Title: Analytic waveform models

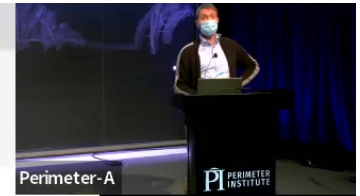
Speakers: Riccardo Sturani

Collection: Gravitational Waves Beyond the Boxes II

Date: April 07, 2022 - 1:15 PM

URL: https://pirsa.org/22040039

The importance of being Modelled: matched-filtering



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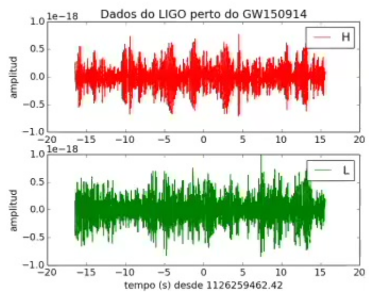

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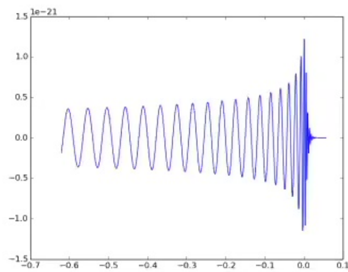

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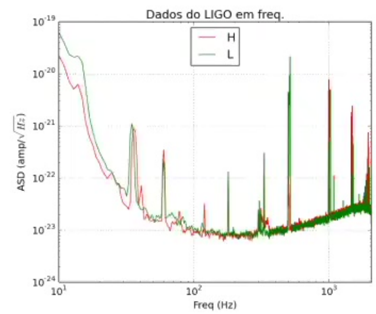

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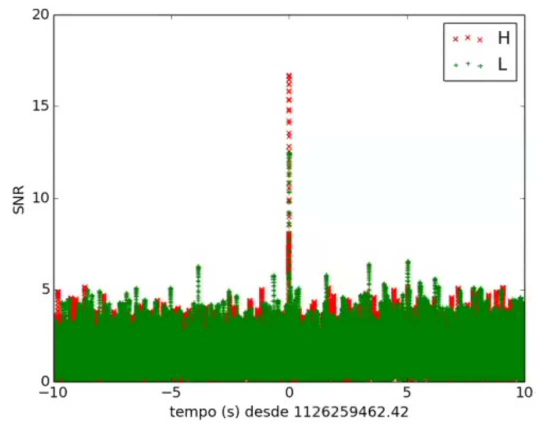


Data from https://losc.ligo.org/events/GW150914/

Fundamental GR: inspiral analytic model

Inspiral $h = A\cos(\phi(t)) \qquad \frac{\dot{A}}{A} \ll \dot{\phi}$

Circular orbits:

$$v \equiv (G_N M \omega)^{1/3} = (G_N M \pi f_{GW})^{1/3} \qquad \eta \equiv \frac{m_1 m_2}{(m_1+m_2)^2}$$

$$E(v) = -\frac{1}{2}\eta M v^2 \left(1 + \#(\eta, S_i/m_i^2)v^2 + \#(\eta, S_i/m_i^2)v^4 + \ldots\right)$$

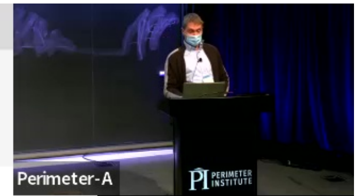
$$P(v) \equiv -\frac{dE}{dt} = \frac{32}{5G_N} v^{10} \left(1 + \#(\eta, S_i/m_i^2)v^2 + \#(\eta, S_i/m_i^2)v^3 + \ldots\right)$$

E(v)(P(v)) known up to 4(3.5)PN

$$\frac{1}{2\pi}\phi(t) = \frac{1}{2\pi}\int^t \omega(t')dt' = -\int^{v(t)} \frac{\omega(v)dE/dv}{P(v)} dv$$

$$\sim \int \left(1 + \#(\eta, S_i/m_i^2)v^2 + \ldots + \#(\eta, S_i/m_i^2)v^6 + \ldots\right) \frac{dv}{v^6}$$

PN Coefficients (tail, absorption $\sim v^8$, memory tidal (NS) $\sim v^{10}$)



Taylor (analytic) approximation schemes:

From $\dot{v} = \frac{P(v)}{dE/dv}$

$$T1: \quad \dot{v} = -\frac{P(v)}{dE/dv} \qquad \dot{\phi} = \frac{v^3}{G_N M}$$

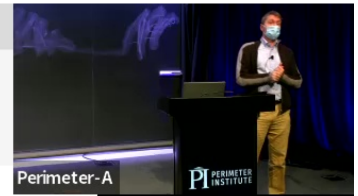
$$T2: \quad \frac{dt}{dv} = -\frac{dE/dv}{P(v)} = \frac{5 G_N M}{32\eta v^9}\left[1 + v^2 + \ldots\right] \qquad \frac{d\phi}{dv} = -\frac{v^3}{G_N M}\left.\frac{P(v)}{dE/dv}\right|_{Tay}$$

$$T3: \quad v(t)\ \mathrm{from}\, t(v) \qquad \phi(v(t))$$

$$T4: \quad \dot{v} = \frac{32\eta}{5 G_N M} v^9\left[1 + v^2 + \ldots\right] \qquad \dot{\phi} = \frac{v^3}{G_N M}$$

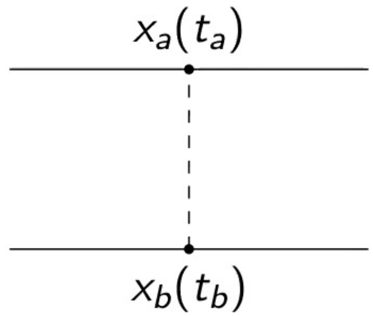
$$T5: \quad \dot{v} = \frac{32\eta}{5 G_N M} v^9\left[1 + v^2 + \ldots\right]^{-1} \qquad \phi(v(t))$$

See review by S. Isoyama, RS, H. Nakano arXiv:2012.01350

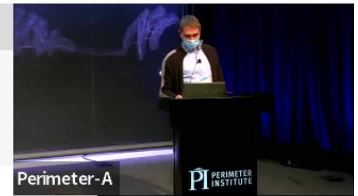


The conservative dynamics

Out of different ways of computing perturbatively the 2-body energy (potential): e.g. 1PM $O(G_N^1)$ or Newtonian potential: consider gravity coupled to particle world-lines:
$-m_a \int d\tau = -m_a \int \sqrt{-g_{\mu\nu} dx^\mu dx^\nu} \sim \frac{1}{2} T^{\mu\nu} h_{\mu\nu} + \ldots$



$$V_{PM}^{(1)}(x_a^\mu - x_b^\mu) = G_N T^a_{\mu\nu} T^b_{\rho\sigma} \Delta^{\mu\nu,\rho\sigma} \int \frac{d^4k}{(2\pi)^4} \underbrace{\frac{e^{ik^\mu(x_{a\mu} - x_{b\mu})}}{|\mathrm{k}|^2 - k_0^2 + \epsilon\,\mathrm{terms}}}_{appropriate\ \ Green's\ \ function}$$



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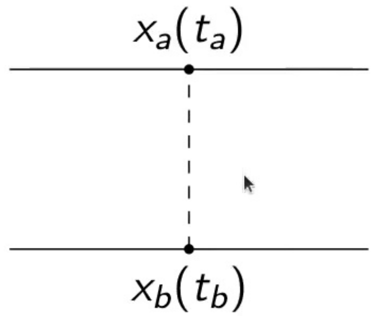

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PM at higher orders

It is a formidable task to go to higher order: almost complete result so far at 4PM $O(G_N^4)$ (3rd order beyond Newtonian) with "particle physics" approaches

1. *Scattering amplitude* method by Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, Zeng, arXiv:2112.10750
2. *EFT+Boundary2Bound* by Cho, Kälin, Porto arXiv:2112.03976

and up to 4PM $O(G_N^4)$ (with partial results up to G_N^7) via the

- "syncretic" *TuttiFrutti* method initiated by

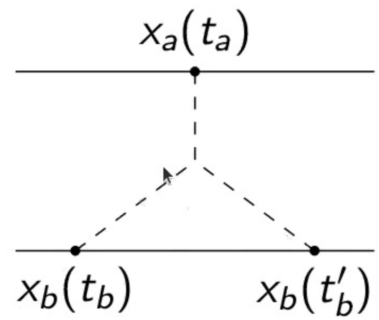
  Bini, Damour, Geralico, PRL '20

  which mixes results from all apprx. methods, including *self-force*, whose expanions small param is $q \equiv m_1/m_2$ $(\eta \equiv m_1 m_2/(m_1+m_2)^2)$

This talk mostly about post-Newtonian approximation to 2-body motion in GR in EFT flavour, aka Non-Relativistic General Relativity initiated by Goldberger& Rothstein PRD '05 (name inspired by NRQCD)

PM gets complicated

At higher order things rapidly complicate



$$
\begin{aligned}
V^{(2PM)} \supset\; & G_N^2 m_1 m_2^2 \int d^4 p\, e^{ip_\mu (x_a^\mu(t_a) - x_b^\mu(t_b))} \frac{p^\alpha p^\beta}{p^2} \int d^4 k \frac{e^{ik^\mu (x_b^\mu(t_b) - x_b^\mu(t'_b))}}{(p-k)^2 k^2} \\
=\; & G_N^2 m_1 m_2^2 \int d^4 p\, e^{ip_\mu (x_a^\mu(t_a) - x_b^\mu(t_b))} \frac{p^\alpha p^\beta}{p^2} \Delta(p^\mu (x_{2\mu}(t_b) - x_{2\mu}(t'_b)))
\end{aligned}
$$

These kinds of conservative diagrams computed up to 4PM order:

- Do not involve radiation to infinity
- Do not involve internal radiative modes, only modes propagating with the "speed of thought", as Eddington put it (Proc. Royal Society of London (1922), series A, 102, 268-282)

PN simplifies: Near

- Near Zone: consider $|\mathrm{k}| \gg k_0$, with $\frac{k_0^2}{|\mathrm{k}|^2} \sim v^2$

$$V_{PN-Near} = \int \frac{dk_0}{2\pi} \frac{d^3k}{(2\pi)^3} e^{ik_0(t_a-t_b)} \frac{e^{i\mathrm{k}\cdot(\mathrm{x}_a-\mathrm{x}_b)}}{|\mathrm{k}|^2} \left(1 + \frac{k_0^2}{|\mathrm{k}|^2} + \cdots \not{\epsilon}\right)$$

k_0 dependence factorizes $\rightarrow \int dk_0 k_0^{2n} e^{ik_0 t_{ab}} \sim \frac{d^{2n}}{dt^{2n}} \delta(t_{ab})$
Near-Zone approximation $V_{PM} \rightarrow V_{PN-Near}$ under control for $k_0^2 < |\mathrm{k}|^2$
Effects for $k_0 \simeq |\mathrm{k}|$ are missed: internal "gravitons" going on-shell

Green's functions

$$G_F = \frac{1}{k_0^2 - |k|^2 + i\epsilon} \qquad G_{A,R} = \frac{1}{(k_0 \mp i\epsilon)^2 - |k|^2}$$

Disregarding imaginary terms in direct space (i.e. for non-rad. modes)

$$G_F = \frac{1}{2}(G_R + G_A)$$

In general: $Im(G_R + G_A) = 0$, $Im(G_f) \propto \delta(k_0 - |k|)$

Near zone conservative dynamics

The potential V (via Feynman Green's function):

$$V \propto \int dk_0 d^3k \frac{e^{-ik_0 t_{12}+i\vec{k}\cdot(\vec{x}_1(t_1)-\vec{x}_2(t_2))}}{k^2-k_0^2-i\epsilon} = \int dk_0 d^3k \frac{e^{-ik_0 t_{12}+i\vec{k}\cdot\vec{x}_{12}}}{k^2}\left(1+\frac{k_0^2}{k^2}+\dots\right)$$

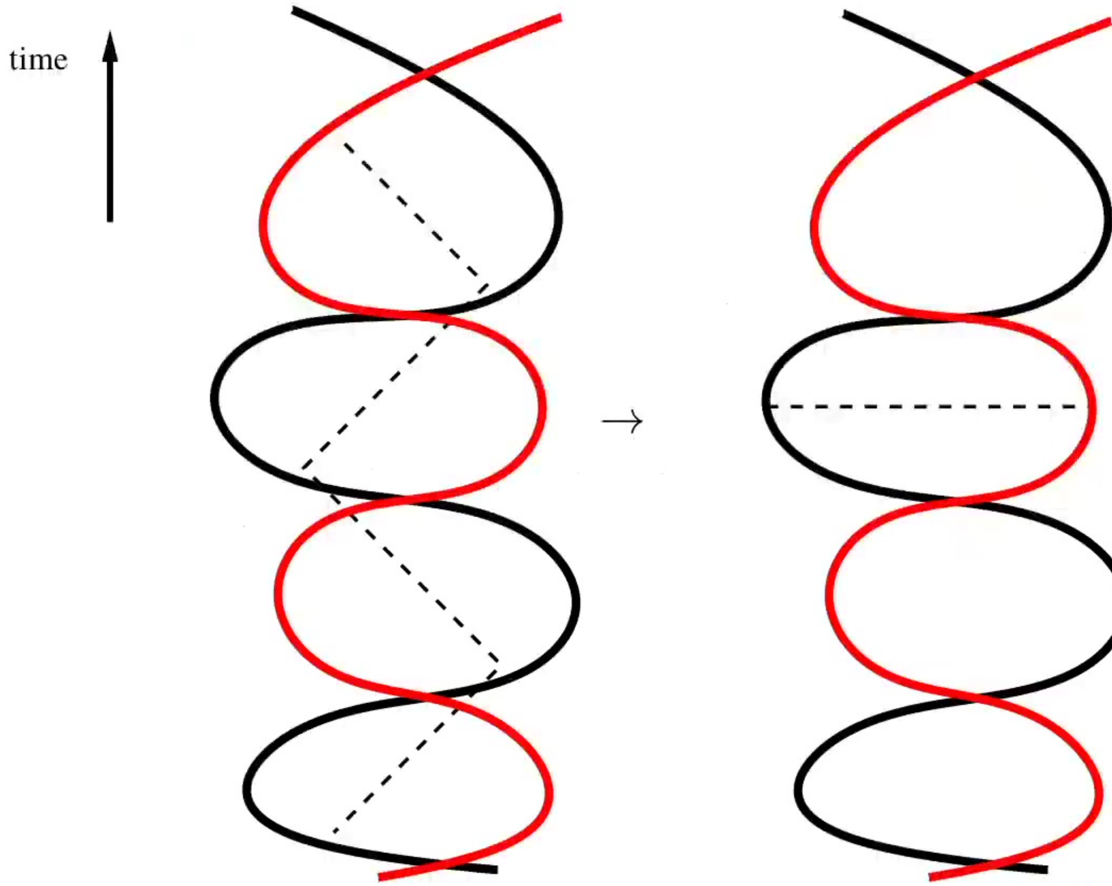
$$= \delta(t_1-t_2)\int d^3k \frac{e^{i\vec{k}\cdot\vec{x}_{12}}}{k^2}\left(1+\frac{\partial_{t_1}\partial_{t_2}}{k^2}+\dots\right)$$

$$= \int d^3k \frac{e^{i\vec{k}\cdot\vec{x}_{12}}}{k^2}\left(1-\frac{\vec{k}\cdot\vec{v}_1\vec{k}\cdot\vec{v}_2}{k^2}+\dots+\frac{\vec{k}\cdot\frac{d^{n-1}\vec{v}_1}{dt^{n-1}}\vec{k}\cdot\frac{d^{n-1}\vec{v}_2}{dt^{n-1}}}{k^{2n}}\right)$$

"Breaking" the propagator enormous simplification, but introduces spurious divergences:
Near zone amplitude integrands clearly bad behaved for $k\to 0$ at high PN-order
Straightforward fix: add the contribution of far-zone, for demonstration see e.g.

Manohar+ '07, Jentzen '12, Foffa, RS '21

Trading knowledge over the full trajectory with knowledge of all derivativ[es] of the trajectory at equal time (PN approximation)

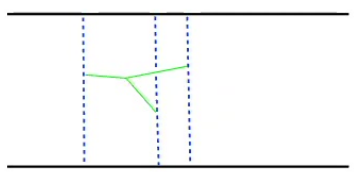


$$+\dot{v} \sim \frac{v^2}{r}$$
$$+\ddot{v} \sim \frac{v^3}{r^2}$$
$$+\ldots$$

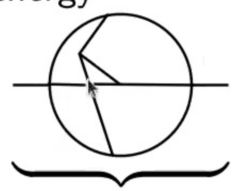
EFT and amplitude: tale of a happy marriage

The main obstruction to scalability of the PN calculation program is the computation of master integrals

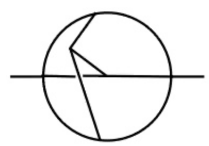
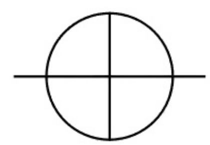
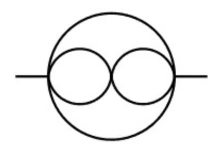
E.g. in the static 4PN sector (i.e. $G_N^5 v^0$) one meets

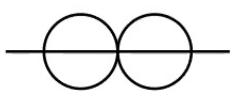


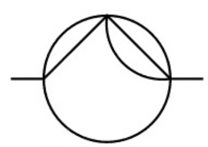
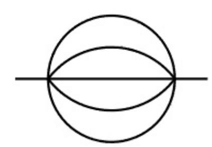
which can be translated in terms of 1 particle self-energy



$$\int_{k_{1,2,3,4}} \frac{Num}{k_1^2\, k_2^2\, k_3^2\, k_4^2\, k_{12}^2\, k_{34}^2\, \hat{k}_{24}^2\, q_{13}^2\, q_{14}^2}$$

which can be reduced  $= c_1$  $+ c_2$ 

$+ c_3$ 

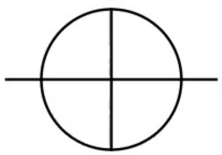
$+ c_4$  $+ c_5$ 

in terms of planar 4-loop self-energy diagrams in gauge theory

Reduction in terms of master integrals

No new master integrals at 5PN, 4PN ones did it all

Foffa, Mastrolia, RS, Sturm '17



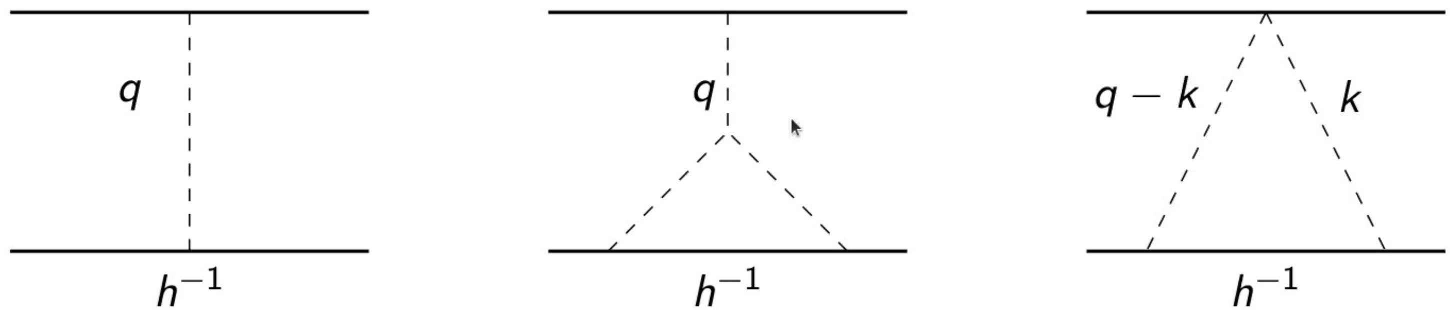
$$= \frac{e^{2\varepsilon\gamma_E}}{s^{2-2\varepsilon}(4\pi)^{4+2\varepsilon}}\left\{\frac{1}{2\varepsilon^2}-\frac{1}{2\varepsilon}-4+\frac{\pi^2}{24} -\varepsilon\left[9-\pi^2\left(\frac{13}{8}-\log 2\right)-\frac{77}{6}\zeta_3\right]+\mathcal{O}\left(\varepsilon^2\right)\right\}$$

Numerical result obtained via $\mathrm{Summertime}$ by Lee& Mingulov
analytic result via PSLQ algorithm, fitting trascendentals to numerical result

Confirmed up to $O(\varepsilon^0)$ by Damour, Jaranowski '18

Classical loops

Feynman diagrams satisfy $L = I - V + 1$, with $I = I_h + I_m$ Green's$\to \hbar$, vertices $\to \hbar^{-1}$ scaling is:
$\hbar^{I_h - V} = \hbar^{L - I_m - 1}$ as massive particles are non-dynamical source/sink



All classical graphs in $q/m \to 0$, with q exchanged momentum also for loop momenta $k/m \to 0$ *before* integration

Method of regions

Method of regions: Internal "graviton" momentum expanded according to scaling:

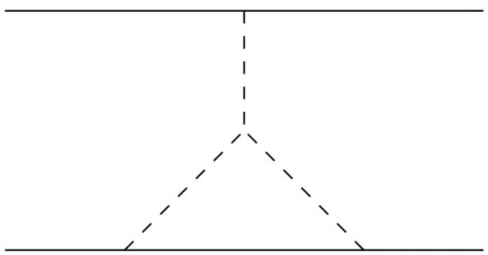
hard	(m, m)	quantum	×
soft	$(\vert\vec{q}\vert, \vert\vec{q}\vert)$	quantum	×
potential	$(v/r, 1/r)$	classical	✓
radiation	$(v/r, v/r)$	classical	✓

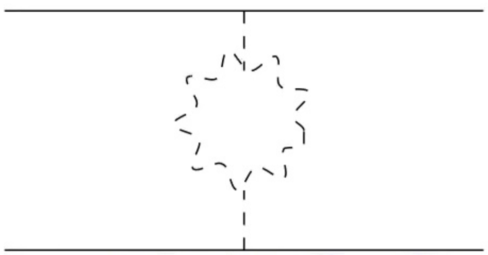
and then integrated over the *full* region of momenta

Only potential and radiation "gravitons" exchanged in classical processes: theory in terms of world lines selects diagrams that do not send source off-shell

Potential "graviton" $\rightarrow$ small change in energy wrt momentum, dominate classically

Ex. of classical/quantum connected diagrams





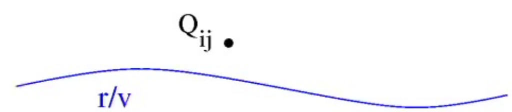
Fundamental aspects of the EFT formalism

NRGR & PN approximation to GR: Small expansion parameter v, related to metric perturbation $v^2 \sim \frac{GM}{r}$

Near zone, $D \sim r$

Describe conservative dynamics

Far zone, $D \gtrsim \lambda = r/v$



conservative + dissipative

EFT framework pioneered by W. Goldberger and I. Rothstein, PRD '06

PN Simplifies: Far

Near zone does not desrcribe all, necessary to include radiative modes

- Far Zone, expand the numerator:

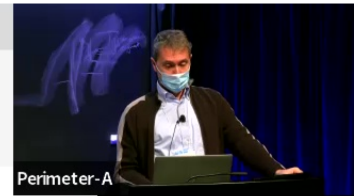
$$V_{PN-Far} \propto \int \frac{d^4k}{(2\pi)^4} e^{ik_0 t_{12}} \sum_n \frac{(i\mathrm{k}\cdot \mathrm{x}_{ab})^n}{n} \frac{1}{|\mathrm{k}|^2 - k_0^2 \pm iak_0 (A,R)}$$

$$G_{A,R} = -\frac{1}{4\pi}\frac{\delta(t\pm r)}{r} \qquad \tilde{G}^*_{A,R}(k_0) = \tilde{G}_{A,R}(-k_0)$$

1. From individual world-lines to multipole expansions $Q_{i_1\dots i_n}$, $L_{i_1\dots i_n}$ with small parameter approximation $\mathrm{k}\cdot\mathrm{x}_{ab} \sim \frac{v}{r}\times r = v$
2. Time-symmetric process determined by $G_R + G_A$ (see later in this talk)
3. Longitudinal modes are present in Far Zone too, sourced by M, P_i, L_i

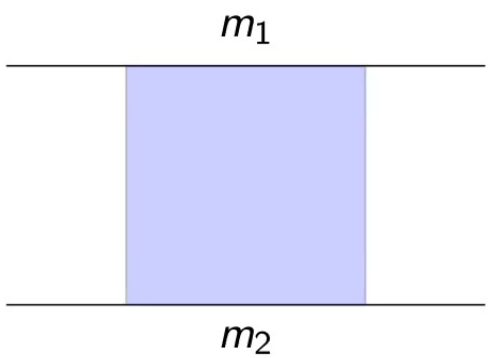
- Old friend of particle physicists: method of regions

PN approximation for compact binary systems

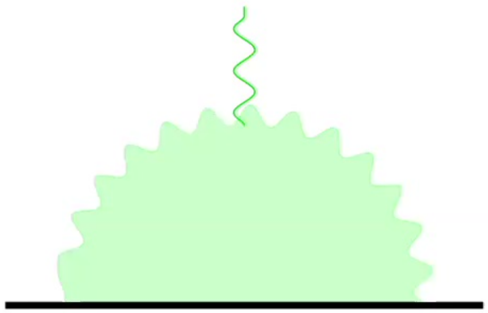


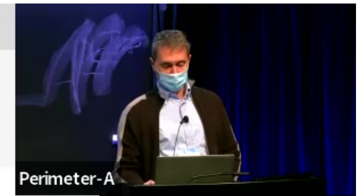
	Near	Far
World-line	$-m_a \int dt + \int dx^\mu S^{ij}\omega_{\mu ij}$ $\int dt \left(Q_{ij}(S)R_{0i0j} + R^2_{\mu\nu\rho\sigma} + \ldots\right)$	$\int d^4x \Big(Eh_{00} + \frac{1}{2}S^{ij}h_{0i,j}$ $+Q_{ij}\underbrace{E^{ij} + O_{ijk}}_{v}E^{ij,k} + J_{ij}B_{ij}\ldots\Big)$
Bulk	$\frac{1}{16\pi G}\int d^4x \left[R - \frac{1}{2}\left(g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}\right)^2\right]$	
5PN	Foffa, RS, '19, '20 (partial) Blümlein+ '20	Almeida, Foffa, RS+ '21 Blümlein '21
6&7 PN	×	Blanchet,RS+ '20 (logs)

Near vs. Far zone graphs (topology)



And 1 pt diagrams → radiation





Controlling the approximation

$$
\begin{aligned}
I &\equiv \frac{e^{i\mathrm{k}\cdot\mathrm{x}}}{\mathrm{k}^2 - k_0^2 + \epsilon} \\
N &\equiv \frac{e^{i\mathrm{k}\cdot\mathrm{x}}}{\mathrm{k}^2} \sum_{n\geq 0} \left(\frac{k_0^2}{\mathrm{k}^2}\right)^n \\
F &\equiv \frac{1}{\mathrm{k}^2 - k_0^2 + \epsilon} \sum_{r\geq 0} \frac{(i\mathrm{k}\cdot\mathrm{x})^r}{r!} \\
D &\equiv \frac{1}{\mathrm{k}^2} \sum_{n,r\geq 0} \left(\frac{k_0^2}{\mathrm{k}^2}\right)^n \frac{(i\mathrm{k}\cdot\mathrm{x})^r}{r!}
\end{aligned}
$$

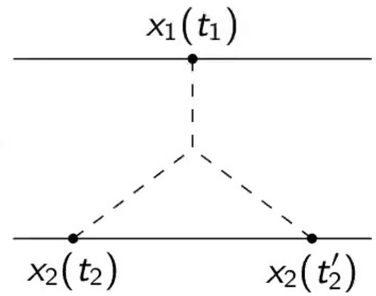
Using scale separation $\frac{v}{r} < \kappa < \frac{1}{r}$ and dim. reg.: $\int_{\mathrm{k}} |\mathrm{k}|^\alpha = 0$, $N : \kappa < k$, $F : k < \kappa$

(Manohar+ PRD '07, Jantzen JHEP '12, Foffa RS '21)

$$
\begin{aligned}
\int_{\mathrm{k}} I - (N + F) &= \int_N (I - N - F) + \int_F (I - N - F) + \underbrace{\int_k D}_{=0} \\
&= \int_N \Big[\underbrace{I - N}_{=0} - \underbrace{(F - D)}_{=0}\Big] + \int_F \Big[\underbrace{I - F}_{=0} - \underbrace{(N - D)}_{=0}\Big]
\end{aligned}
$$

(IR) Divergences in the Near zone

IR divergences due to the splitting into Near and Far



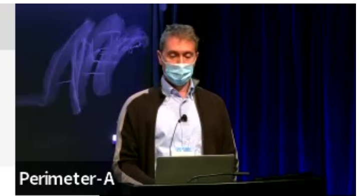
In the full theory:

$$\begin{aligned} V &\supset \int dt_{1,2,2'}\, d^4p\, e^{ip_\mu(x_1^\mu(t_1)-x_2^\mu(t_2))}\frac{p^\alpha p^\beta}{p^2}\int d^4k\,\frac{e^{ik^\mu(x_2(t_2)-x_2(t_2'))}}{(p-k)^2k^2} \\ &= \int dt_{1,2,2'}\, d^4p e^{ip_\mu(x_1^\mu(t_1)-x_2^\mu(t_2))}\frac{p^\alpha p^\beta}{p^2}\Delta(p^\mu(x_2(t_2)-x_2(t_2'))) \end{aligned}$$

after near/far breaking:

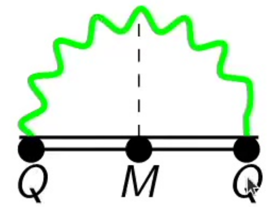
$$\int dt\, d^3p\, e^{i\vec{p}(\vec{x}_1-\vec{x}_2)}\frac{p^ip^j}{|\mathrm{p}^2|}\int d^3k\,\frac{1}{|\mathrm{k}|^2|\mathrm{p}-\mathrm{k}|^2}\left(1+\ldots+\frac{\omega^4}{|\mathrm{k}|^4}+\ldots\right)$$

$$=\int dt\, d^3p e^{i\vec{p}\cdot\vec{x}_{12}}\frac{p^ip^j}{|\mathrm{p}|^3}\left\{1+\ldots+\frac{1}{|\mathrm{p}|^4}\left[(\vec{p}\cdot\vec{v}_1)^3(\vec{p}\cdot\vec{v}_2)^3+\ldots+\underbrace{\vec{p}\cdot\dot{\vec{a}}_1\vec{p}\cdot\dot{\vec{a}}_2}_{IR\ \ divergence}\right]\right\}$$



Compensating divergences Near - Far

Far diagram with triple interacion



$$V \supset G M \dddot{Q}_{ij}^2 \left(\frac{1}{\epsilon_{UV}} + \log(\mu\omega) \right) - G M m_1 m_2 r^2 \dot{a}_1^i \dot{a}_{2i} \left(\frac{1}{\epsilon_{IR}} + \log(\mu r) \right)$$

- Theory at short and large distances have compensating spurious divergences, finite terms derived straightforwardly
- No far zone IR divergences
- From far zone alone → leading UV logs in the Energy function at all orders via Renormalization group flow (see later in this talk)

Goldberger+ '13, Blanchet,RS+ '20

- Near zone UV divergences canceled by local counterterms:

$$G^2 m_a^3 \int d\tau \left(a^\mu \dot{v}_\mu + R_{\mu\nu} v^\mu v^\nu \right)$$

$a^\mu = 0 = R_{\mu\nu}$ on the equations of motion, Foffa, RS+ '19

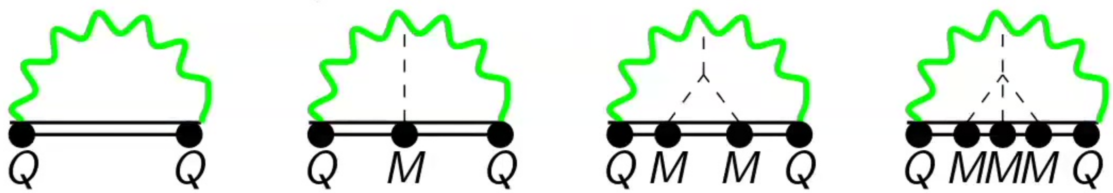
What if one had only the Far theory? Divs. + Logs

- In PN approximation Log terms arise from tail processes at 4PN order, non-local (but causal) effective term in conservative dynamics $(x \sim v^2 \sim Gm/r \equiv \gamma)$:

$$\mathcal{L} = \frac{M\eta}{2} v^2 + \ldots + \frac{2G^2M}{5} \dddot{Q}_{ij}(t) \int d\tau \log(\tau) \ddddot{Q}_{ij}(t-\tau) \ldots$$
$$\rightarrow E_{circ} = -\frac{M\eta x}{2}\left(1 + \ldots + \frac{448}{15}\eta x^5 \log x + \ldots\right)$$

which turns local on circular orbits

- Expansion in $GM\omega = \frac{GM}{r} \times r\omega \sim v^3$:



Rad Reac 2.5PN

LO tail: Blanchet, Damour PRD ('88)

Leading Logs at all orders

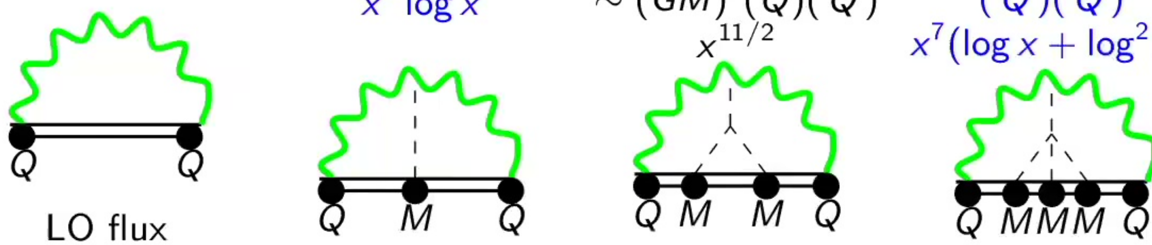
Real part E:

Self-E$\sim (\ddot{Q})(\dddot{Q}) \sim 0$

tail$^1 \sim GM(\dddot{Q})^2 \log t$ — $x^4 \log x$

tail$^2 \sim (GM)^2(\dddot{Q})(\overset{....}{Q}) \log \sim (GM)^2(\dddot{Q})(\overset{....}{Q})$ — $x^{11/2}$

tail$^3 \sim (GM)^3 (\log + \log^2) \times (\overset{....}{Q})(\overset{....}{Q})$ — $x^7(\log x + \log^2 x)$



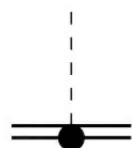
Imaginary $\frac{dE}{dt}$:

LO flux (via optical theorem)

tail$^1 \sim \pi x^{3/2}$

tail$^2 \sim x^3 log(x)$

tail$^3 \sim ? \times x^{9/2}$



Other insertions possible:
but do not contribute to leading E-logs: $\eta^2 x^{3n+1} \log^n x$ from tail^{2n-1}
Renormalization group enables to compute all leading logs

Far UV divergences

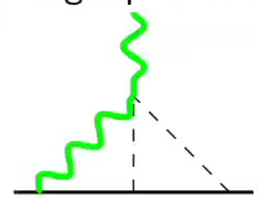
Suppose one had the Far zone theory only: the UV divergence is not compensated by the NZ but it can be renormalized:

- drop the divergence (absorb it with a local counterterm)
- impose $\mu-$independence Goldberger, Ross, Rothstein PRD '14

$$\frac{d\mathcal{L}_{tail}}{d\log\mu}=0 \implies \frac{dM}{d\log\mu}=-\frac{2G^2M}{5}\left(2Q_{ij}^{(1)}Q^{(5)}-2Q_{ij}^{(2)}Q_{ij}^{(4)}+\left(Q_{ij}^{(3)}\right)^2\right)$$

which can be solved by short-circuiting with analog equation for

$$\frac{dQ_{ij}}{d\log\mu}=\frac{214}{105}(GM)^2\,\ddot{Q}_{ij}(t,\mu)$$



Goldberger, Ross PRD '09
(see also Anderson+ '82!)
Adding analogous formula for J (Bernard, Blanchet, Faye, Marchand, Phys. Rev. D97 (2018)) and taking orbital average:

$$M(\mu)=M(\mu_0)-MG^2\sum_{n\geq1}\frac{(2\log(\mu/\mu_0))^n}{n!}\left(\beta_I G^2M^2\right)^{n-1}\langle Q_{ij}^{(n+2)}Q_{ij}^{(n+2)}\rangle$$

$$L(\mu)=L(\mu_0)-\frac{12MG^2}{5}\sum_{n\geq1}\frac{(2\log(\mu/\mu_0))^n}{n!}\left(\beta_I G^2M^2\right)^{n-1}\langle Q_{ij}^{(n+1)}Q_{ij}^{(n+2)}\rangle$$

Not quite there for E_{circ}: need for $dE = \omega dL$

Using $Q_{ij}(M,\mu)$ one has (leading log part of) $M(M_0, v, \gamma)$, $L(L_0, v, \gamma)$, adding $dE = \omega dL$ one can compute $r(v)$ on circular orbits: $Energy(r,v) \to E_{circ}(x)$ $(x \equiv (GM\omega)^{2/3})$

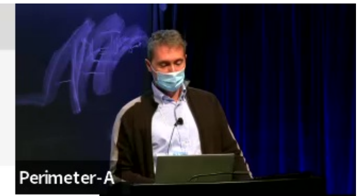
$$\gamma \equiv \frac{GM}{r} = x\left[1 + \frac{32\eta}{15}\sum_{n\geq 1}\frac{3n-7}{n!}(4\beta_I)^{n-1}x^{3n+1}(\log x)^n\right]$$

$$E = -\frac{m\eta x}{2}\left[1 + \frac{64\eta}{15}\sum_{n\geq 1}\frac{6n+1}{n!}(4\beta_I)^{n-1}x^{3n+1}(\log x)^n\right]$$

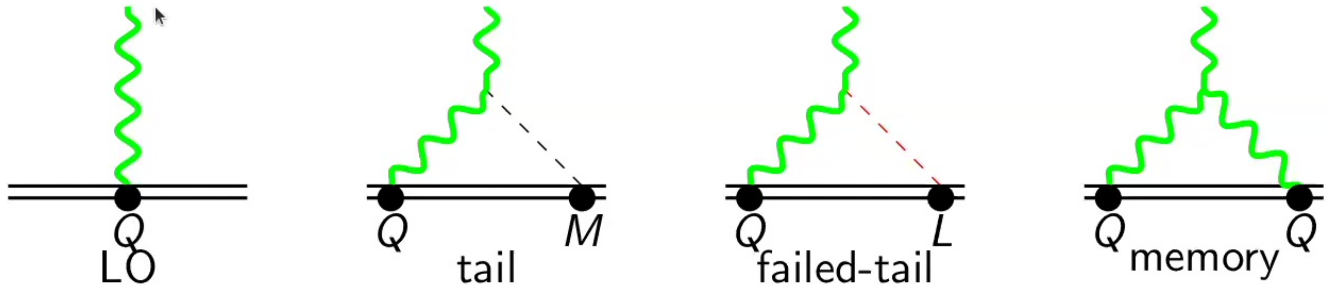
$$J = \frac{m^2\eta}{\sqrt{x}}\left[1 - \frac{64\eta}{15}\sum_{n\geq 1}\frac{3n+2}{n!}(4\beta_I)^{n-1}x^{3n+1}(\log x)^n\right]$$

Remarkably $E(x)$ agrees 22PN order $x^{3n+1}(\log x)^n$ (up to $n=7$), expanded self-force result by Kavanagh, Ottewill, Wardell, PRD (2015)

$$E_{circ} = -\frac{M\eta}{2}x\left(1 + \frac{16\eta x^2}{15\beta_I}\left[\left(1 + 24\beta_I x^3\log x\right)x^{4\beta_I x^3} - 1\right]\right) \qquad \beta_I = -\frac{214}{105}$$



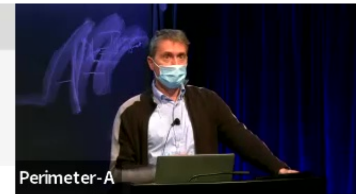
More far zone diagrams



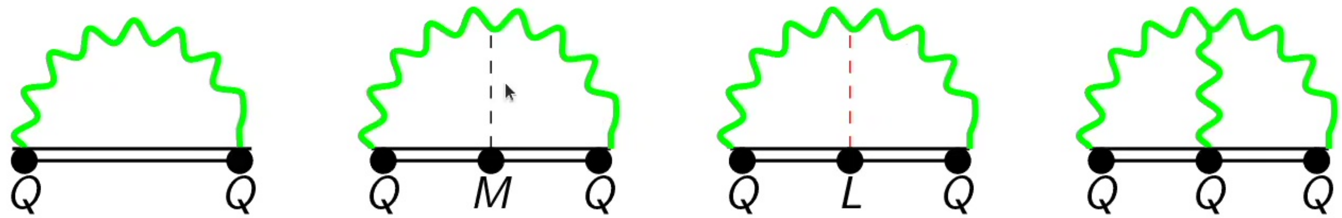
$$h = \frac{G_N}{r}\ddot{Q}\left[1 + G_N \left(\underbrace{\omega M}_{v^3} + \underbrace{\omega^2 L}_{v^5} + \underbrace{\omega^3 Q}_{v^5} \right)\right]$$

- Tail: $\sim G_N M \int_\infty^t dt' \log(t-t')\ddddot{Q}(t') \simeq$
  $\frac{1}{2P}\ddot{Q}(t-2P) + \int_{t-2P}^t dt' \ddddot{\log}(t-t')Q(t') - \int_{-\infty}^{t-2P} \frac{dt'}{t-t'^2}\ddot{Q}(t')$
  hence the name "tail"
  Local contribution to the Flux (h is affeced by a phase terms)
- Failed-tail: $\sim G_N M \dddot{Q}_{ik} L_{jk}$, local, no divergence
- Memory: $\sim G_N M \int_{-\infty}^t \dddot{Q}(t')\dddot{Q}(t')$
  Contribution from arbitrarily past time is not supressed, Flux $\sim \dot{h}^2$ is local

Christodoulou non-linear memory PRL '91

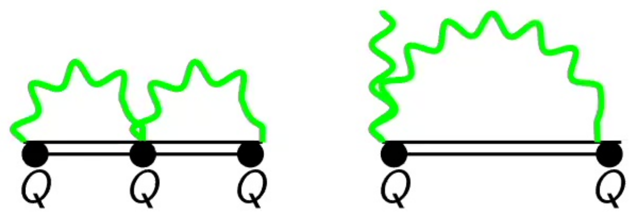


Conservative counterparts of far-zone processes



$$V_{eff} \sim G_N \left[\underbrace{\ddot{Q}\dddot{Q}}_{v^5} + G_N \left(\underbrace{M \dddot{Q}\dddot{Q} \log(t)}_{v^8} + \underbrace{L_{ij} \dddot{Q}_{ik} \ddddot{Q}_{jk}}_{v^{10}} + \underbrace{\omega^8 Q^3}_{v^{10}} \right) \right]$$

Plus the insertion of a new topology $\sim Q^3$ (self-interaction squared)



which are v^5 wrt the LO radiation reaction effect

Almeida, Foffa, RS '21, Blümlein, Maier, Marquard, Schäfer '21

Problem in wrapping up the 5PN dynamics

Tail and memory diagrams are what is left to compute at 5PN oder, however they fail to pass a consistency check
Actually computations are simpler in terms of *scattering angle* rather than potentials.
E. g. transferred momentum

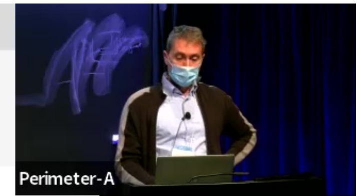
$$
\begin{aligned}
|\Delta p| \quad = \quad & \frac{G_N m_1 m_2}{b}\Bigg[q^{(1PM)} + \frac{G_N}{b}\left(m_1 q_1^{(2PM)} + m_2 q_2^{(2PM)}\right) \\
& + \frac{G_N^2}{b^2}\left(m_1^2 q_{11}^{(3PM)} + m_1 m_2 q_{12}^{(3PM)} + m_2^2 q_{22}^{(3PM)}\right)\Bigg] + O(G_N^4)\,,
\end{aligned}
$$

scattering angle χ

$$
\begin{aligned}
\chi \quad = \quad & \eta\frac{G_N E}{b}\Bigg[X^{(1PM)} + \frac{G_N M}{b}X^{(2PM)} + \frac{(G_N M)^2}{b^2}\left(X_{11}^{(3PM)} + \eta X_{12}^{(3PM)}\right) \\
& + \frac{(G_N M)^3}{b^3}\left(X_{111}^{(3PM)} + \eta X_{112}^{(4PM)}\right)\Bigg] + O(G_N^5)\,,
\end{aligned}
$$

Damour PRD '20

implying that *n*-order Self-Force results determine (2n+2)PM results
As memory $\sim G_N^2\omega^8 Q^3 \sim Mv^{12} \sim MG_N^4 v^4$ (usign e.o.m.) one has that 1SF order can determine memory G_N^4 terms at oder η^2, but it does not match



Summary body dynamics expansions (spin-less)

Post-Minkowskian expansion parameter is $G_N M/r$, vs PN expansion

$$\mathcal{L} = -Mc^2 + \frac{\mu v^2}{2} + \frac{GM\mu}{r} + \frac{1}{c^2}[\ldots] + \frac{1}{c^4}[\ldots]$$

Terms known so far

		N	1PN	2PN	3PN	4PN	5PN	...
0PM	1	v^2	v^4	v^6	v^8	v^{10}	v^{12}	...
1PM		$1/r$	v^2/r	v^4/r	v^6/r	v^8/r	v^{10}/r	...
2PM			$1/r^2$	v^2/r^2	v^4/r^2	v^6/r^2	v^8/r^2	...
3PM				$1/r^3$	v^2/r^3	v^4/r^3	v^6/r^3	...
4PM					$1/r^4$	v^2/r^4	v^4/r^4	...
5PM						$1/r^5$	v^2/r^5	...
...							$1/r^6$	...

Conclusion

- Field theory methods to solve the 2-body problem in GR have proven very effective, they are under intense use as efficient tools for computations from first-principle
- There is a problem at the moment in including the memory process into the dynamics (5PN, 4PM, 2SF)
- For future developments going to higher order will lead to new master integrals, stumbling block for any perturbative method (PN, PM, scattering angles. . .)