longrightarrow C^\infty(P)$$

$$\{f, g\} = \sum_i \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q^i}$$

$$\{\mathcal{H}, \cdot\} = X_{\mathcal{H}} = \sum_i \underset{\dot{q}}{\boxed{\frac{\partial \mathcal{H}}{\partial p_i}}}\frac{\partial}{\partial q^i} \underset{\dot{p}}{\boxed{-\frac{\partial \mathcal{H}}{\partial q^i}}}\frac{\partial}{\partial p_i}$$



Symplectic Geom

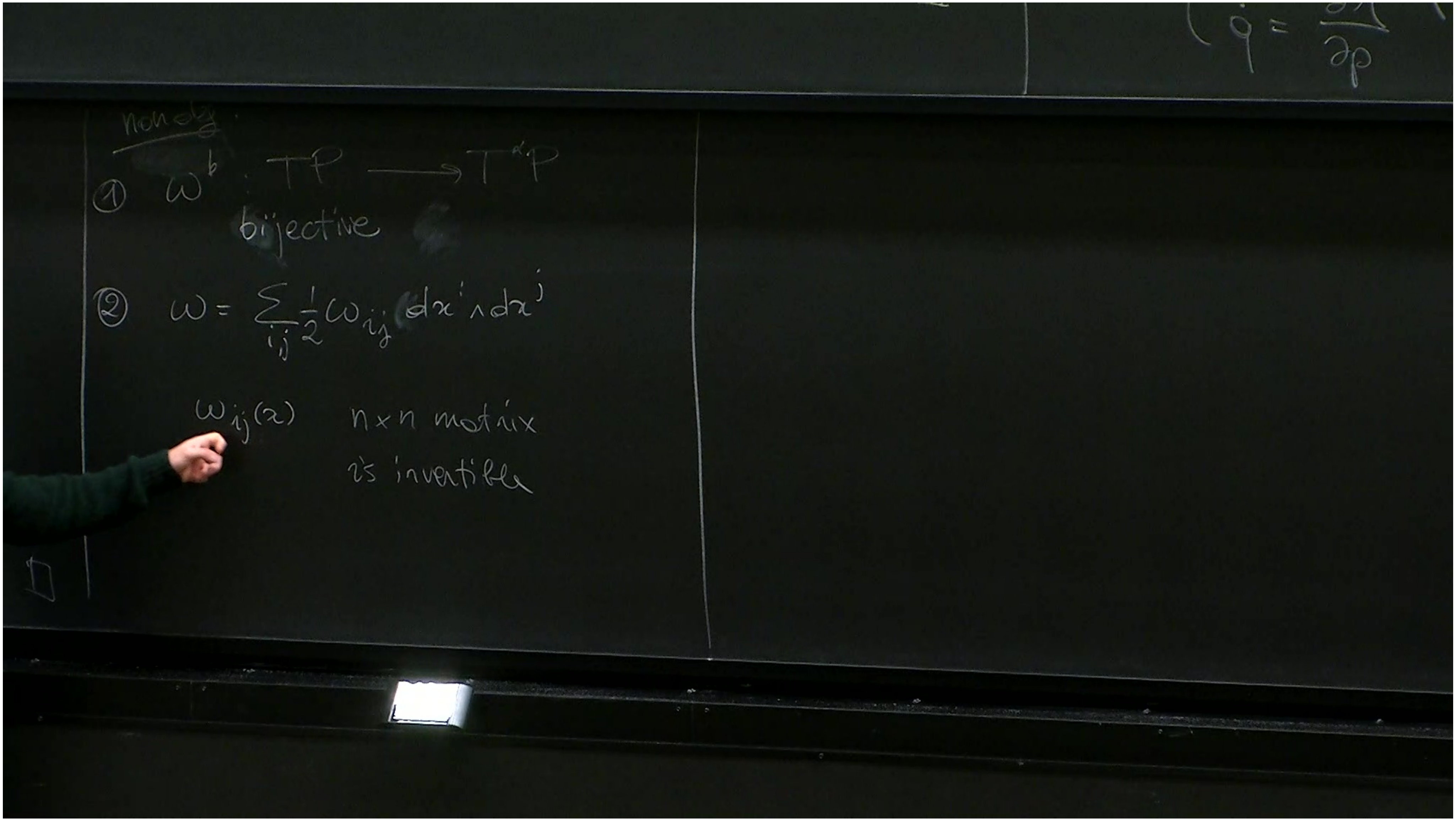
DEF [Sympl. mfd]

Let P be a $\underline{2n}$-dim mfd,

$\omega \in \Omega^2(P)$

Then (P, ω) is a SYMPL MFD iff

- ω is non deg.
- ω is closed $d\omega = 0$



$(\dot{q} = \frac{\partial}{\partial p}$

nondeg:

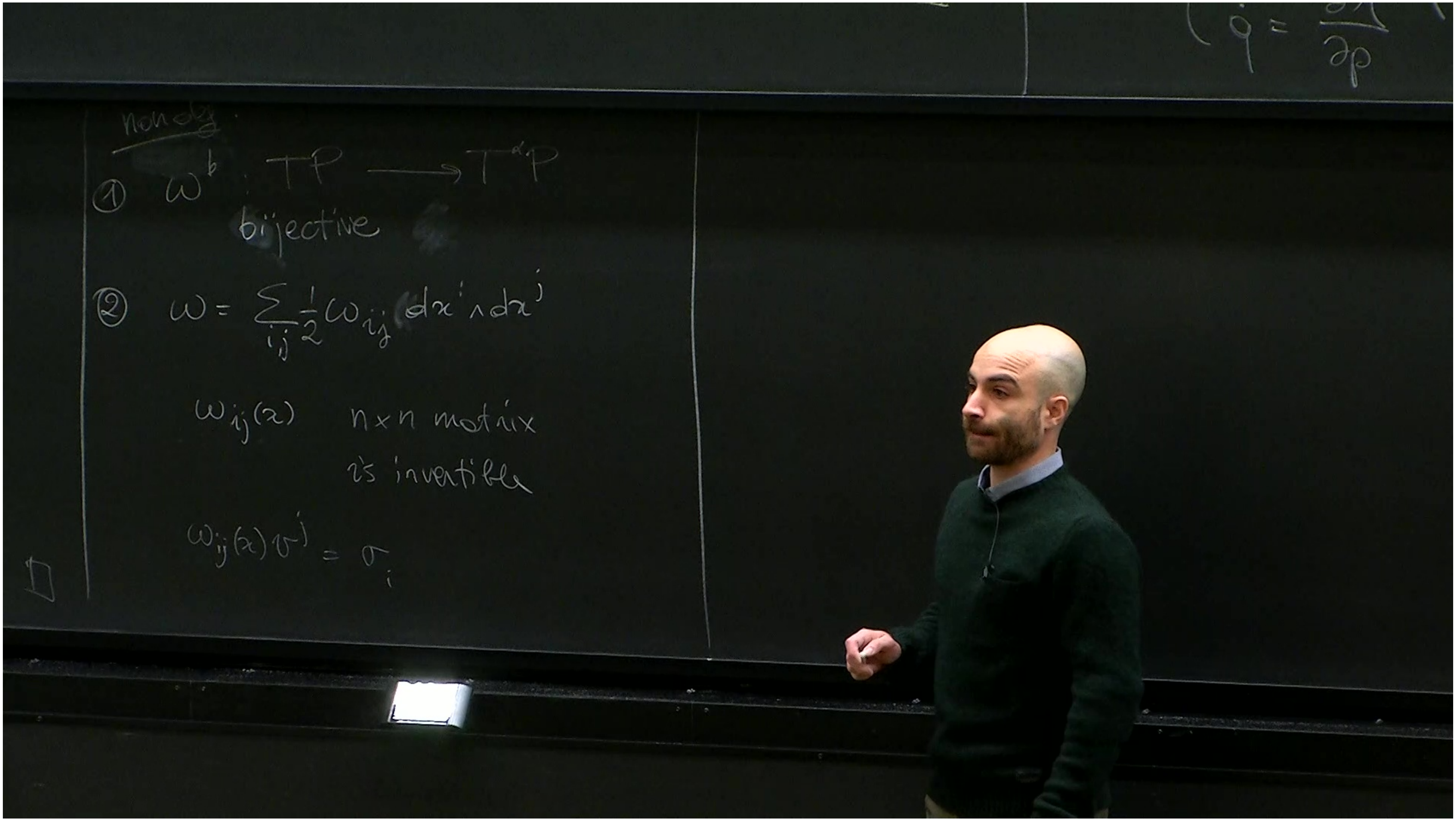
① $\omega^\flat : TP \longrightarrow T^*P$

bijective

② $\omega = \sum_{i,j} \frac{1}{2}\omega_{ij}\, dx^i \wedge dx^j$

$\omega_{ij}(x)$ $n \times n$ matrix

is invertible



$\left(\dot{q} = \frac{\partial}{\partial p} \right.$

nondeg:

① $\omega^\flat : TP \longrightarrow T^*P$

bijective

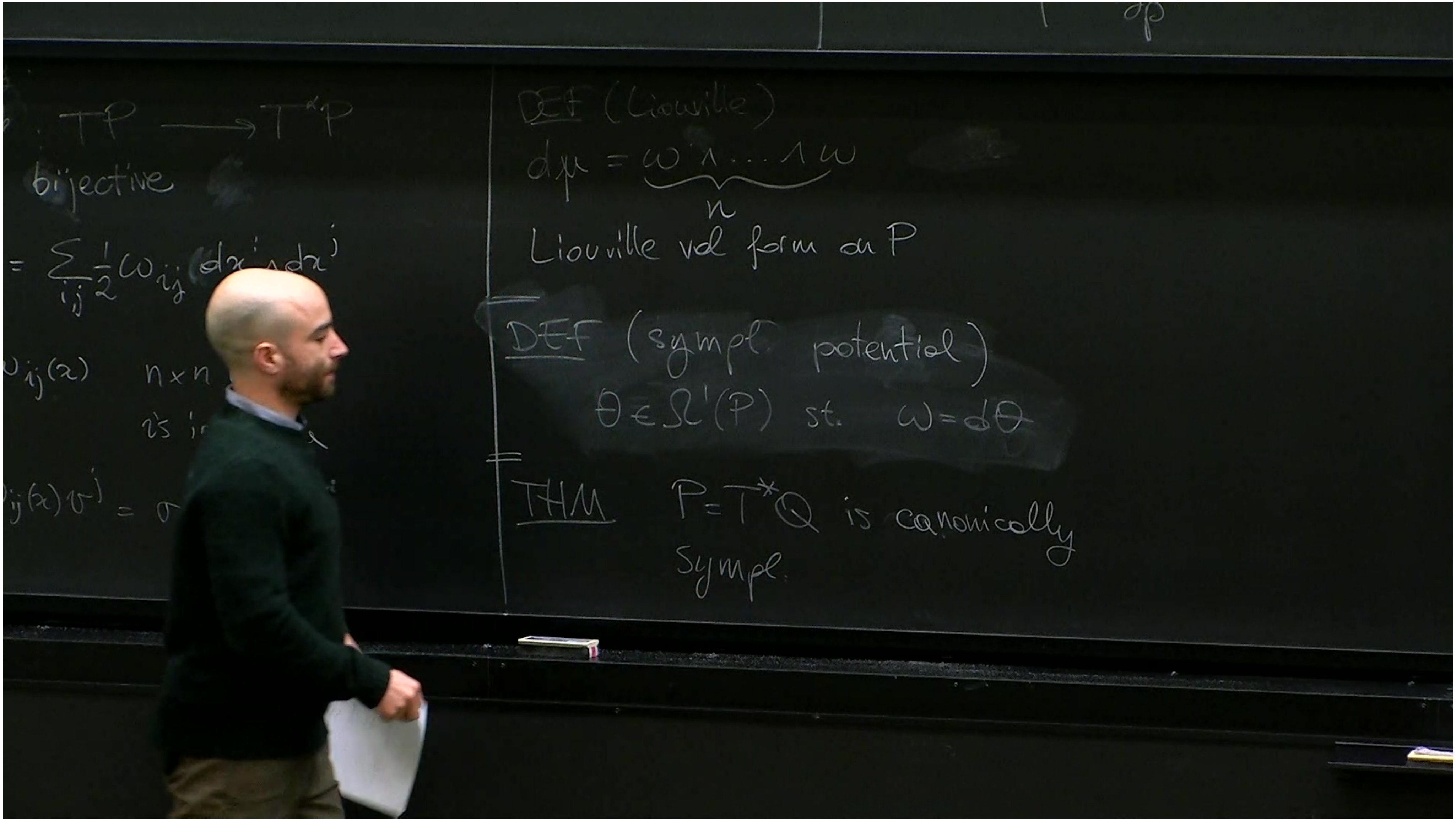
② $\omega = \sum_{i,j} \frac{1}{2} \omega_{ij} \, dx^i \wedge dx^j$

$\omega_{ij}(x)$ n × n matrix

is invertible

$\omega_{ij}(x) v^j = \sigma_i$

□



$: TP \longrightarrow T^*P$

bijective

$= \sum_{ij} \frac{1}{2} \omega_{ij} dx^i \wedge dx^j$

$\omega_{ij}(x) \quad n \times n$

is i

$\omega_{ij}(x) v^j = 0$

DEF (Liouville)

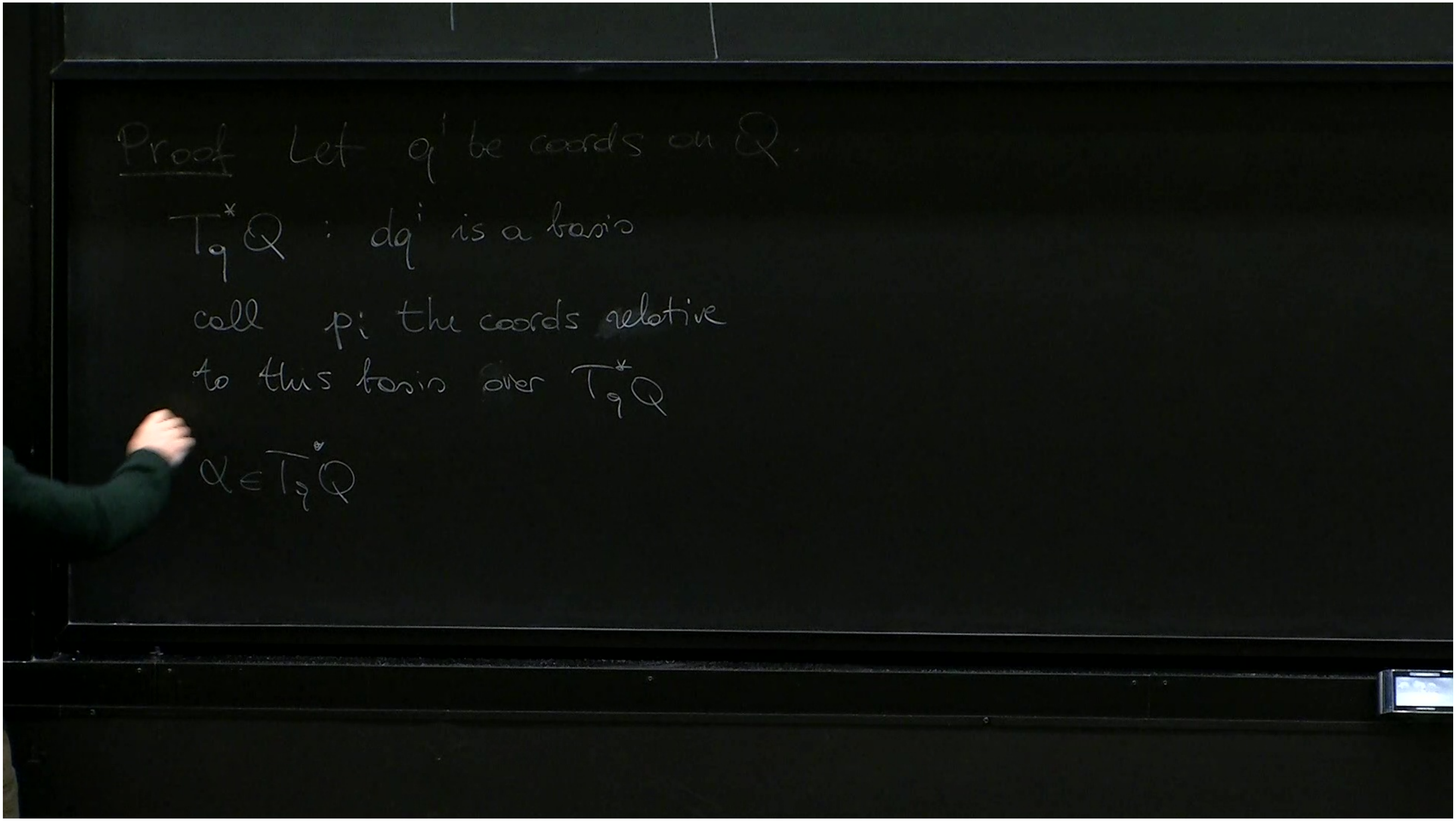
$d\mu = \underbrace{\omega \wedge \ldots \wedge \omega}_{n}$

Liouville vol form on P

DEF (sympl. potential)

$\theta \in \Omega^1(P)$ st. $\omega = d\theta$

THM $P = T^*Q$ is canonically sympl.

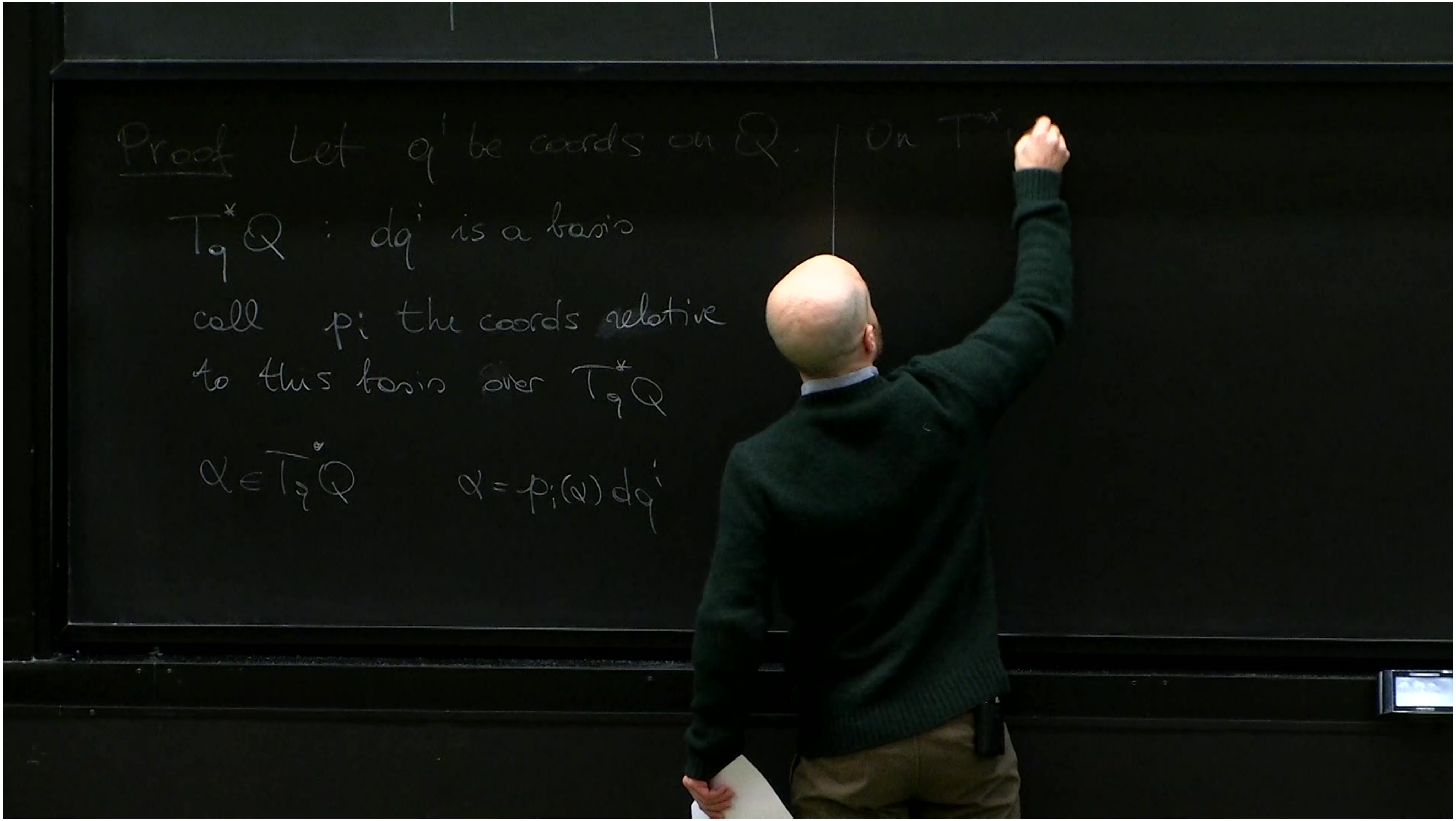


Proof Let q^i be coords on Q.

$T^*_q Q$: dq^i is a basis

call p_i the coords relative to this basis over $T^*_q Q$

$\alpha \in T^*_q Q$

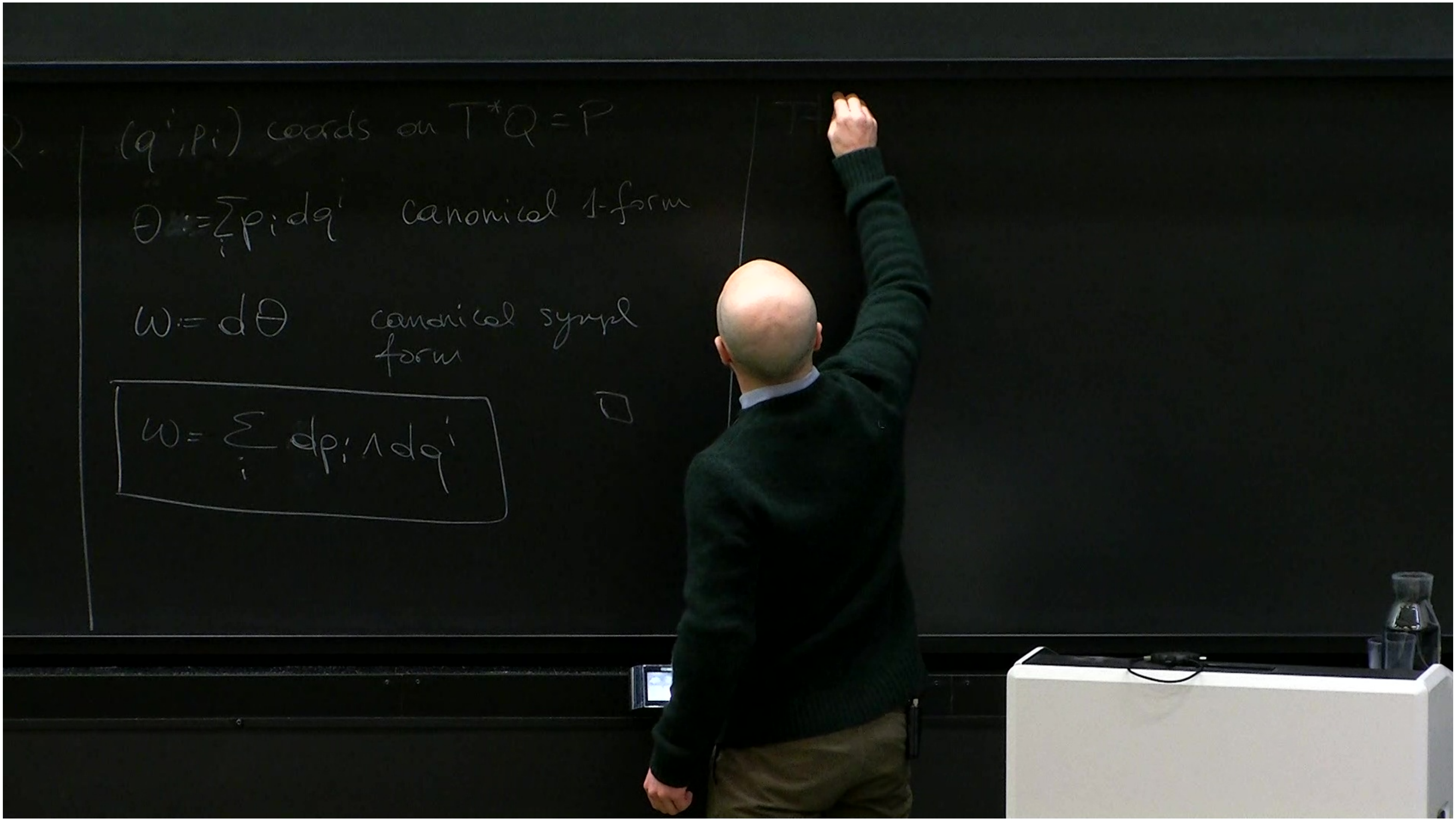


Proof Let q^i be coords on Q. | On T^*

T_q^*Q : dq^i is a basis

call p_i the coords relative to this basis over T_q^*Q

$\alpha \in T_q^*Q$ $\quad \alpha = p_i(\alpha)\, dq^i$



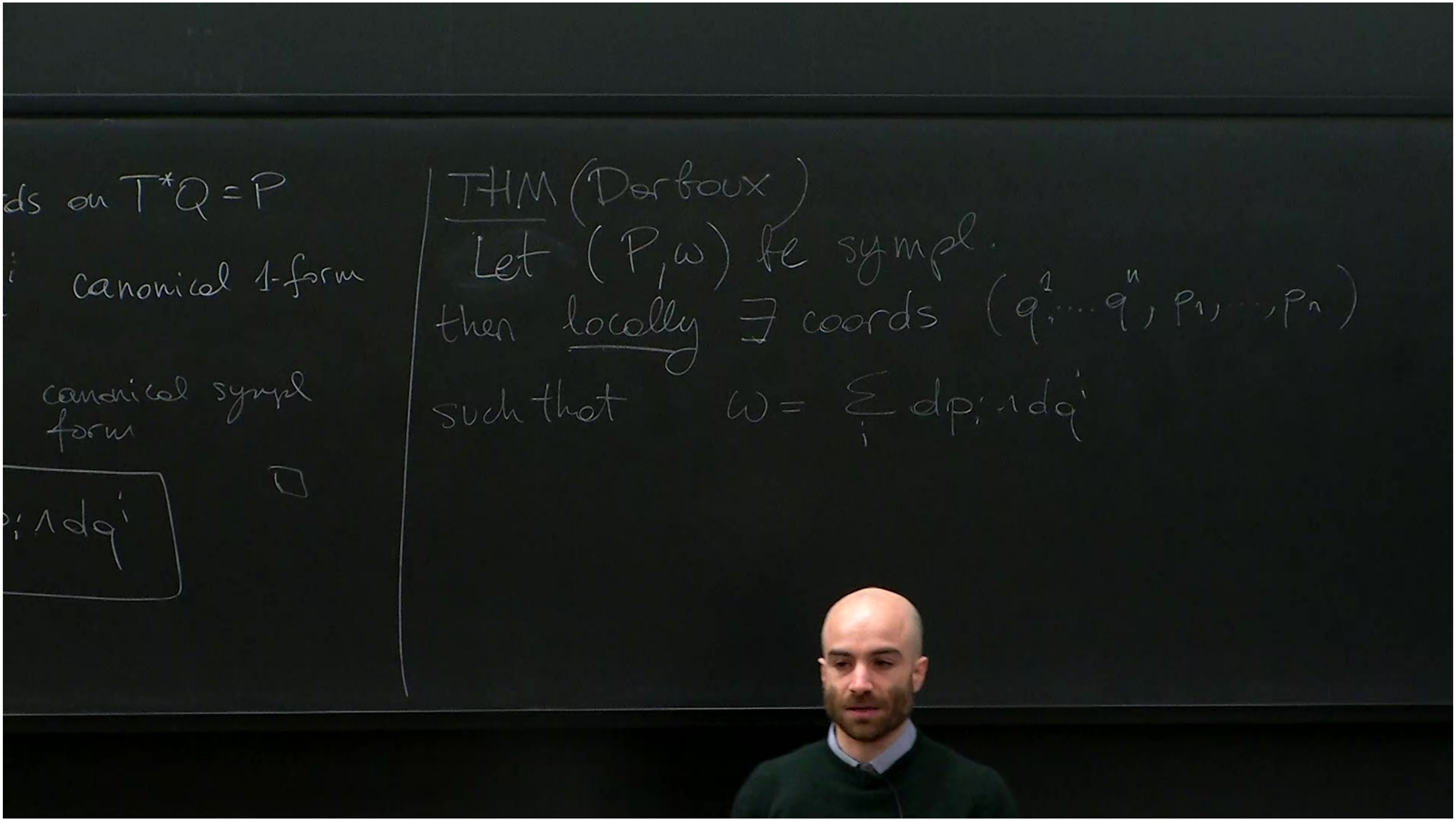
(q^i, p_i) coords on $T^*Q = P$

$\Theta := \sum_i p_i \, dq^i$ canonical 1-form

$\omega := d\Theta$ canonical sympl form

$$\boxed{\omega = \sum_i dp_i \wedge dq^i}$$

$\square$



…ds on $T^*Q = P$

…^{i} canonical 1-form

canonical sympl form

$\boxed{\ldots_i \wedge dq^i}$

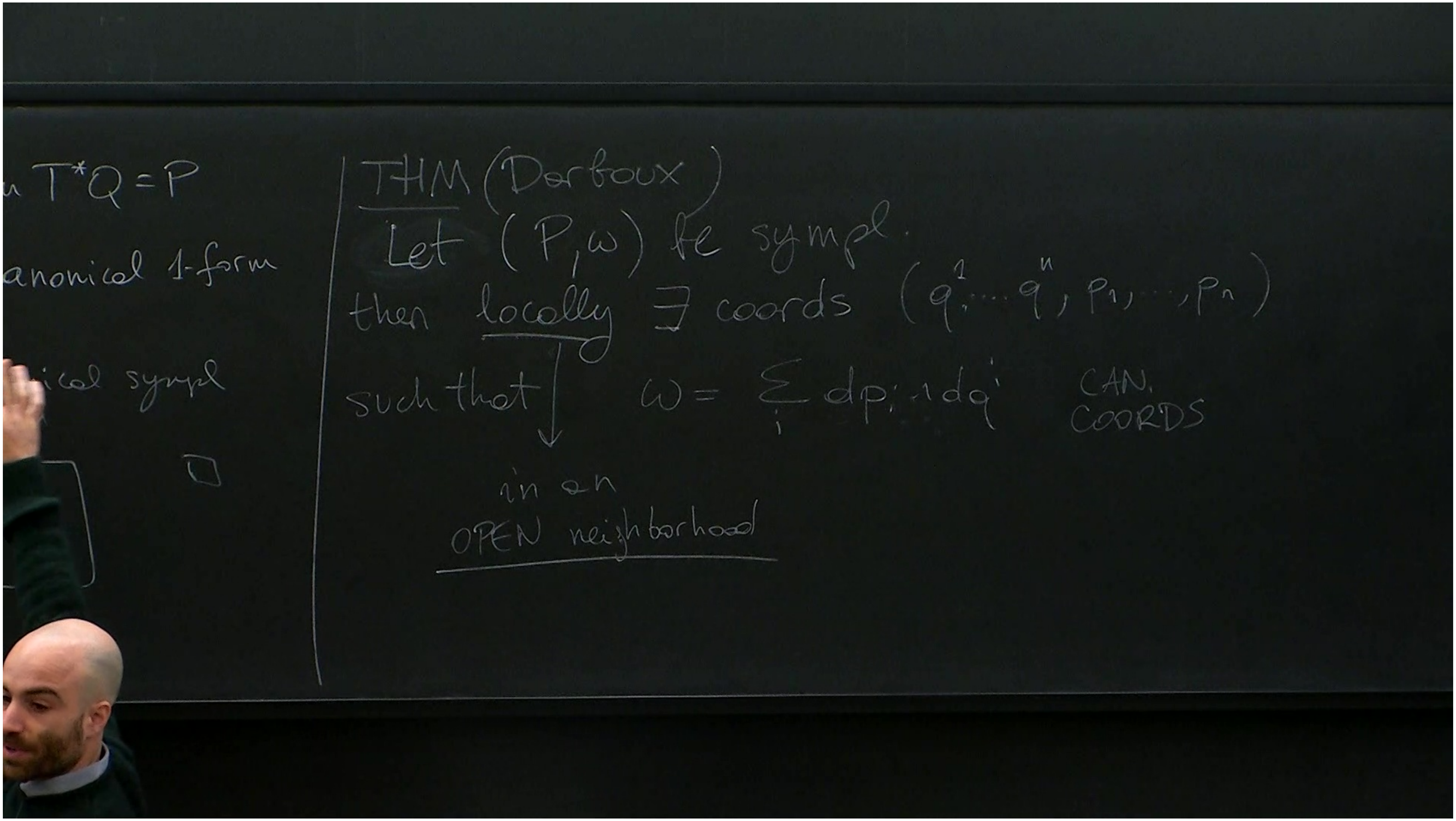
$\square$

THM (Darboux)

Let (P,ω) be sympl.

then **locally** $\exists$ coords $(q^1,\dots,q^n,p_1,\dots,p_n)$

such that $\omega = \sum_i dp_i \wedge dq^i$



n $T^*Q = P$

anonical 1-form

ical sympl

$\square$

THM (Darboux)

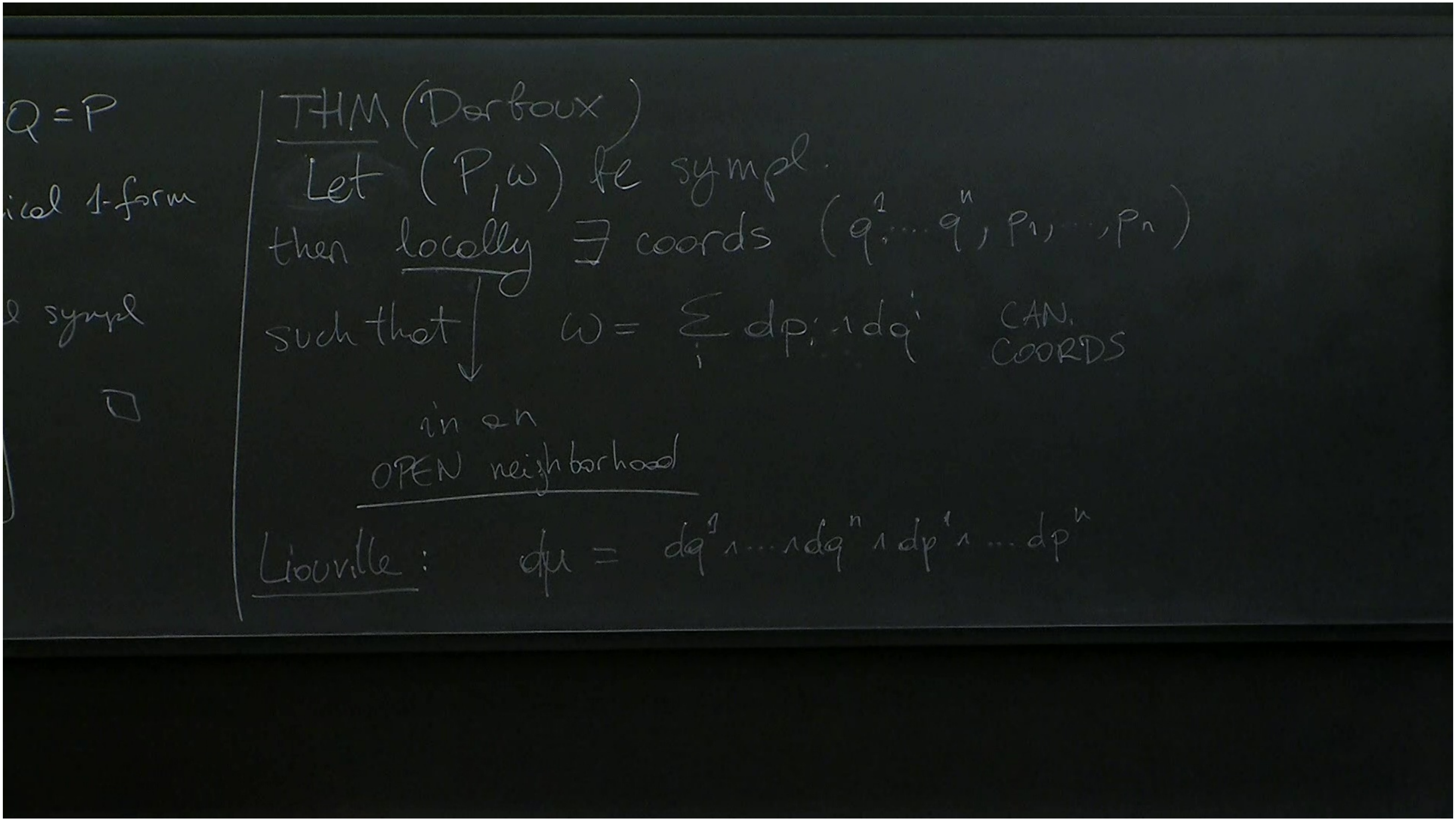
Let (P,ω) be sympl.

then **locally** $\exists$ coords $(q^1, \cdots, q^n, p_1, \cdots, p_n)$

such that $\omega = \sum_i dp_i \wedge dq^i$ CAN. COORDS

in an

OPEN neighborhood



$Q = P$

ical 1-form

l sympl

$\square$

THM (Darboux)

Let (P, ω) be sympl.

then locally $\exists$ coords $(q^1, \dots, q^n, p_1, \dots, p_n)$

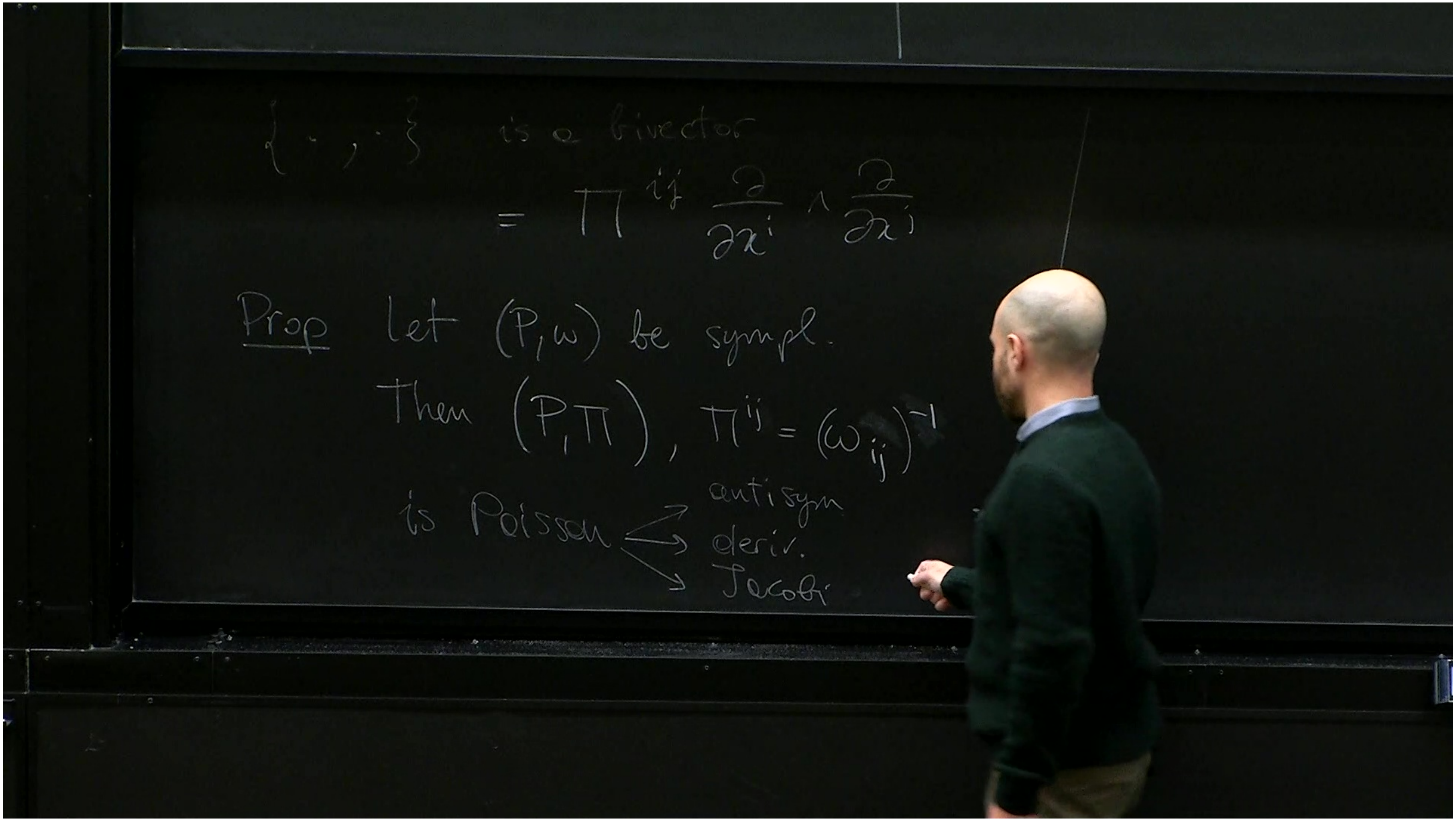
such that $\omega = \sum_i dp_i \wedge dq^i$ CAN. COORDS

$\downarrow$

in an

OPEN neighborhood

Liouville: $d\mu = dq^1 \wedge \dots \wedge dq^n \wedge dp^1 \wedge \dots dp^n$



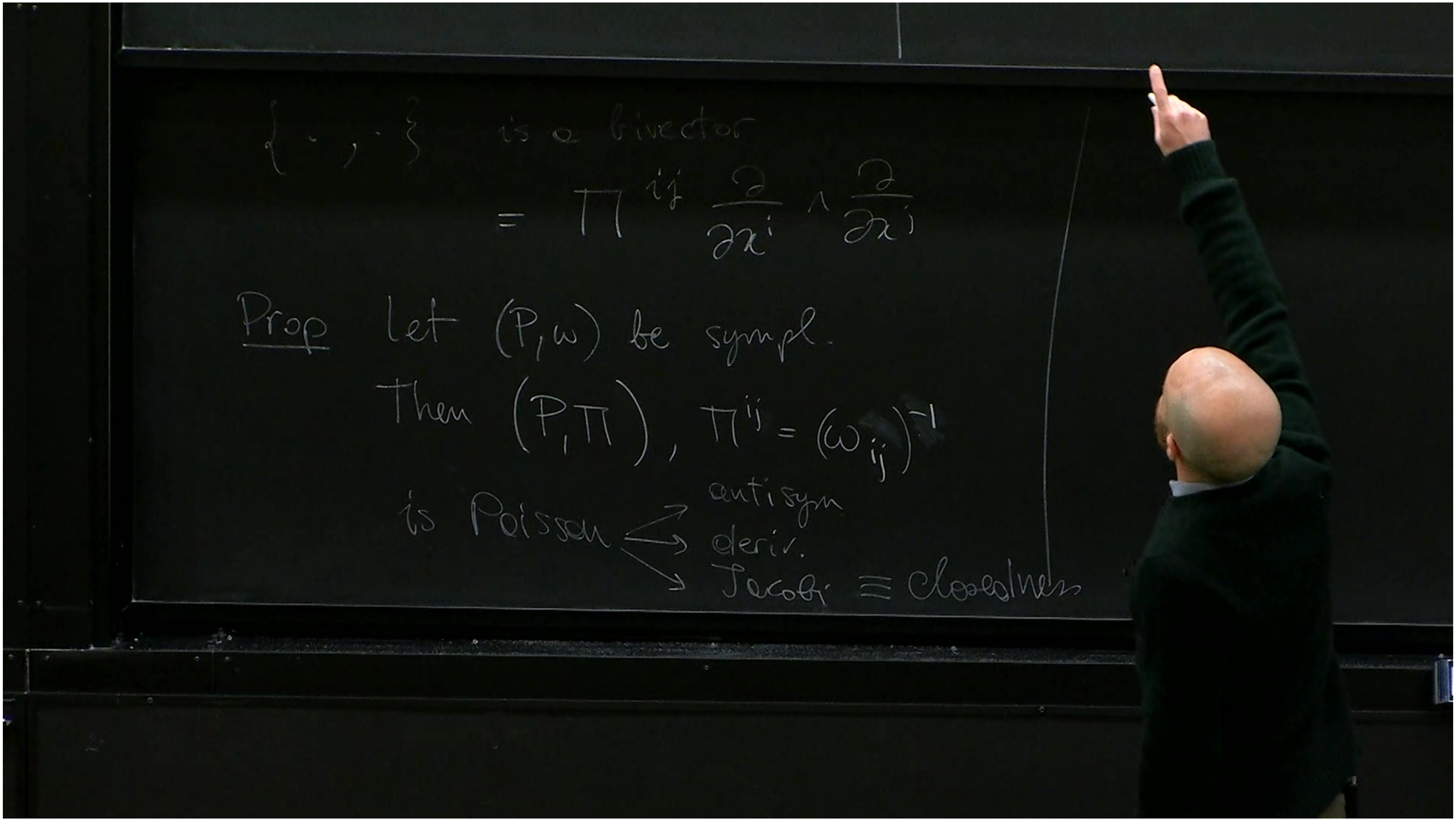
$\{\cdot\,,\cdot\}$ is a bivector

$$= \Pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$$

Prop let (P, ω) be sympl.

Then (P, Π), $\Pi^{ij} = (\omega_{ij})^{-1}$

is Poisson $\to$ antisym
$\to$ deriv.
$\to$ Jacobi



$\{\cdot,\cdot\}$ is a bivector

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Prop Let (P,ω) be sympl.

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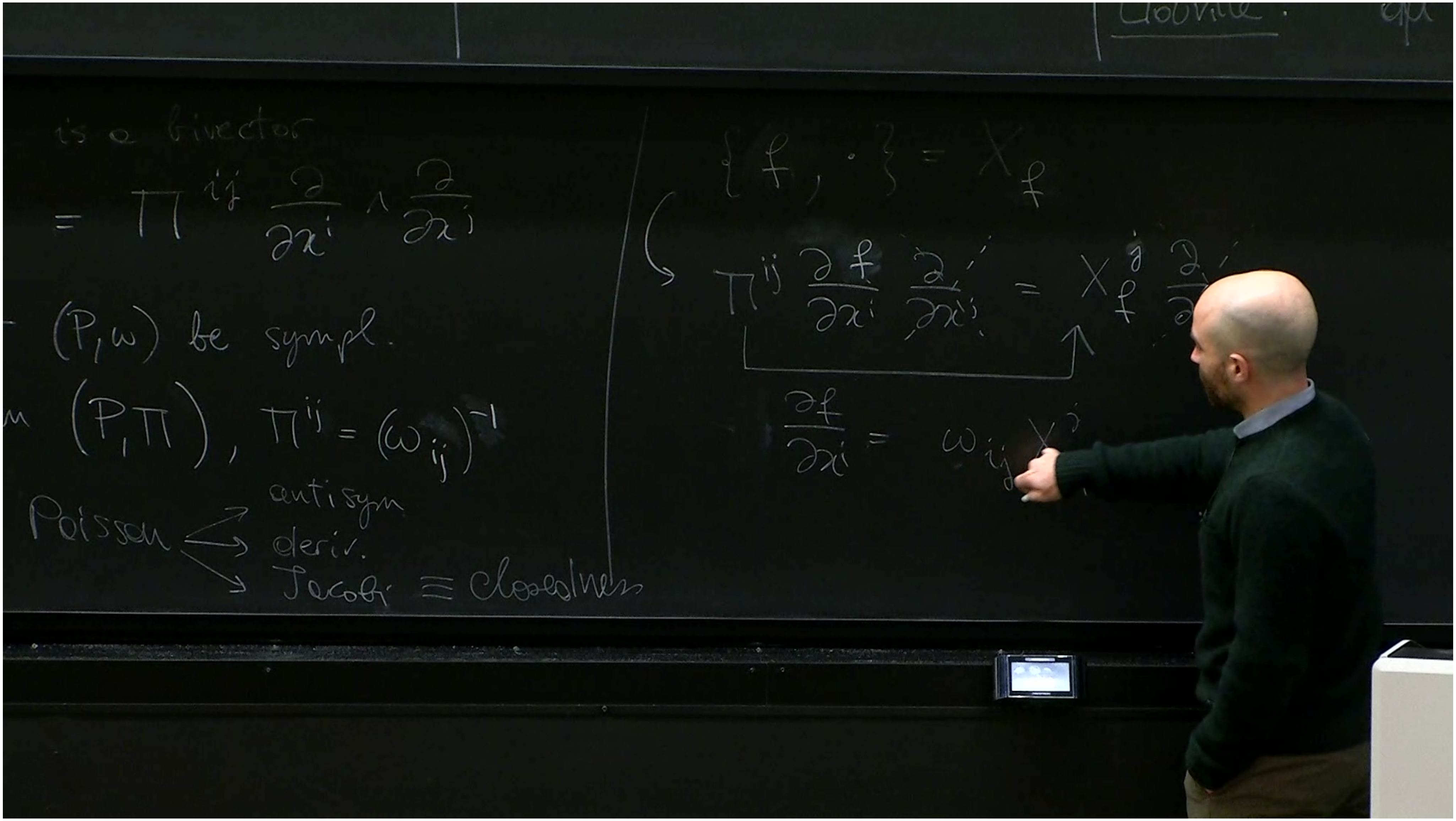
is Poisson → antisym

→ deriv.

→ Jacobi ≡ closedness



Liouville :

is a bivector

$= \Pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$

(P, ω) be sympl.

(P, Π), $\Pi^{ij} = (\omega_{ij})^{-1}$

Poisson → antisym

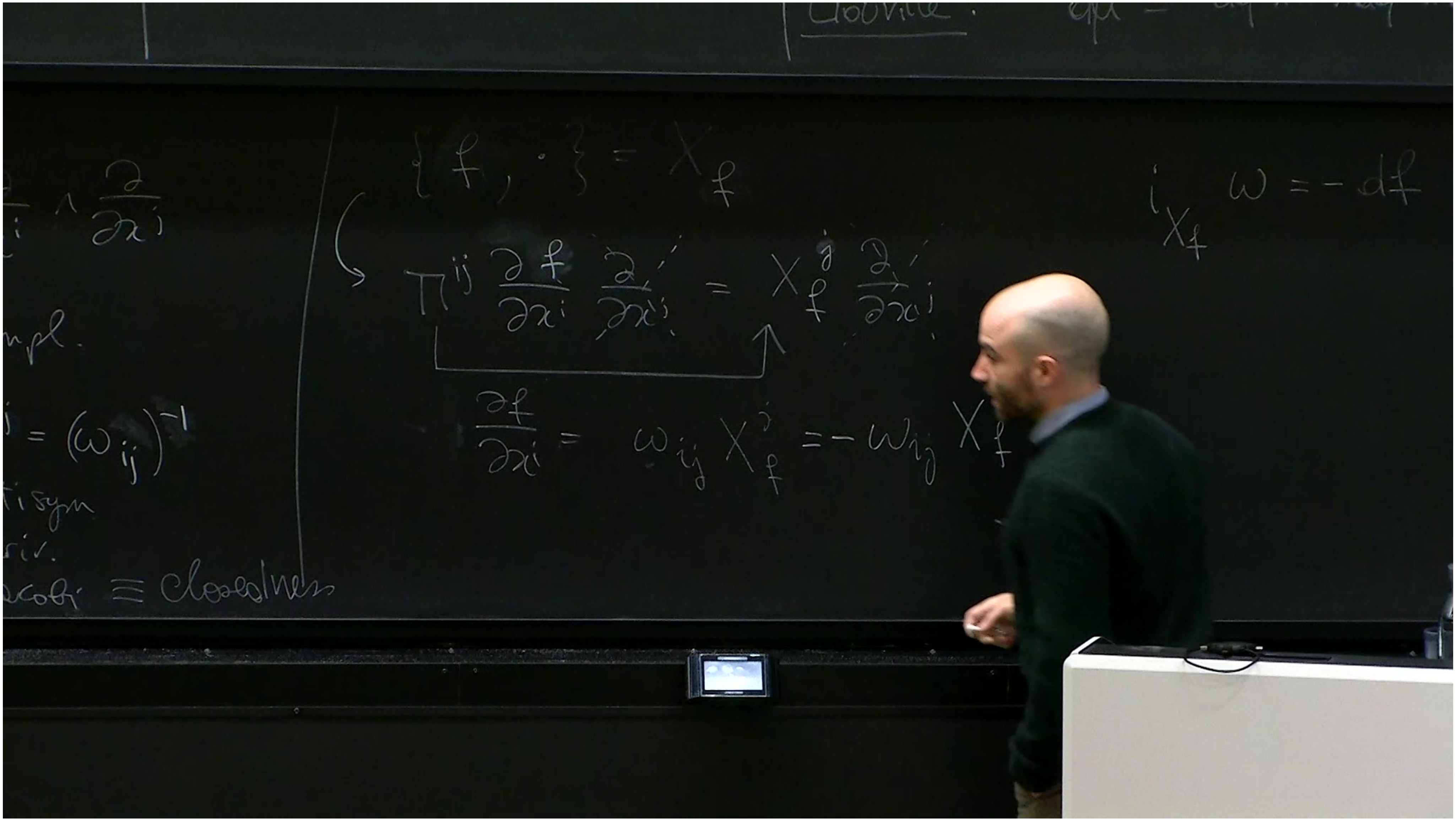
→ deriv.

→ Jacobi ≡ closedness

$\{f, \cdot\} = X_f$

$\Pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} = X_f^j \frac{\partial}{\partial}$

$\frac{\partial f}{\partial x^i} = \omega_{ij} X^j$



Liouville :

$\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$

mpl.

$= (\omega_{ij})^{-1}$

tisym

riv.

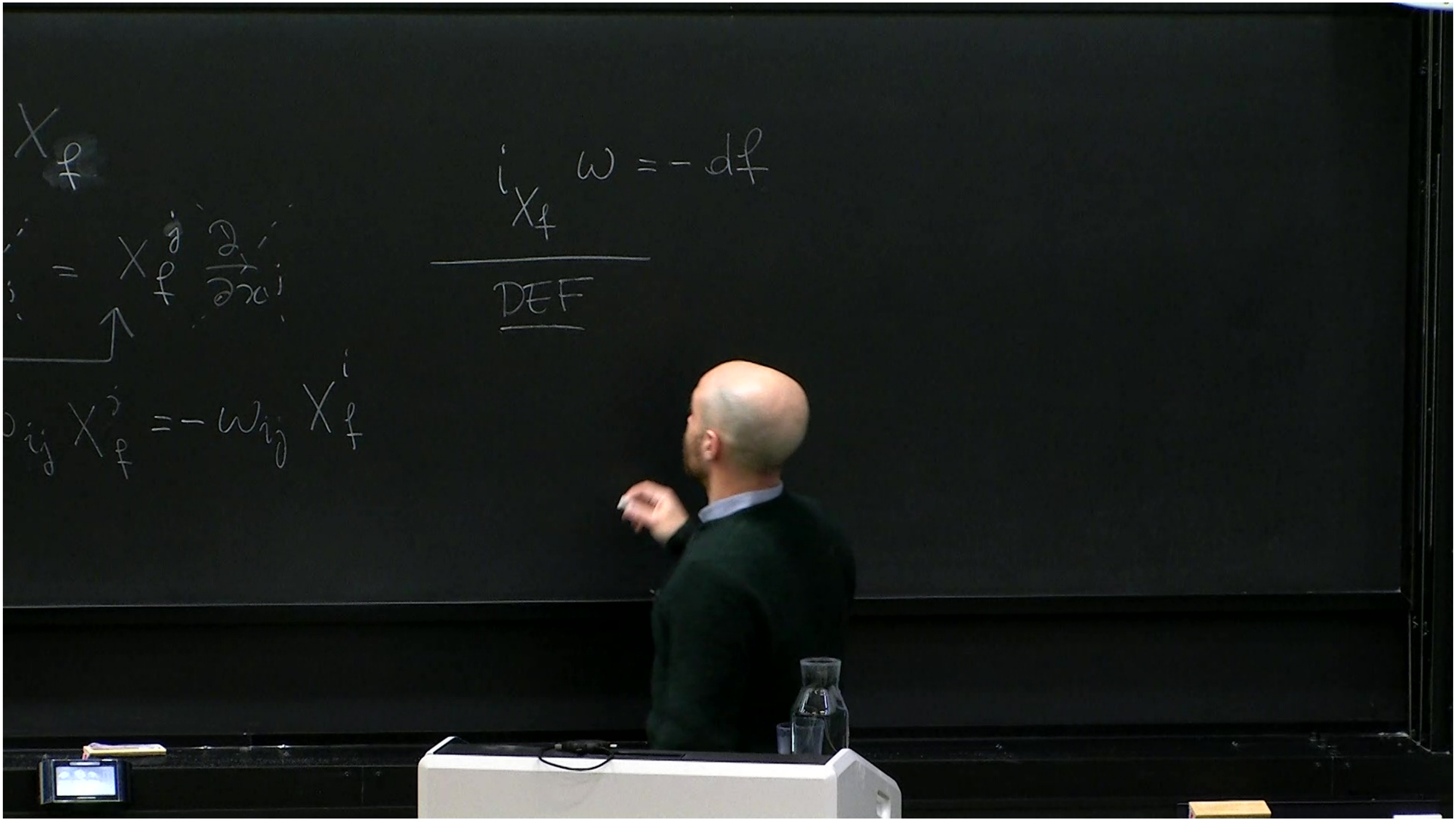
acobi $\equiv$ closedness

$$\{f, \cdot\} = X_f$$

$$\underbrace{\Pi^{ij} \frac{\partial f}{\partial x^i}}\frac{\partial}{\partial x^j} = X_f^j \frac{\partial}{\partial x^j}$$

$$\frac{\partial f}{\partial x^i} = \omega_{ij} X_f^j = -\omega_{ij} X_f$$

$$i_{X_f}\, \omega = -df$$



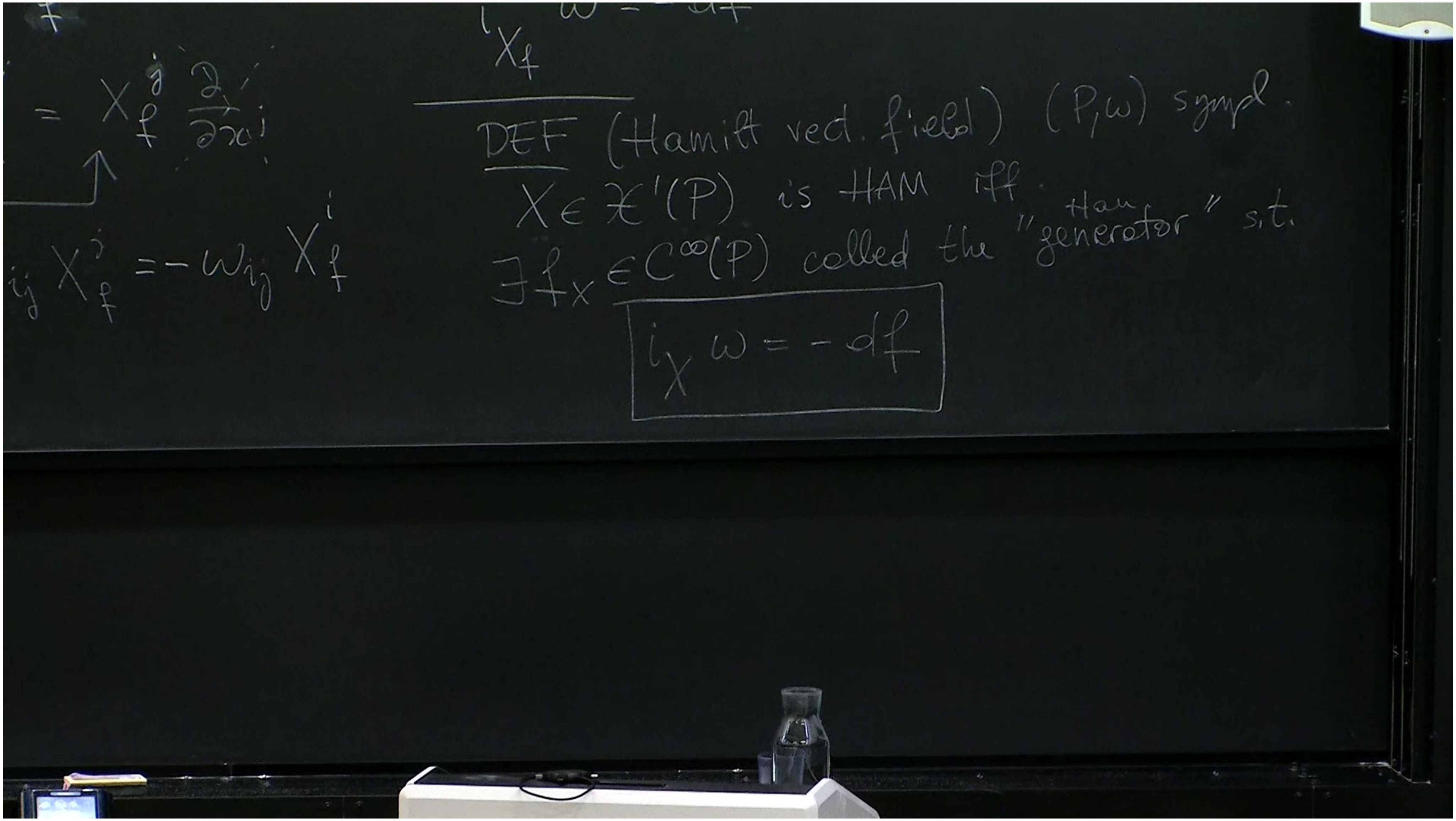
X_f

$= X_f^j \frac{\partial}{\partial x^j}$

$_{ij} X_f^j = -\omega_{ij} X_f^i$

$i_{X_f} \omega = -df$

<u>DEF</u>



$i_{X_f}\omega = -df$

$= X_f^j \frac{\partial}{\partial x^j}$

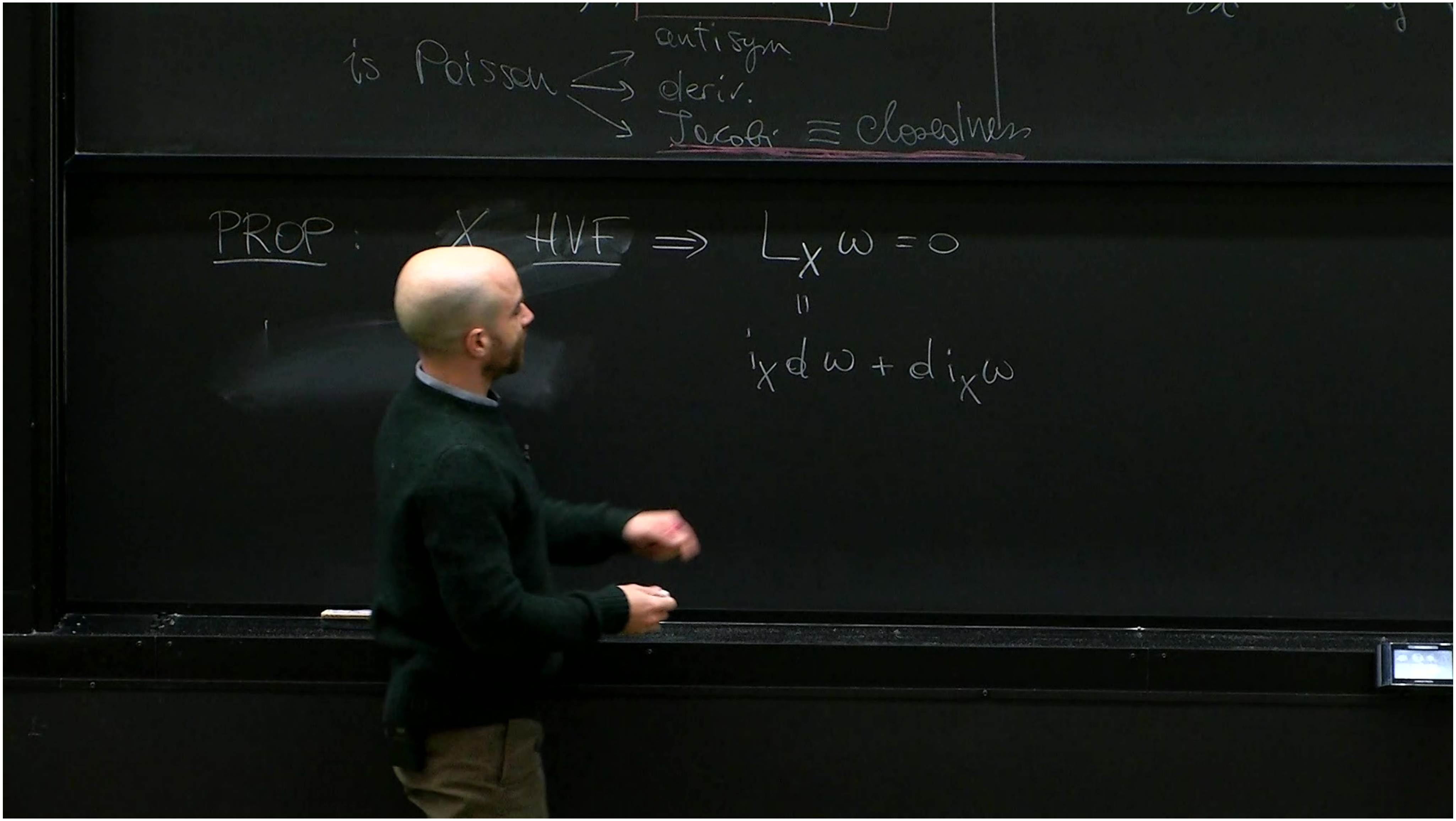
$_{ij} X_f^j = -\omega_{ij} X_f^i$

DEF (Hamilt vect. field) (P,ω) sympl.

$X \in \mathfrak{X}^1(P)$ is HAM iff

$\exists f_X \in C^\infty(P)$ called the "$\overset{\text{Ham}}{\text{generator}}$" s.t.

$$\boxed{i_X \omega = -df}$$



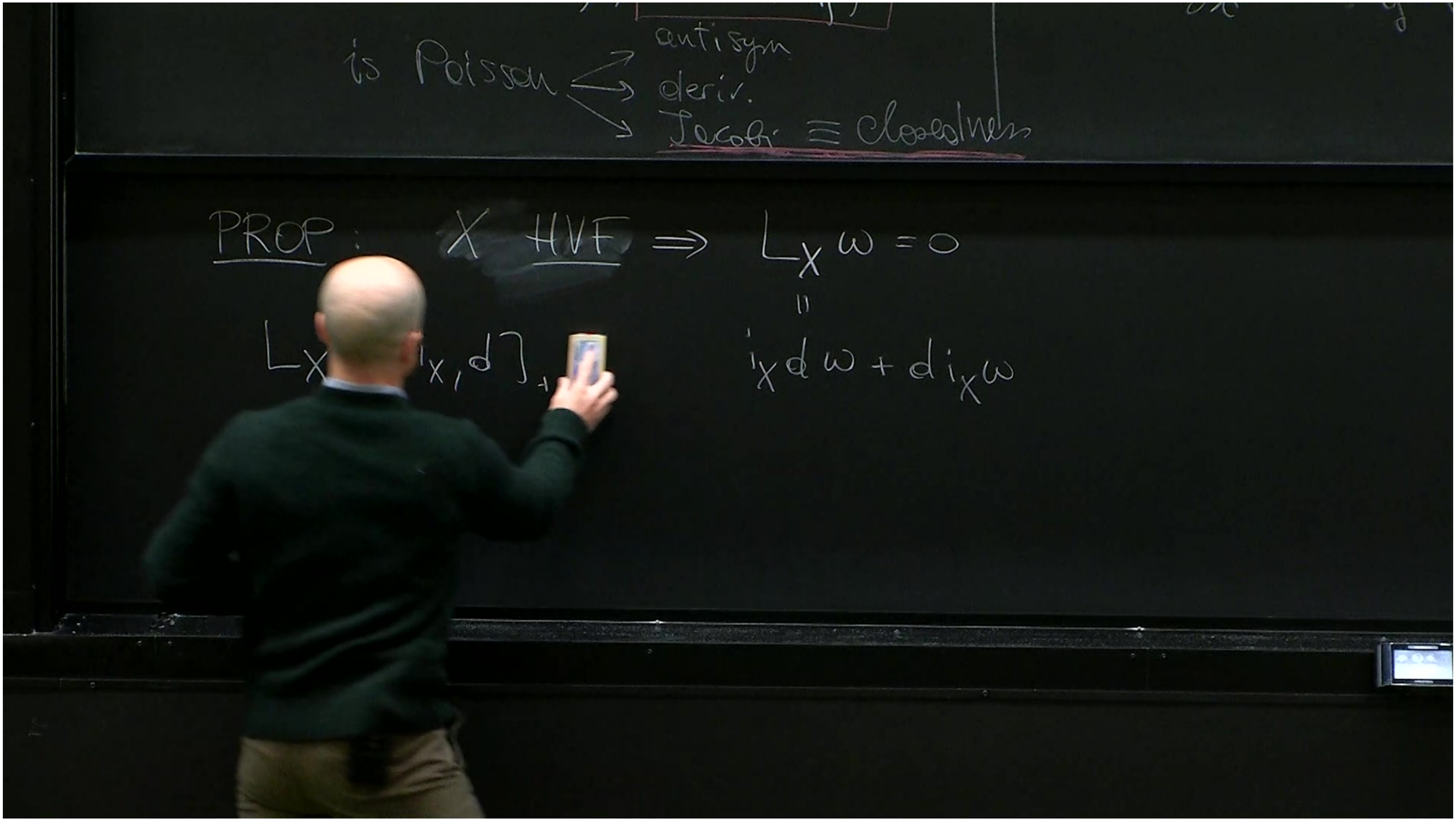
is Poisson → antisym

→ deriv.

→ Jacobi ≡ closedness

PROP: X HVF $\Longrightarrow$ $L_X \omega = 0$

$$\begin{aligned} L_X \omega &= 0 \\ \| \\ i_X d\omega &+ d i_X \omega \end{aligned}$$



is Poisson → antisym

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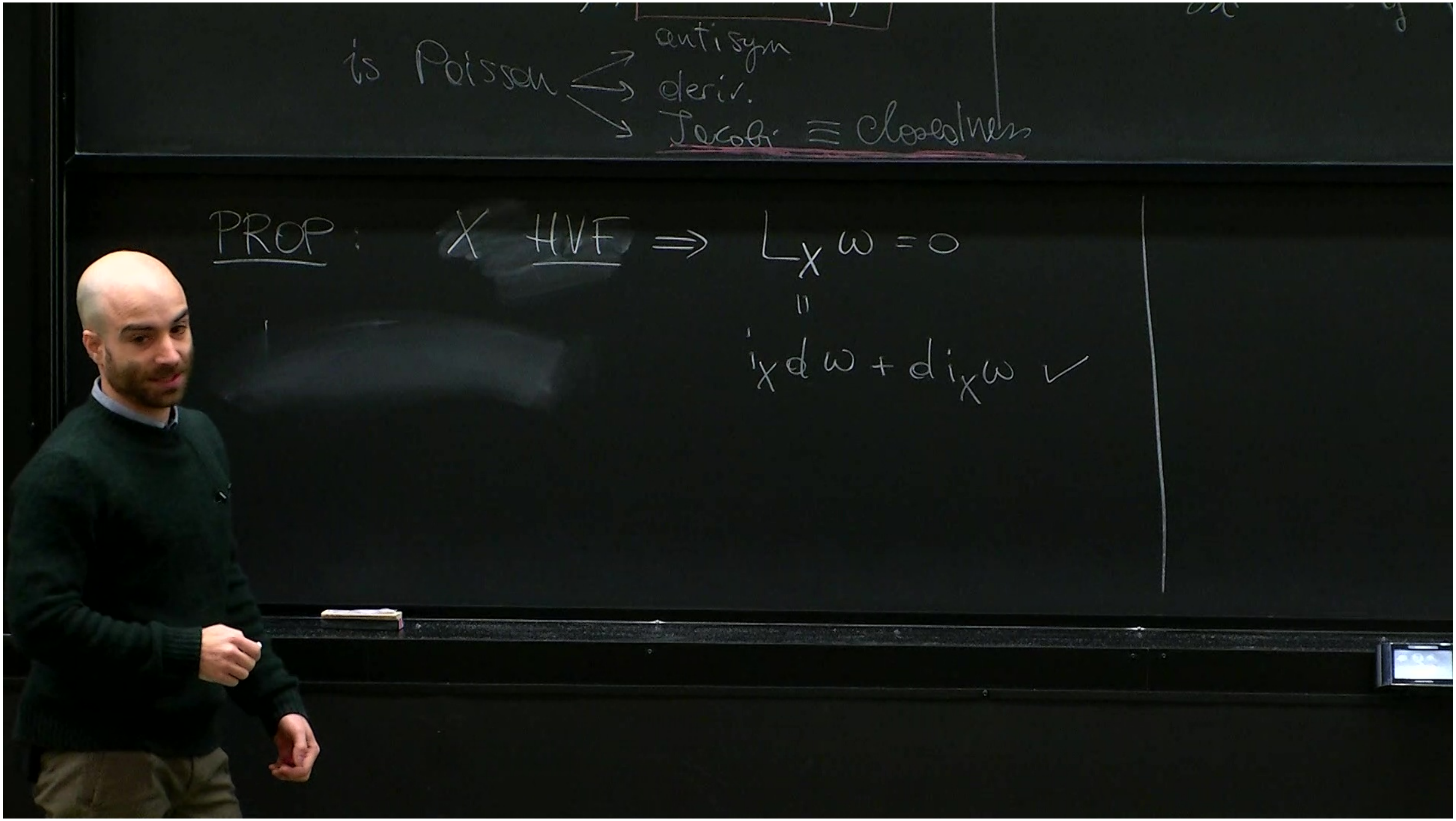
PROP: X HVF $\Longrightarrow$ $L_X \omega = 0$

L_X $i_X, d]_+$ $\qquad$ $= i_X d\omega + d i_X \omega$



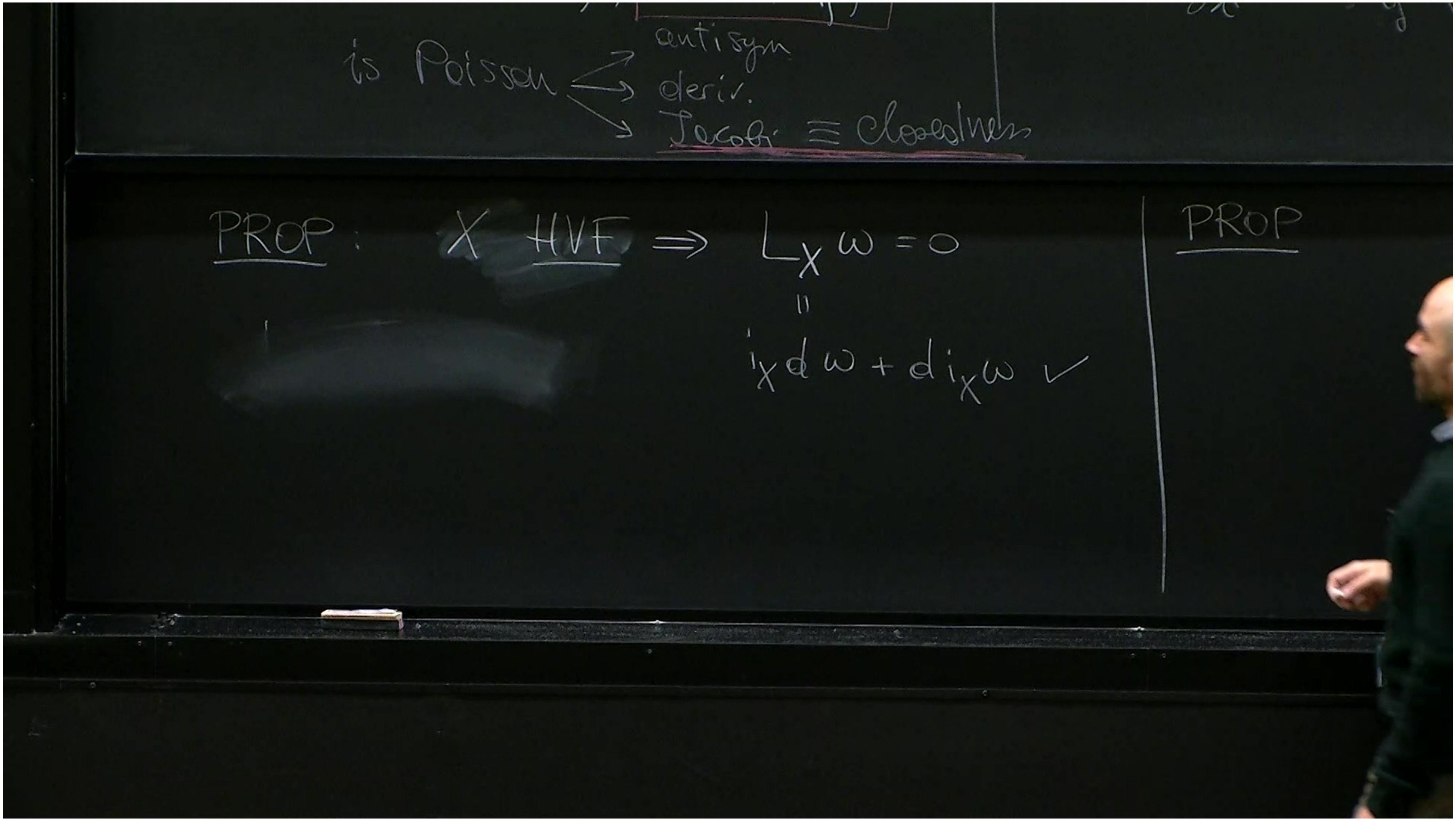
is Poisson $\rightarrow$ antisym

$\rightarrow$ deriv.

$\rightarrow$ Jacobi $\equiv$ closedness

PROP : X HVF $\Longrightarrow$ $L_X \omega = 0$

$$L_X \omega = i_X d\omega + d i_X \omega \checkmark$$

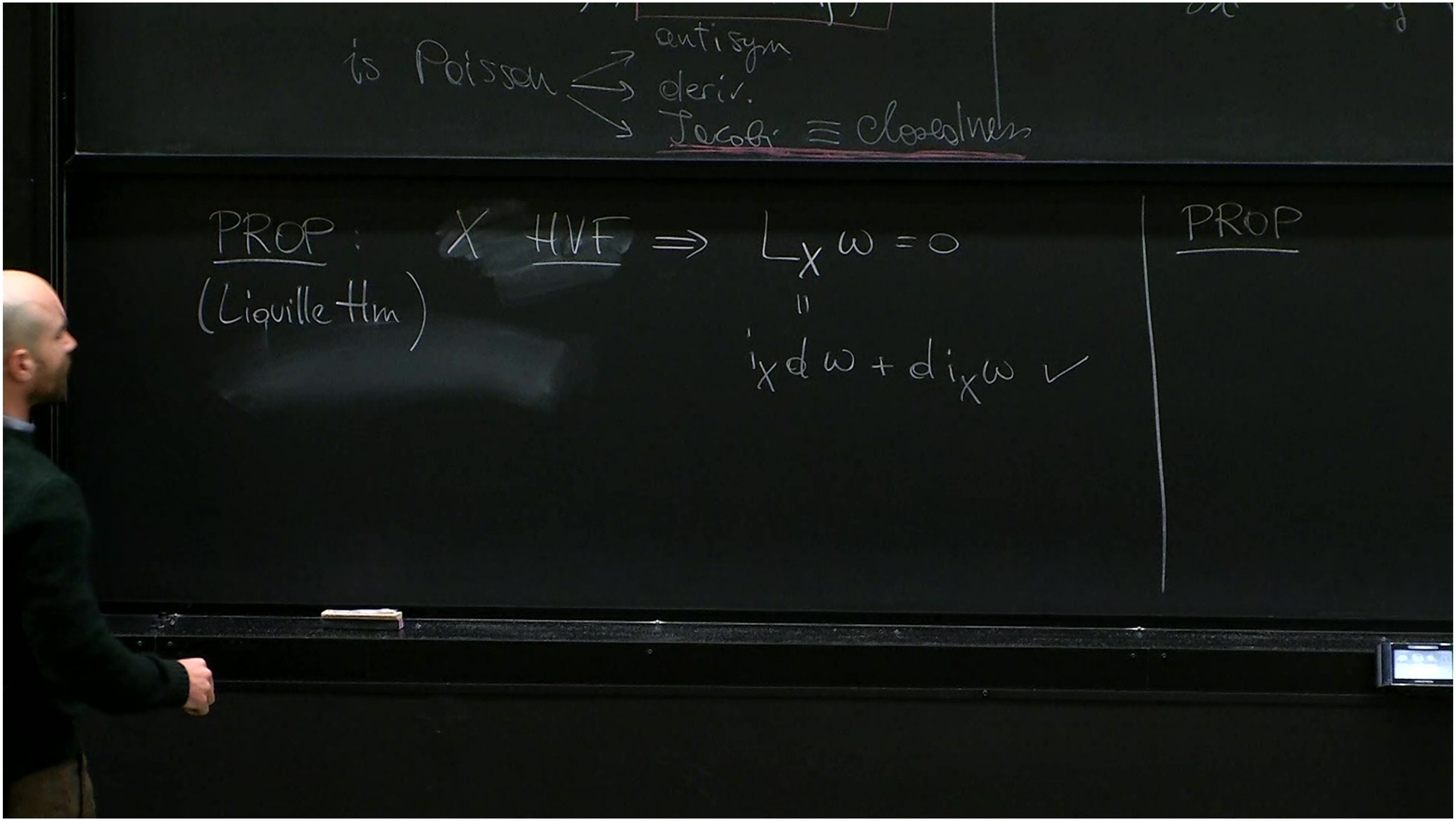


is Poisson $\begin{cases} \to \text{antisym} \\ \to \text{deriv.} \\ \to \text{Jacobi} \equiv \text{closedness} \end{cases}$

PROP: X HVF $\Longrightarrow L_X \omega = 0$

$$\begin{gathered} L_X \omega \\ \shortparallel \\ i_X d\omega + d i_X \omega \ \checkmark \end{gathered}$$

PROP

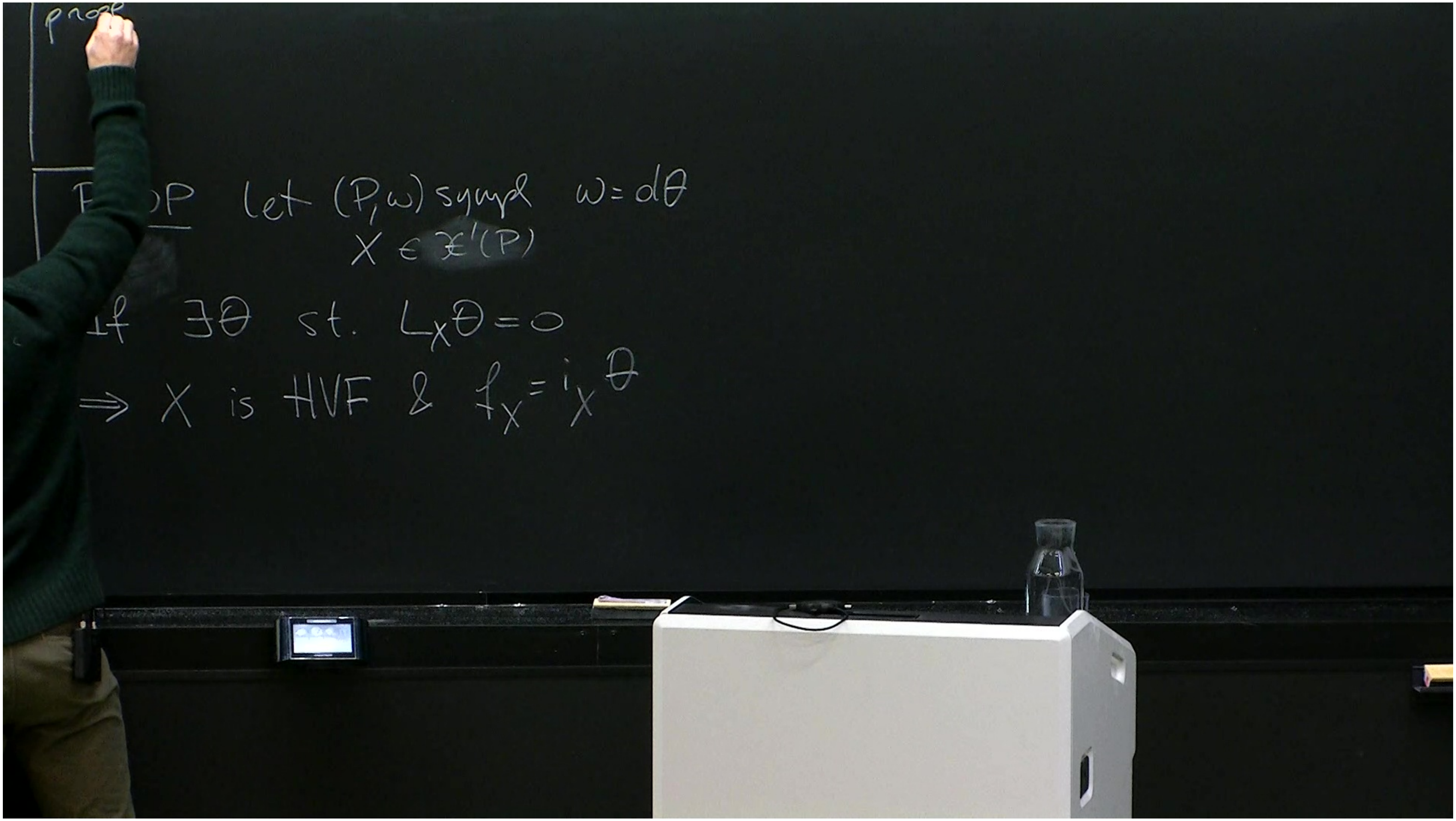


is Poisson → antisym
→ deriv.
→ Jacobi ≡ closedness

PROP: X HVF $\Longrightarrow L_X \omega = 0$
(Liquille thm)

$$L_X \omega = 0$$
$$\parallel$$
$$i_X d\omega + d i_X \omega \checkmark$$

PROP



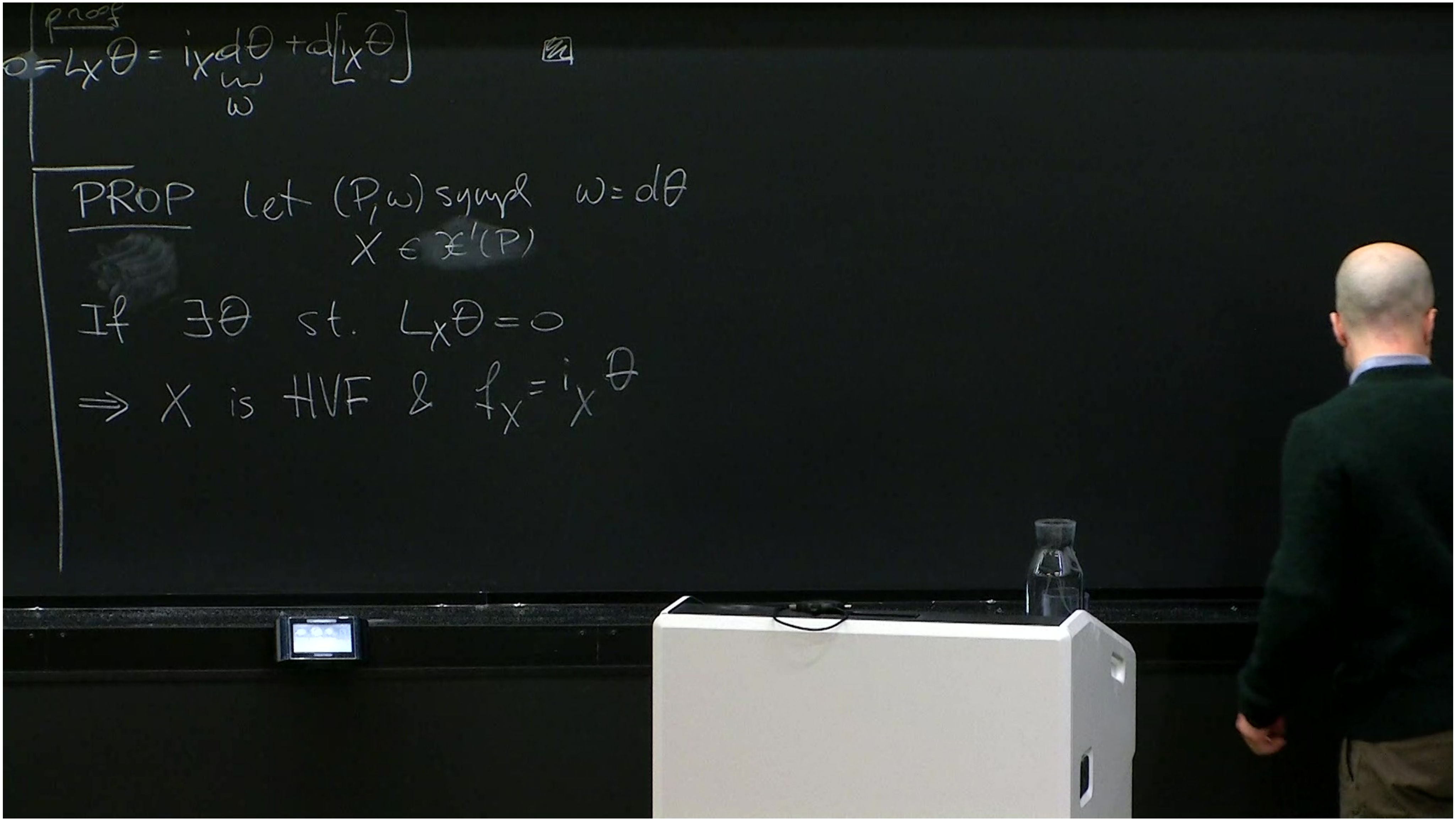
proof

PROP let (P,ω) sympl $\omega = d\theta$

$X \in \mathfrak{X}^1(P)$

If $\exists \theta$ st. $L_X \theta = 0$

$\Longrightarrow X$ is HVF & $f_X = i_X \theta$



proof

$0 = L_X\theta = i_X \underbrace{d\theta}_{\omega} + d[i_X\theta]$ ▨

PROP let (P,ω) sympl $\omega = d\theta$

$X \in \mathfrak{X}^1(P)$

If $\exists\theta$ st. $L_X\theta = 0$

$\Rightarrow X$ is HVF & $f_X = i_X\theta$