Title: Epstein-Glaser renormalisation in constructing factorisation algebras in QFT

Speakers: Katarzyna Rejzner

Series: Mathematical Physics

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Abstract: In this talk I will review the recent progress in constructing factorisation algebras that represent QFT models on Lorentzian manifolds. I will also discuss their relation to local nets of algebras in AQFT. Renormalization in these models is done using the Epstein-Glaser approach. This is joint work with O. Gwilliam, E Hawkins and B. Visser.

Zoom Link: https://pitp.zoom.us/j/93242972298?pwd=Z0taa3hBSzJ5VWtpQWlaTVRkbTBaUT09

Epstein-Glaser renorm. in constructing fact. alg. in QFT

Def: QFT is a BD algebra $\mathcal{A}$ in fact. alg.

$FA(M, Ch(\underline{\underline{Nuc}}_\hbar))$, i.e.

$\forall\, U \subset M$ assign the chain complex $\mathcal{A}(U)$ with a differential s, and with:

•) a $\hbar$-lin. commutative product $\cdot_T$

·) $\hbar$-linear 1-shifted Poisson bracket $\{\cdot,\cdot\}$

s.t.: $$s(X \cdot_T Y) = sX \cdot_T Y + (-1)^{|X|} X \cdot_T sY + i\hbar \{X, Y\}$$

(·) BV differential Δ ("comes from" s and $\cdot_T$)

Rem: Here $\cdot_T$ is the time-ordered product (encodes the combinatorics of path integral)

GR 2020 (CMP)

→ no renormalisation

→ Input from physics:

M – Lorentzian spacetime, globally hyperbolic
($M \cong \Sigma \times \mathbb{R}$, w/ Σ Cauchy surface)

$\mathcal{E}$ – configuration space, $\mathcal{E} = \Gamma(E \to M)$
↳ vector bundle over M

classical observables → smooth functionals on $\mathcal{E}$.

$\mathcal{F}_{reg}$ - regular functionals: functionals F s.t.:

$$F^{(n)}(\varphi) \in \Gamma_c((E^*)^{\boxtimes n} \to M^n) \subset \Gamma'(E^{\boxtimes n} \to M^n)$$

$\varphi \in \mathcal{E}$ distributions

$\mathcal{F}_{loc}$ - local functionals: $F(\varphi) = \int_M f(j^k(\varphi))\, d\mu$

$\mathcal{F}_{ml}$ - multilocal functionals: sums of pointwise products in $\mathcal{F}_{loc}$

$$(F \cdot G)(\varphi) = F(\varphi) \cdot G(\varphi)$$

Take multivector fields: $\mathcal{F}_{reg} = \mathcal{O}_{reg}(T^*[-1]\mathcal{E})$

The bracket is the Schouten bracket

·) $F \in \mathcal{F}_{reg}$, X vector field $\{X, F\} = \partial_X F$

·) X, Y, $\{X, Y\} = [X, Y]$

Free scalar field: dynamics given 1-form $dS \in \Gamma(T^*\mathcal{E})$

$$dS(\varphi) = \underbrace{(\Box + m^2)}_{P}\varphi$$

differential

$\underline{\underline{s}}\, X = \iota_{dS} X$

normally hyp.-op.

Products.

For P, $\exists!$ retarded / advanced Green functions $G^{R/A}$

$$G^C = G^R - G^A \Rightarrow G^C(x,y) = -G^C(y,x)$$

causal propagator / commutator function

$$G^D = \frac{1}{2}(G^R + G^A) \qquad G^D(x,y) = G^D(y,x)$$

Dirac propagator

$$(X \cdot_T Y)(\varphi) \doteq m \circ e^{i\hbar \langle G^D, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \rangle} X \otimes Y$$

$$(X \star Y)(\varphi) \doteq m \circ e^{\frac{i\hbar}{2} \langle G^C, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \rangle} X \otimes Y$$

Properties of $\cdot_T$:

1) $X \cdot_T Y = \mathcal{T}(\mathcal{T}^{-1} X \cdot \mathcal{T}^{-1} Y)$, $\quad \mathcal{T} = e^{\frac{i\hbar}{2} \langle G^D, \frac{\delta^2}{\delta\varphi^2} \rangle}$

time-ordering operator

2) $\mathcal{T}^{-1} \circ s \circ \mathcal{T} = s - i\hbar \triangle$

G^D is a Green function for P

normally hyp.-op. $\longrightarrow$ P | $=$

$$(X \cdot_{\mathcal{T}} Y)(\varphi) \doteq m \circ e^{i\hbar \langle G^D, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \rangle} X \otimes Y$$

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write elements of $\mathfrak{O}_{reg}(T^*[-1]\mathcal{E})$ as $(\varphi, \varphi^\ddagger)$

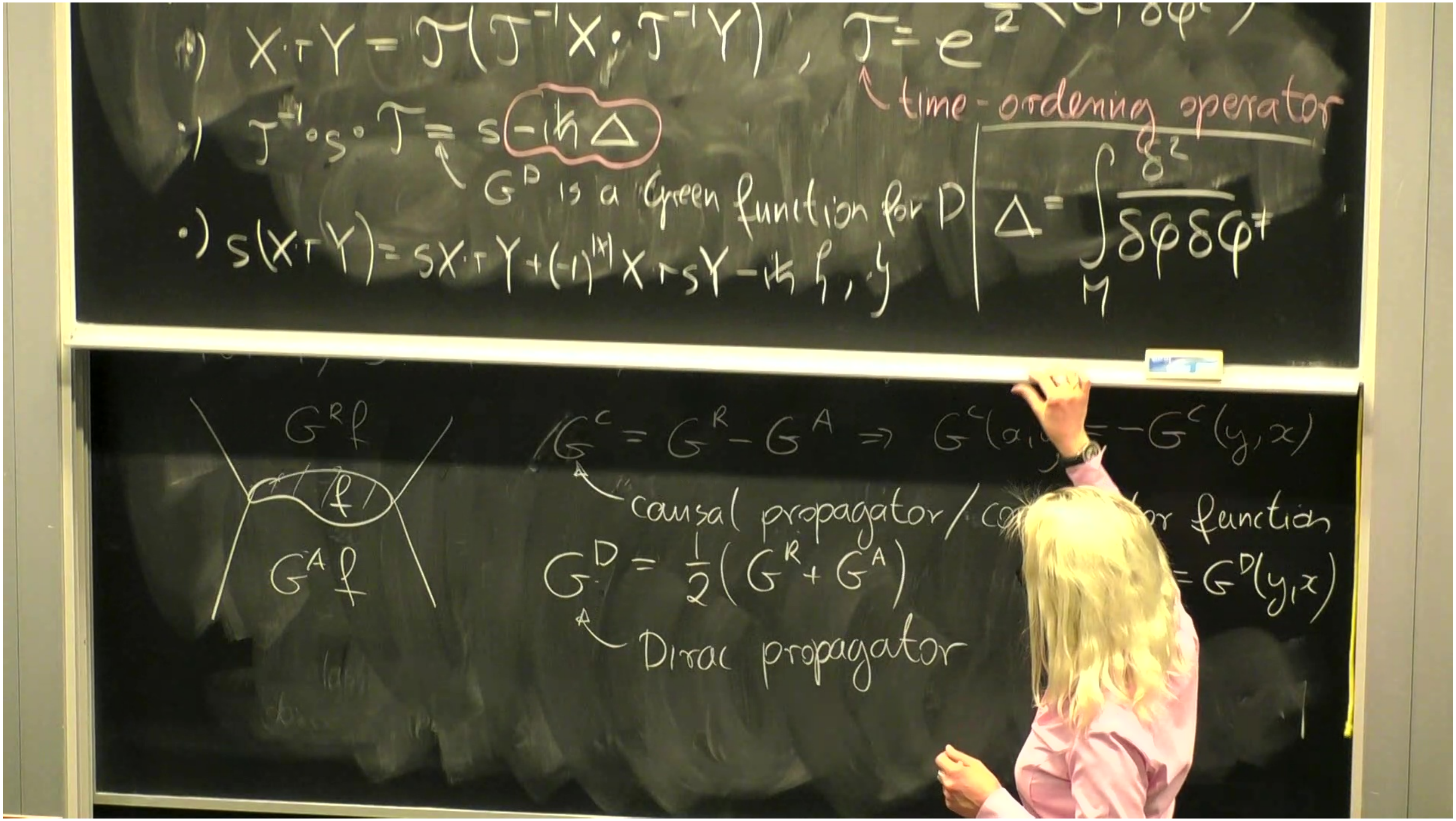
Properties of $\cdot_{\mathcal{T}}$:

- $X \cdot_{\mathcal{T}} Y = \mathcal{T}(\mathcal{T}^{-1} X \cdot \mathcal{T}^{-1} Y)$, $\mathcal{T} = e^{\frac{i\hbar}{2} \langle G^D, \frac{\delta^2}{\delta\varphi^2} \rangle}$

time-ordering operator

- $\mathcal{T}^{-1} \circ s \circ \mathcal{T} = s - i\hbar \Delta$

G^D is a Green function for P | Δ


$X \cdot_T Y = T(T^{-1}X \cdot T^{-1}Y)$, $T = e$
time-ordering operator
$T^{-1} \circ s \circ T = s - i\hbar\Delta$
G^D is a Green function for D
$s(X \cdot_T Y) = sX \cdot_T Y + (-1)^{|X|} X \cdot_T sY - i\hbar \{ , \}$
$\Delta = \int_M \frac{\delta^2}{\delta\varphi\,\delta\varphi^\ddagger}$
$G^R f$
f
$G^A f$
$G^C = G^R - G^A \Rightarrow G^C(x,y) = -G^C(y,x)$
causal propagator /
function
$G^D = \frac{1}{2}(G^R + G^A)$
$= G^D(y,x)$
Dirac propagator

$$(X \cdot_T Y)(\varphi) \doteq m \circ e^{i\hbar \langle G^D, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \rangle} X \otimes Y$$

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write elements of $\mathcal{O}_{reg}(T^*[-1]\mathcal{E})$ as $(\varphi, \varphi^\ddagger)$

Properties of $\cdot_T$:

∘) $X \cdot_T Y = \mathcal{T}(\mathcal{T}^{-1} X \cdot \mathcal{T}^{-1} Y)$, $\mathcal{T} = e^{\frac{i\hbar}{2} \langle G^D, \frac{\delta^2}{\delta\varphi^2} \rangle}$

time-ordering operator

∘) $\mathcal{T}^{-1} \circ s \circ \mathcal{T} = s - i\hbar \Delta$

G^D is a Green function for P

∘) $s(X \cdot_T Y) = sX \cdot_T Y + (-1)^{|X|} X \cdot_T sY - i\hbar \{\cdot, \cdot\}$

$$\Delta = \int_M \frac{\delta^2}{\delta\varphi \, \delta\varphi^\ddagger}$$

<u>Alternatively</u>: take $\{\cdot,\cdot\}$, $\hat{S} = S - i\hbar\Delta$, $\cdot$

(also gives a BD alg)

<u>Rem.</u> $X \cdot_T Y = \begin{cases} X \star Y, & \text{if } \operatorname{supp} X \quad \operatorname{supp} Y \qquad X \gtrsim Y \\ Y \star X, & \text{if } Y \gtrsim X \end{cases}$

We can reconstruct $\star$ from $\cdot_T$ if $\mathfrak{A}$ is locally constant in time direction (time-slice axiom)

I want to do everything on $\mathcal{F}_{loc}$ and $\mathcal{F}_{mc}$

$G^D, \underline{G^S}$ too singular for τ and $\star$ to be well-def.

Fix $\star$ by defining $\underline{G^+ = \frac{i}{2}\boxed{G} + \underbrace{H}_{\text{symmetric}}}$

s.t. ·) WF G^+ is "better" (so that $\star$) is well-defined
·) H bisolution for P
·) G^+ is of positive type

$\Bigg\}$ G^+ is a 2-point function of a Hadamard state

I want to do everything on $\mathcal{F}_{loc}$ and $\mathcal{F}_{mc}$

G^D, G^c too singular for $\cdot_T$ and $\star$ to be well-def.

1) Fix $\star$ by defining $G^+ = \frac{i}{2} \boxed{G^c} + \underbrace{H}_{\text{symmetric}}$

s.t. • WF G^+ is "better" (so that $\star$) is well-defined

• H bisolution for P

• G^+ is of positive type

G^+ is a 2-point function of a Hadamard state

2) Define Feynman propagator: $G^F = iG^D + H$

→ use this for $\cdot_T$

We can reconstruct $\star$ from $\cdot_T$ ...

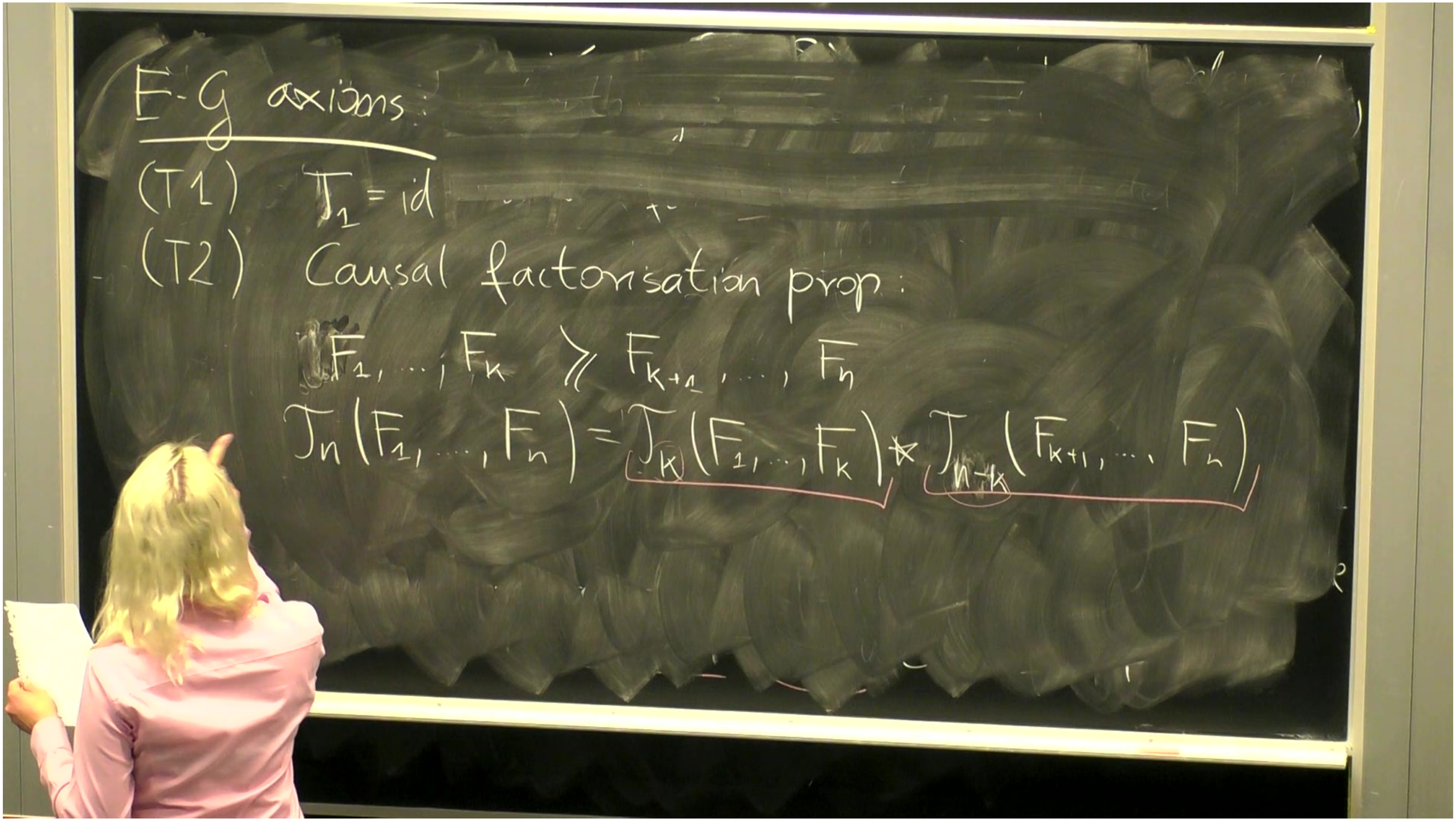
in time direction (time-slice axiom)

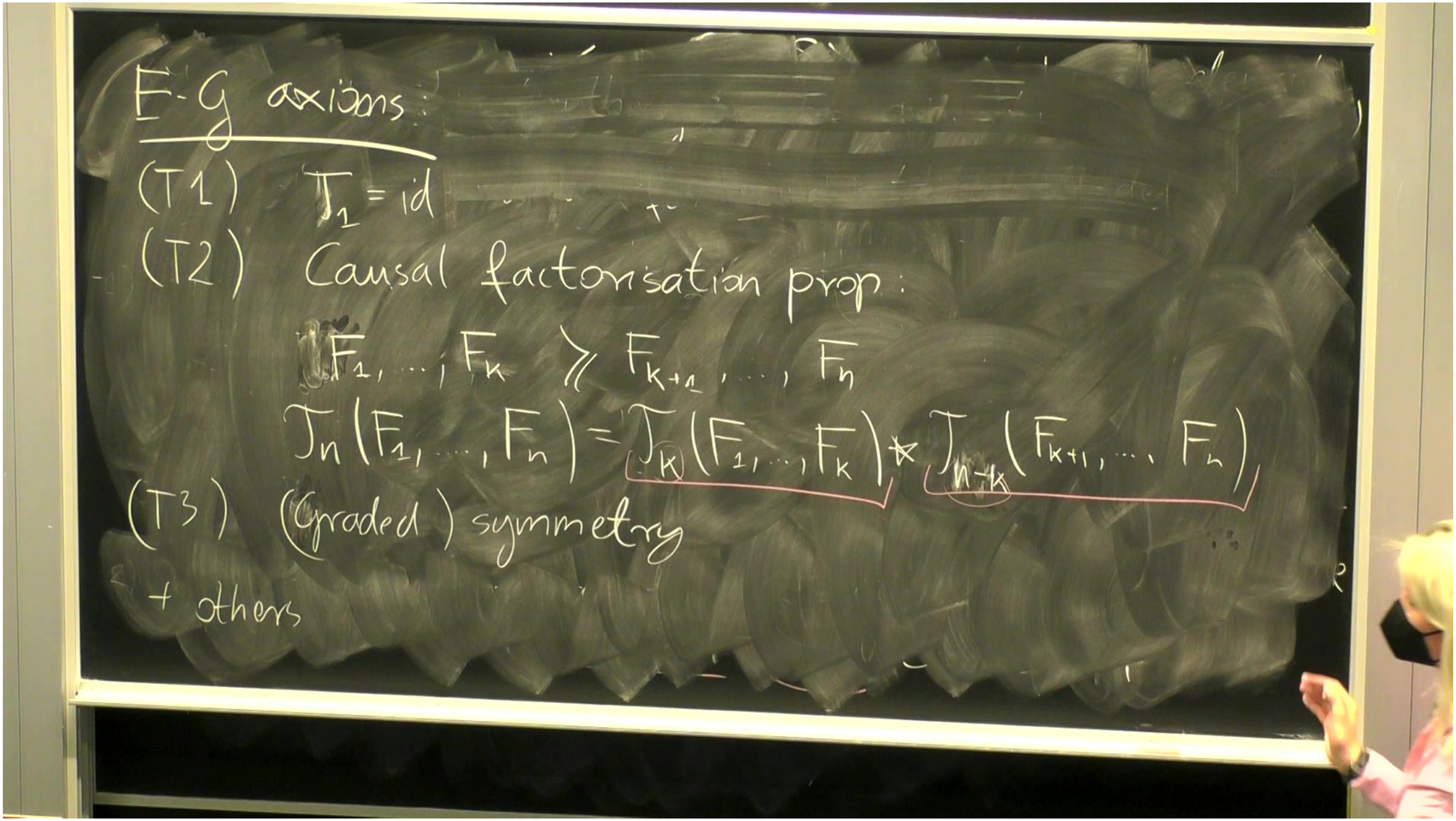
$F_1, \ldots, F_n \in \mathcal{F}_{loc}$ with pairwise disjoint supports:

$F_1 \cdot_T \ldots \cdot_T F_n \equiv \mathcal{T}_n(F_1 \otimes \ldots \otimes F_n)$ is well-def.

Renorm. problem: Extend $\mathcal{T}_n$, $n \in \mathbb{N}$ to arguments with coinciding supports.

→ Epstein-Glaser


E-G axioms:
(T1) $T_1 = id$
(T2) Causal factorisation prop:
$F_1, \dots, F_k \gtrsim F_{k+1}, \dots, F_n$
$T_n(F_1, \dots, F_n) = T_k(F_1, \dots, F_k) \star T_{n-k}(F_{k+1}, \dots, F_n)$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(T3) (Graded) symmetry
+ others

$\beta : \mathcal{F}_{mc} \to \bigoplus_{n=1}^{\infty} \mathcal{F}_{reg,0}^{\otimes n}$

from E-G (not unique!)

Define renormalised $T \doteq \bigoplus_{n=1}^{\infty} T_n \circ \beta$

$X \cdot_T Y \doteq T(T^{-1}X \cdot T^{-1}Y)$

renormalised BV Laplacian

Note: $\hat{s} \doteq T^{-1} \circ s \circ T X = s - i\hbar \tilde{\triangle} X$

this depends on renorm. freedom.

$S + V$

$$R_V(F) \doteq \left(e_T^{\frac{i}{\hbar}V}\right)^{\star -1} \star \left(e_T^{\frac{i}{\hbar}V} \cdot_T F\right)$$

interacting obs. for int. V

$$\hat{s}_V = R_V^{-1} \circ s \circ R_V = s_V - i\hbar \tilde{\Delta}_V$$