Title: The Noether Theorems: Then and Now

Speakers: Karen Uhlenbeck

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Abstract: The 1918 Noether theorems were a product of the general search for energy and momentum conservation in Einstein's newly formulated theory of general relativity. Although widely referred to as the connection between symmetry and conservation laws, the theorems themselves are often not understood properly and hence have not been as widely used as they might be. In the first part of the talk, I outline a brief history of the theorems, explain a bit of the language, translate the first theorem into coordinate invariant language and give a few examples. I will mention briefly their historical importance in physics and integrable systems. In the second part of the talk, I describe why they are still relevant: why George Daskalopoulos and I came to be interested in them for our investigation into the best Lipschitz maps of surfaces of Bill Thurston and the open problems in higher dimensions. I will finish by mentioning two recent papers, one in math and the other in physics, which greatly simplify the derivations of important identities by using the theorems.

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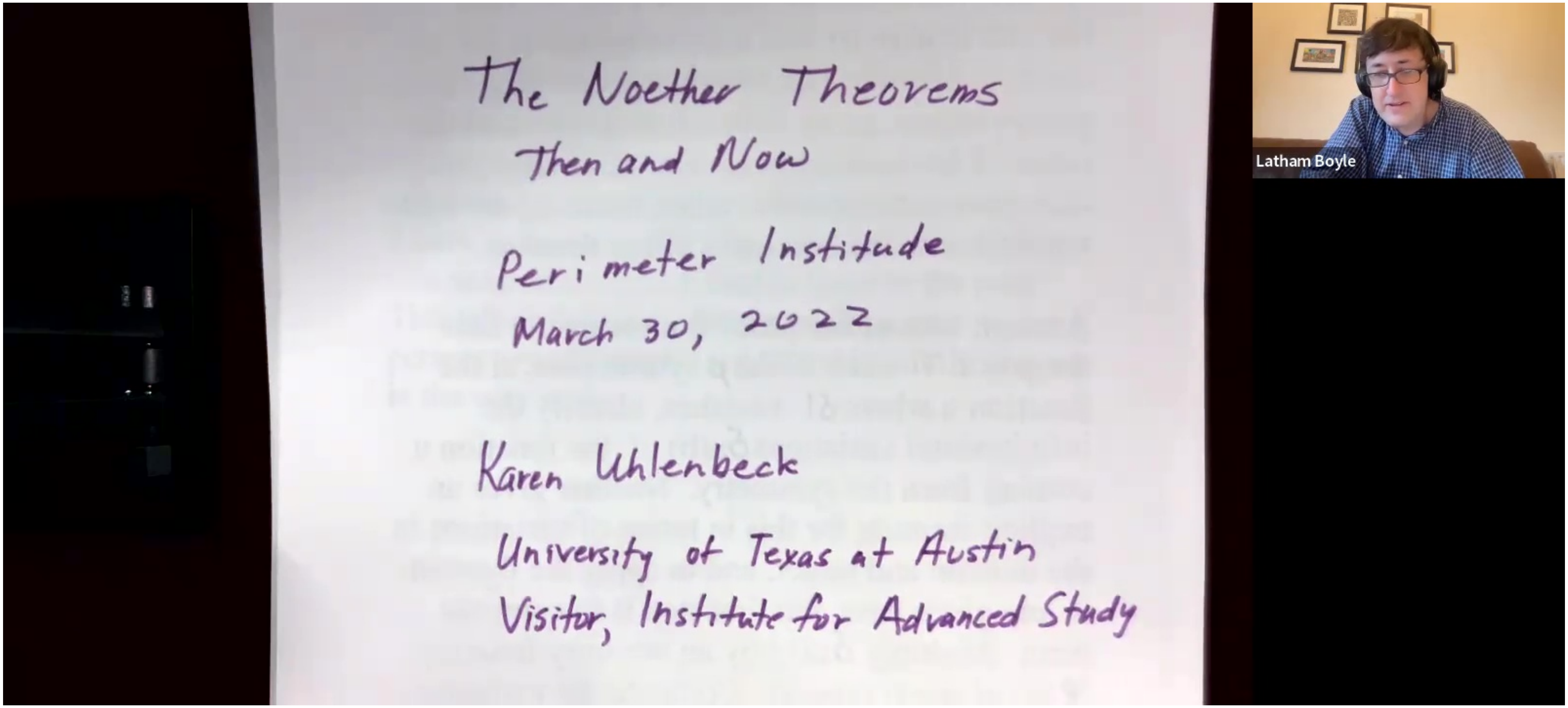
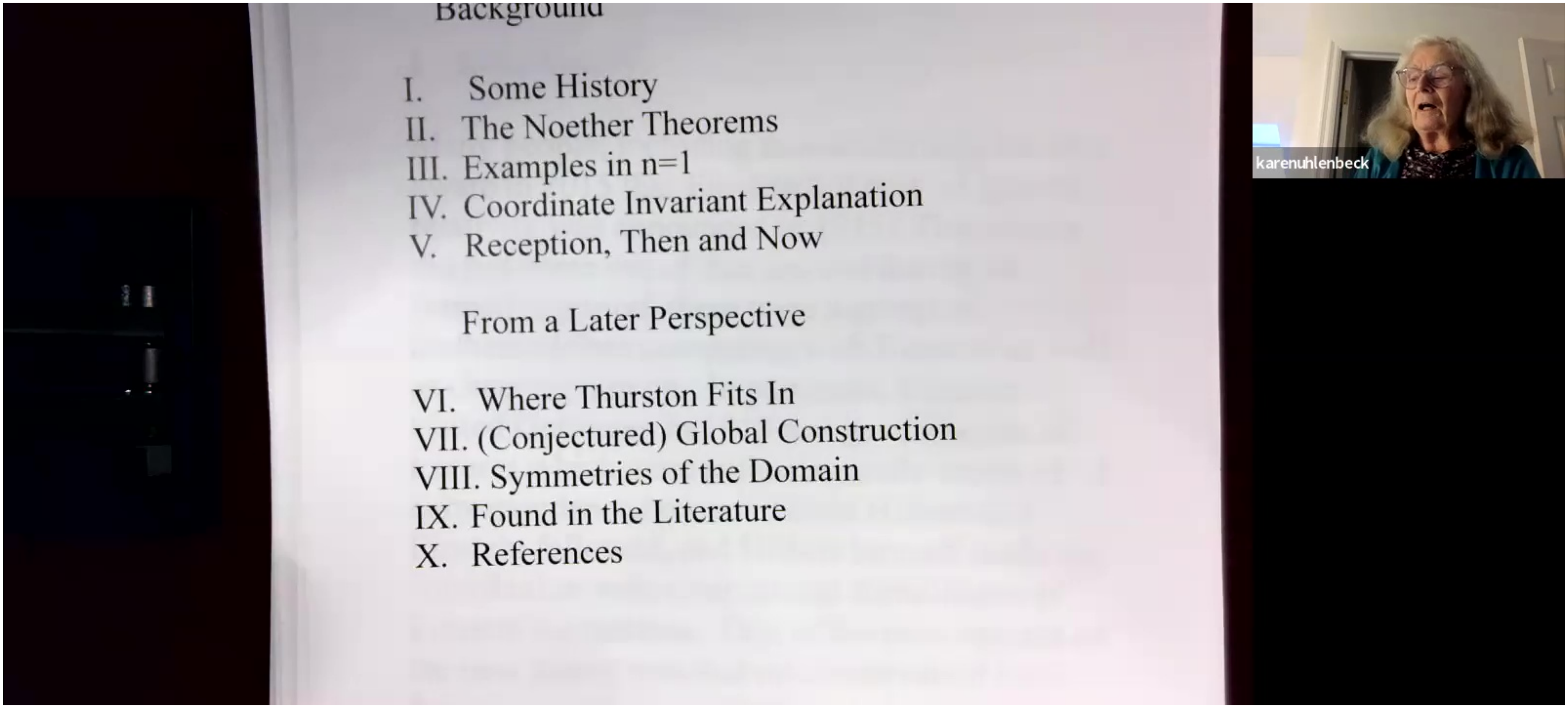

The Noether Theorems
Then and Now
Perimeter Institude
March 30, 2022
Karen Uhlenbeck
University of Texas at Austin
Visitor, Institute for Advanced Study
Latham Boyle

I am pleased at the opportunity to talk on Noether's theorems. Two years ago, I had little interest in the topic, however, in the last two years, my research has lead in a direction which altered my attitude. I need them in my present work, and I came to be sorry that I had not understood their role in several topics I tried to learn in the past. This talk is definitely from a mathematics perspective, although I wil throw in a little physics in the very end.



Background

I. Some History
II. The Noether Theorems
III. Examples in n=1
IV. Coordinate Invariant Explanation
V. Reception, Then and Now

From a Later Perspective

VI. Where Thurston Fits In
VII. (Conjectured) Global Construction
VIII. Symmetries of the Domain
IX. Found in the Literature
X. References

Many people, including non-academics, became aware in 2015 that Einstein's theory of general relativity was announced in 1915. This theory did not come out of thin air, and during its formative period, there were a group of mathematicians competing with Einstein as well as cheering him on. In particular, Einstein visited Gottingen in 1915 and gave a series of lectures which were enthusiastically received. A correspondence between David Hilbert and Einstein followed, and Hilbert himself made the contribution with a variational formulation of Einstein's equations. One of the puzzlements of the new theory was that no conservation laws for energy and momentum could be found for the new equations. These were readily discovered in special relativity, which had made

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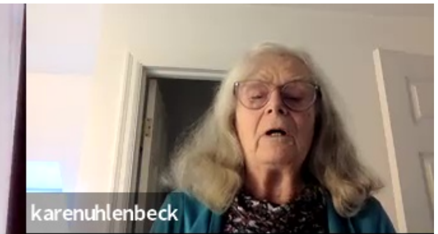
In 1915 Emmy Noether had been invited by Hilbert and Klein to come to Gottingham, and had probably been in the audience for Einstein's lectures. "Miss Noether", as she was referred to at the time, was the daughter of the mathematician Max Noether and sister to another mathematician Fritz Noether. At the time, higher eduction in Germany was not open to women, but she had managed to write her doctoral thesis in 1907 in Erlangen under the supervision of the algebraist Paul Gordon. She had published a number of papers on invariant theory, and it appears that Hilbert simply assigned her the problem of looking into the problem of the lack of conservation laws for energy in general relativity. From surviving

time, higher eduction in Germany was not open to women, but she had managed to write her doctoral thesis in 1907 in Erlangen under the supervision of the algebraist Paul Gordon. She had published a number of papers on invariant theory, and it appears that Hilbert simply assigned her the problem of looking into the problem of the lack of conservation laws for energy in general relativity. From surviving correspondence of the time, it appears that there were preliminary results as early as 1916, but it was not until 1918 that the breakthrough insight which explained the problem came. Her paper "Invariant Variational Problems" was presented in July of 1918 with a dedication to Felix Klein on the 50th anniversary of his doctorate.



familar to applied mathematicians and physicists. My apologies.

"Part I. If the integral I is invariant under a (group) G_ρ, then there are ρ linearly independent combinations among the Lagrangian expressions which become divergences - and conversely, . .. "

"Part II. If the integral I is invariant under a (group) $G_{\rho\infty}$ depending on ρ arbitrary functions and their derivatives up to order k, then there are ρ identities among the Lagrangian expressions and their derivatives up to order k, and conversely.

Some Explanation

The integrals in Noether's theorem are of the very general form

$$I = \iint_{\Omega} F(x, f(x), df(x), \ldots, D_{\alpha} f(x))\, dx^n$$

Here $f = \vec{y} = (y^1, \quad y^m)$ is vector valued and $y^{\ell}_{\alpha} = D_{\alpha} f^{\ell}$ is a multi index derivative indicating the partials up through some fixed order k, as in all the general PDE books I studied as a graduate student. We would now call Fdx a volume valued function on the jet bundle. The function F is classically called the Lagrangian. However, Noether discusses symmetries of the Lagrangian density L = Fdx Some of the first variational

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derived in this fashion are called the Euler-Lagrange equations. The general form of the Euler-Lagrange equations for the most general example of Noether's is the system of equations

$$\sum_{\beta} (-1)^{|\beta|} D_\beta F_{y^\ell_\beta}(\ ,u, du, D_\alpha u) = 0. \qquad \ell=(1,\dots,m)$$

Hilbert's integrand for general relativity is

$$L(g) = R(g)\, d\mu(g).$$

Here g is a metric on space-time, R(g) is the scalar curvature an $d\mu(g)$ the induced volume form. The Euler-Lagrange equations are

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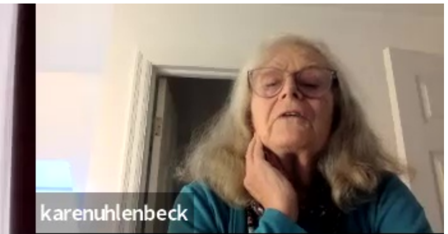
For general relativity, the partials up through order 2 are required, but for all but for most the examples in this talk, it is sufficient to think of L as a function on the derivatives of vector valued functions f. By invariant, she does NOT mean that the problem at hand is invariant, only that the Lagrangian density L = Fdx is. (This confused me when I was a student). In fact, L need only be invariant up to a "divergence" or an exact n form. Hence the theorem is local, not global in nature, and applies as it was originally stated, to locally Euclidean spaces, or manifolds, irrespective of topology or boundary conditions. The modifications needed when the image of f is in a manifold are easily encompassed in the theory, whether by using



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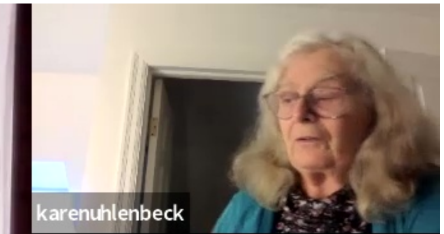
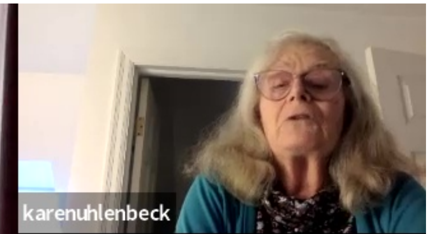


Noether's theorem really uses "infinitesimal" symmetries, or the action of the Lie algebra. For practical purposes, a group action is far easier to identify. Her symmetries involve groups which act on both the domain x and the range of f as needed in general relativity. Our examples use one or the other, meaning we will not deal with the subtleties of her theorem.

A rough idea of her proof is more useful than the proof. To each of the ρ symmetries, at the function u where δI vanishes, identify the infinitesimal variations $\delta u(b)$ of the function u coming from the symmetry. Noether gives an explicit formula for this in terms of variations in



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$$\delta u = \sum_b \varphi(b)\delta u(b).$$

In general for any integral which depends only on first derivatives this should look like

$$\delta I = \iint \sum_b [(V^j(b)\, d_j\varphi(b)) + W(b)\varphi(b)]dx^n$$

However, since $\delta u(b)$ is an infinitesimal symmetry of $L = Fdx^n$, $W(b) = 0$. But if $\delta I = 0$, this means

$$0 = \iint_\Omega \sum_b [V(b)\cdot d\varphi(b)]$$

for all $\varphi(b)$ with compact support. Integrate by parts, and the "fundamental lemma of the calculus of variations" tells you that for each b

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From this it is not hard to guess how to handle more derivatives, and see how part II works, In part II, one knows δI vanishes (up to a divergence) for all f. If $L = F dx^n$ is invariant under δf for all f . Then we do not need the Euler-Lagrange equations to know

$$\operatorname{div} V(b) = 0.$$

It hence is true for all f. Noether called these conservation laws “improper”. For physicists, they are true “off shell” as well as for solutions.

III. Some examples in n = 1

.Conservation of energy in one dimension is the easiest example of part I (we use x = t). Given our simplification, every integral is of the form

$$I = \int_a^b F(t, f(t), \dot{f}(t))\,dt$$

The Euler-Lagrange equations are just the equations, as many as the dimension of the target

$$d/dt\, F_{\dot{y}^k} - F_{y^k} = 0.$$

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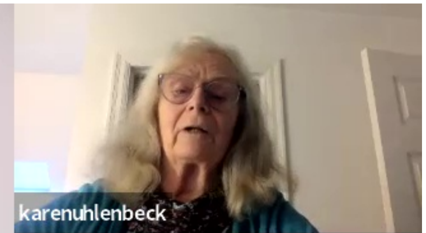
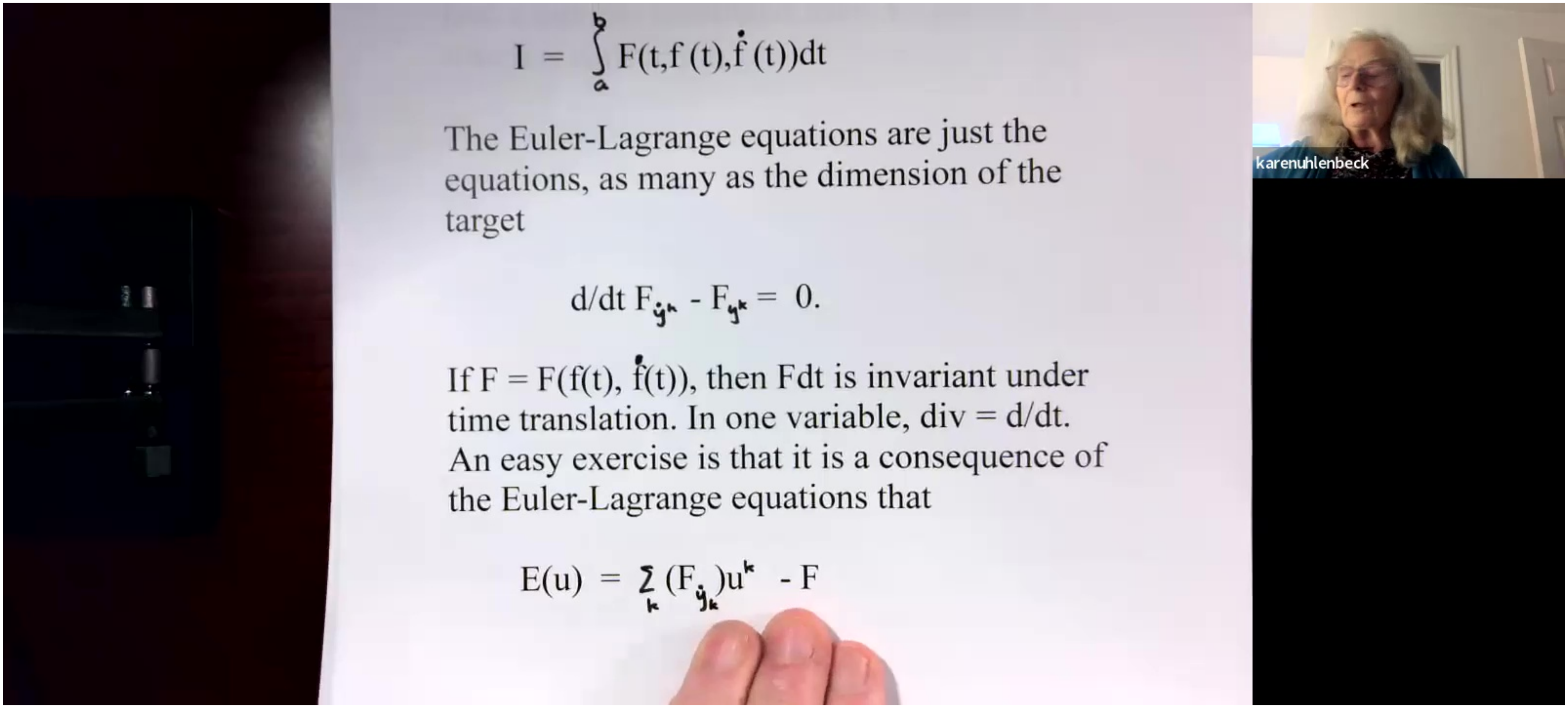
If $F = F(f(t), \dot{f}(t))$, then Fdt is invariant under time translation. In one variable, div = d/dt. An easy exercise is that it is a consequence of the Euler-Lagrange equations that

$$E(u) = \sum_k (F_{\dot{y}_k})u^k - F$$

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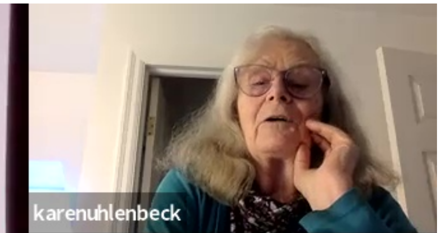
karenuhlenbeck

While we are in n = 1, we might look at an example of part II of the theorem. For a curve u in a manifold, we may choose to minimize length instead of energy. Then we have

$$I = \int_a^b |\dot{f}(t)|_{f(t)}\, dt.$$

It does not matter than the norm depends on f, only that it not depend on t. Using the recipe above, without restricting to u where I vanishes we come out with the mundane identity

$$E = (\dot{f}/|\dot{f}|\,, \dot{f}) - |\dot{f}| = 0.$$



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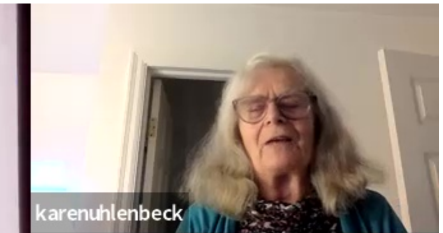
The second theorem explains that this is because the translations are embedded in the group of diffeomorphisms of t, a group which depends on an arbitrary function on the line, which leads to an identity true for all curves, not just those that solve the Euler Lagrange equations. In this



Mills functional.

Noether's theorem is quoted quite a bit in Hamiltonian mechanics. This comes about from the classical Legendre transformation between the Lagrangian and Hamiltonian formulation of mechanics.

But why was this known to Noether's contemporaries as a conservation law? Simply the divergence theorem. In our language, If S_1 and S_2 are two codimension 1 submanifolds bounding a domain Ω

$$\int_{S_1} \theta - \int_{S_2} \theta = \iint_{\Omega} d\theta = 0$$

Notice that to this day it is usual to take S_i as constant time slices in an evolution problem and use the conserved quantity that the induced surface intgral is constant on the time slices.

Hilbert functic

Bianchi identit

IV. Coordinate Invariant Formulation

Suppose the domain variable is x in n dimensions rather than 1 (We will work our way down to 2 dimensions at the end). In part I, Noether asserts that the finite dimensional symmetries lead to vector fields V with div V = 0. This looks impossibly coordinate dependent. Not so. After all, the formulas for the symmetries are coordinate dependent, but the symmetries themselves are not, if we incorporate the volume form dx into our thinking. For all functions with compact support, at critical points

$$I = \iint V \, d\varphi \, dx^n = 0.$$

We write $V \lrcorner dx = \theta$ for θ an n-1 form and get

$$0 = \iint \theta \wedge d\varphi$$

Now use the fa[illegible] hat this is the mathematical definition that d[illegible] 0 as a distribution.

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V. The Reception: Then and Now

The Noether theorems accomplished two things. They showed that many if not all of the known conservation laws in both physics and mathematics arose from symmetries. They also solved the problem assigned to her. Many energy conservation laws, such as those of special relativity, come from translations in time and those of momentum from translations in space. In general relativity these translations are embedded in the group of diffeomorphisms, which leave the Einstein-Hilbert Lagrangian invariant. Hence they give rise to identities, not conservation laws. In 1921 Shouten and Struik published a proof that her identities for the Hilbert functional were equivalent to the

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How were the theorems received? For much of this I rely entirely the reference by Kosmann-Schwarzbach [K-S]. Some attention was paid to them at first, with mention in correspondence among Einstein, Hilbert, Klein and their associates. Her work was referenced in talks and papers in the next few years, but surprisingly little attention was paid to them in the ensuing years until the 1960's. Kosmann-Schwarzbach suggests two reasons.

i). The importance of the variational formulation of problems in physics, and in particular the value of the variational formulation of Einstein's equation, was in

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i). The importance of the variational formulation of problems in physics, and in particular the value of the variational formulation of Einstein's equation, was in doubt.

ii) They did not seem to have applications.

Several revolutions and developments in physics and mathematics changed everything.

i) Group theory in quantum mechanics

This established group theory and symmetry as a fundamental part of the mathematical language physicists use. The group theory was associated with classical field theory, which is often presented in a Lagrangian formulation.

ii) Feynman path integrals

With the introduction of Feynman path integrals in the development of quantum field theory, Lagrangians and their critical points became

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ii) Feynman path integrals

With the introduction of Feynman path integrals in the development of quantum field theory, Lagrangians and their critical points became suddenly central. Groups were already present.

iii) Gauge theory

A little bit later, when gauge theory arose in describing fundamental particles, her discussion

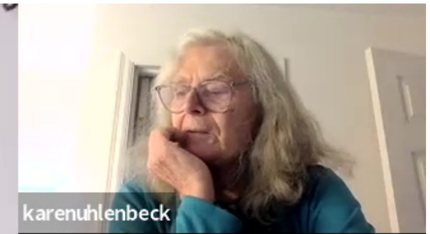
of groups modeled on arbitrary functions was seen as a prophesy out of the past.

iv) Integrable systems

The classical theory of "completely integrable" ordinary differential equations in Hamiltonian mechanics found its way into the development of infinite heirarchies of partial differential equations.

v). Supersymmetry

The importance of these extended symmetries is reflected in the equations that are generated by them.



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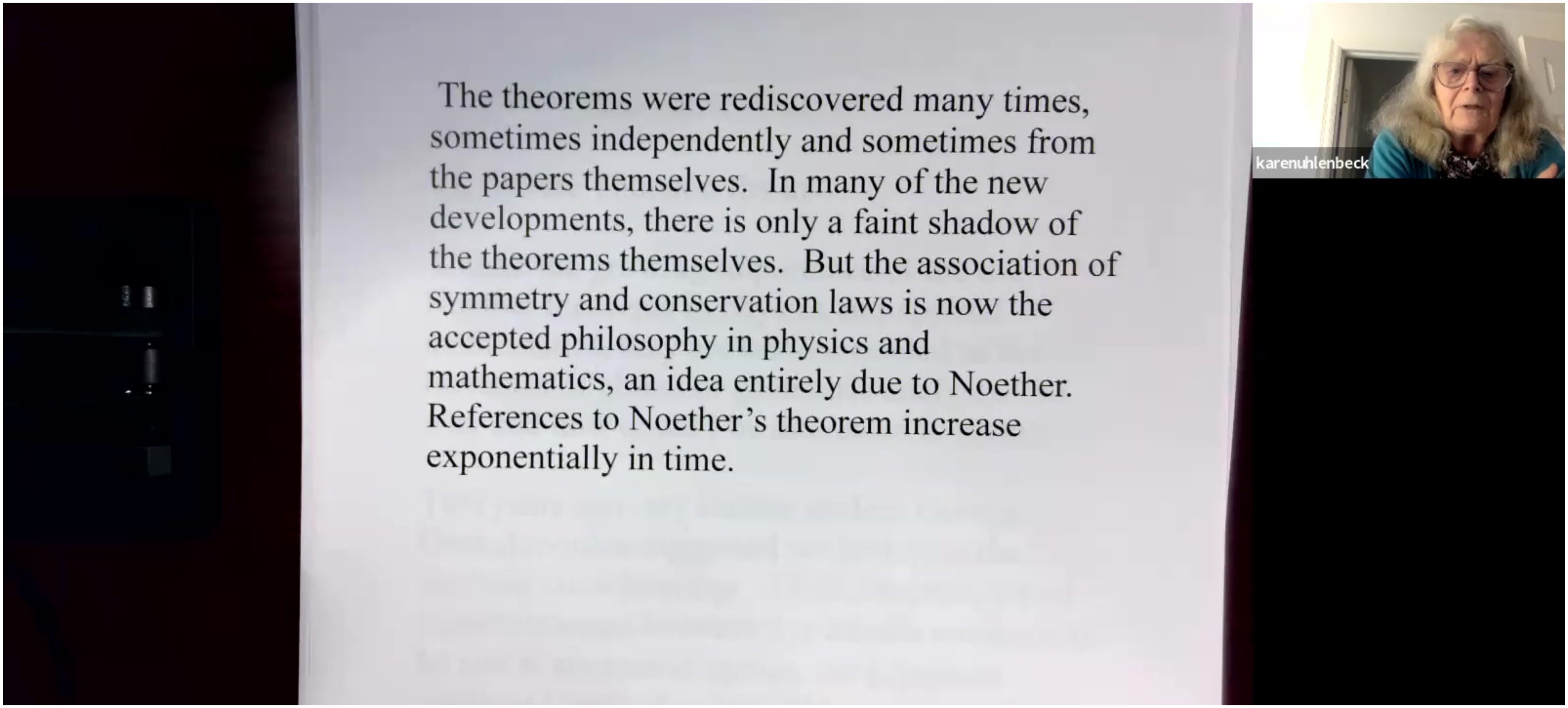
vii) Applied mathematics

I know know about this only because I did a literature search on Math Sci Net. In looking for ways to handle their equations, symmetry seems to be in the top layer of their tool box. See the basic reference by Peter Olver[O].


The theorems were rediscovered many times, sometimes independently and sometimes from the papers themselves. In many of the new developments, there is only a faint shadow of the theorems themselves. But the association of symmetry and conservation laws is now the accepted philosophy in physics and mathematics, an idea entirely due to Noether. References to Noether's theorem increase exponentially in time.

VI. Where Thurston fits in

Unlike the growing importance of the the Noether theorems in physics and applied mathematics, they are seldom quoted in the literature of global or geometric analysis. So why and how could I be interested in them?

Two years ago, my former student George Daskalopoulos suggested we look into the analytic underpinnings of Bill Thurston's best Lipschitz maps between hyperbolic surfaces. If M and N are metric spaces, the Lipschitz constant Lip(f) of a map f:M—>N is defined as

$$\mathrm{Lip}(f) = \max \text{distance}\,(f(x),f(y))/\text{distance}(x,y).$$

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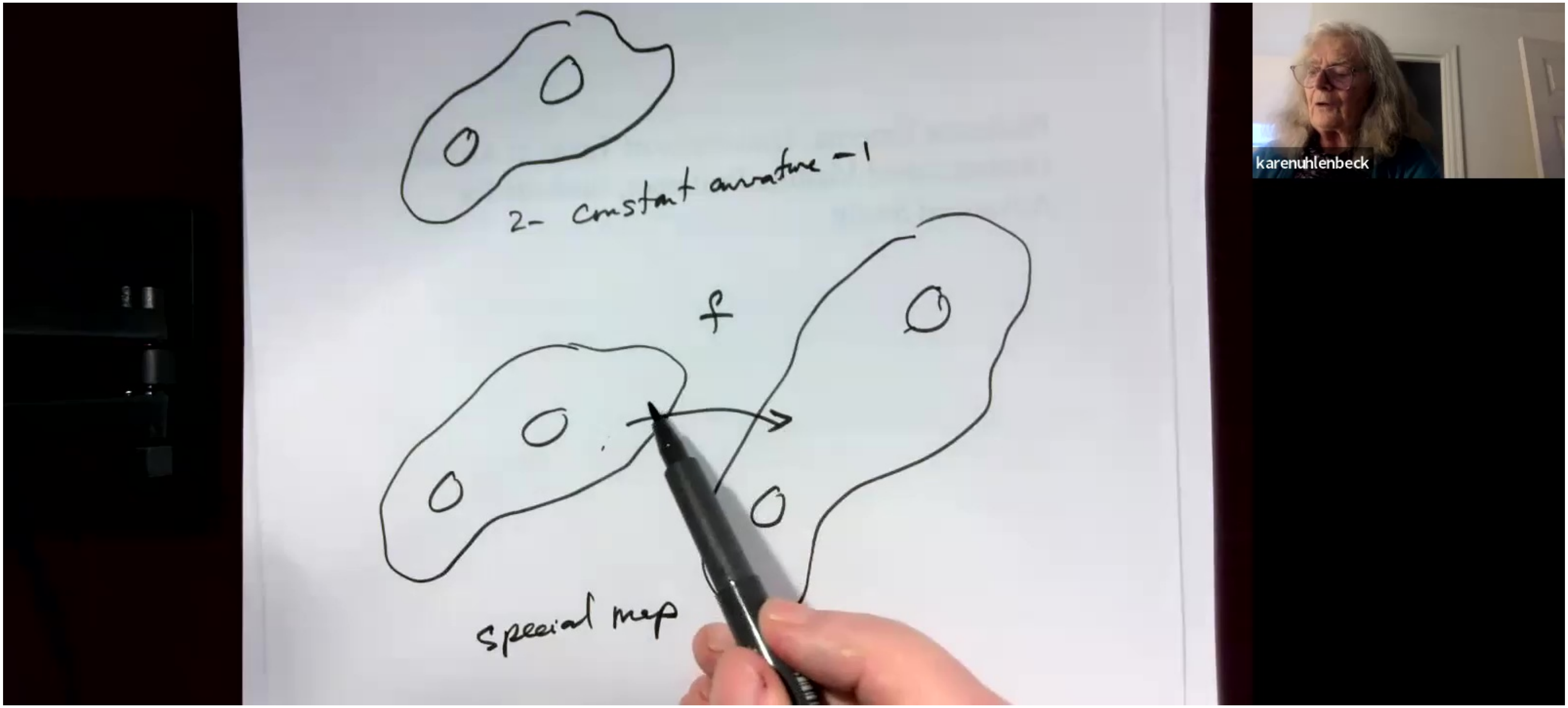
We seek to find u such that Lip(u) <= Lip(f)

for all maps f in some class which contains u. There is a 1995 preprint of Bill Thurston that



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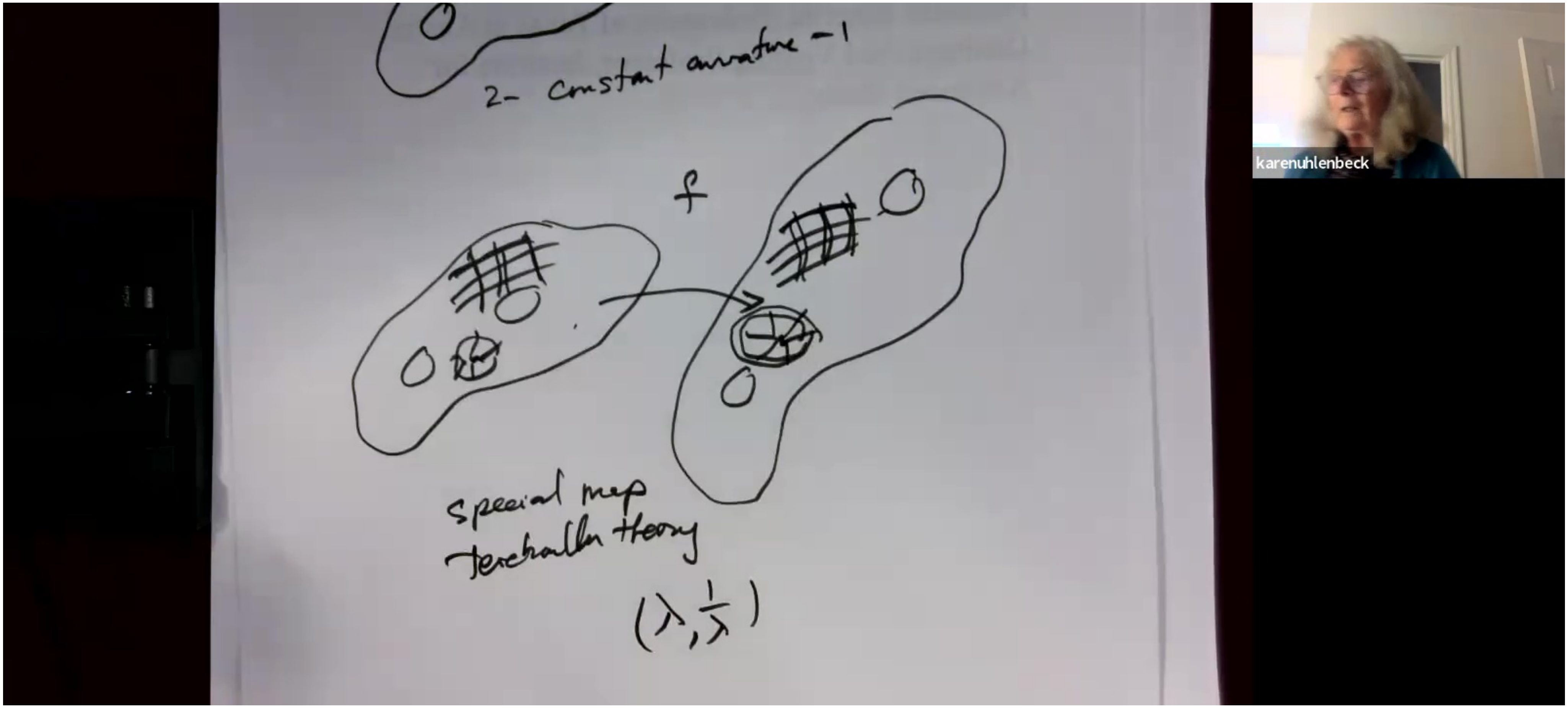

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$(\lambda, \frac{1}{\lambda})$
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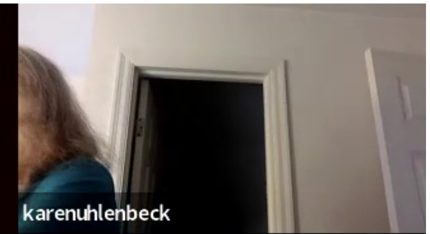

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We seek to find u such that Lip(u) <= Lip(f)

for all maps f in some class which contains u. There is a 1995 preprint of Bill Thurston that proposes to base a Teichmuller theory on the properties of these best Lipschitz maps between hyperbolic surfaces. While the preprint is still in circulation and familiar to topologists, we were aware of no constructions tying this in with modern analysis.

This is a minimization problem, Is it connected with the calculus of variations? Yes, in the sense that the problem can by approximated by integrals and we are able to carry the conserved quantities over into the limit to form transverse measures. There is extensive literature in the case of real valued functions in the form of infinity harmonic functions. As a warm-up case for the Thurston project, we were able to

with the calculus of variations? Yes, in the sense that the problem can by approximated by integrals and we are able to carry the conserved quantities over into the limit to form transverse measures. There is extensive literature in the case of real valued functions in the form of infinity harmonic functions. As a warm-up case for the Thurston project, we were able to transfer this to the case of f:M—> S^1 (or a real line bundle). We study the variational integral

$$J_p(f) = \iint_M |df|^p \, d\mu \quad \text{on } f: M \longrightarrow S^1 .$$

The norm of df is computed in the Riemannian metric and $d\mu$ is the volume form induced by the metric. In the place of boundary conditions, we

class, The unique (up to rotation in the circle) minimum u_p solves the Euler-Lagrange equations

$$D^*|du_p|^{p-2}du_p = 0.$$

The results were very promising, especially after we noticed that D = d on forms and the one-forms

$$*|du_p|^{p-2}\, du_p = \qquad = dv_p$$

were closed with important properties. In particular as p goes to infinity, the limits of v were least gradient functions of bounded variation which define a transverse measure so important in Thurston's work.

When we went to tackle the problem of N a hyperbolic manifold, we found the right integrals whose minima approached best Lipschitz maps, but we did not know how to find anything like the dv and the transverse measure. We then remembered Noether.

In the case f:M—>S^1, the symmetry of the target is the rotations, and without following her recipe., we had found the closed one-form corresponding to this symmetry

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$$dv_p = *|du_p|^{p-2} du_p$$

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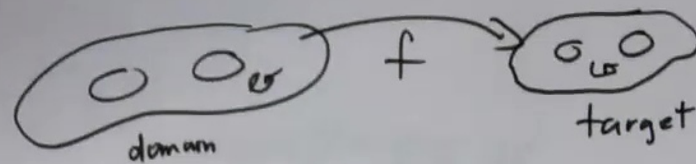
We found that some of the conserved quantities satisfy dual variational problems of their own. We fnd this duality intriquing, and it does not seem to be in the literarure.

VII. Global Construction

The global nature of the general form of Noether's theorem is promising. I state it as a conjecture, with the comment that, in any given situation, it is easier to make the construction from the proof of Noether's theorem than to prove the general conjecture and then figure out how to apply it.

Conjecture 1: Let f:M -> N and L



VIII. The Symmetries of the Domain

The symmetries of the Euclidean group on the domain is the basis for classical concepts of laws of conservation and momentum. Standard textbooks show that for

$$L = F(\vec{f}, d\vec{f})\, dx^{n};$$

associated with every solution of the Euler-Lagrange equations is a symmetric (energy)

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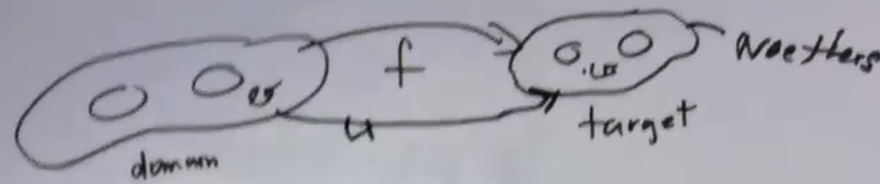

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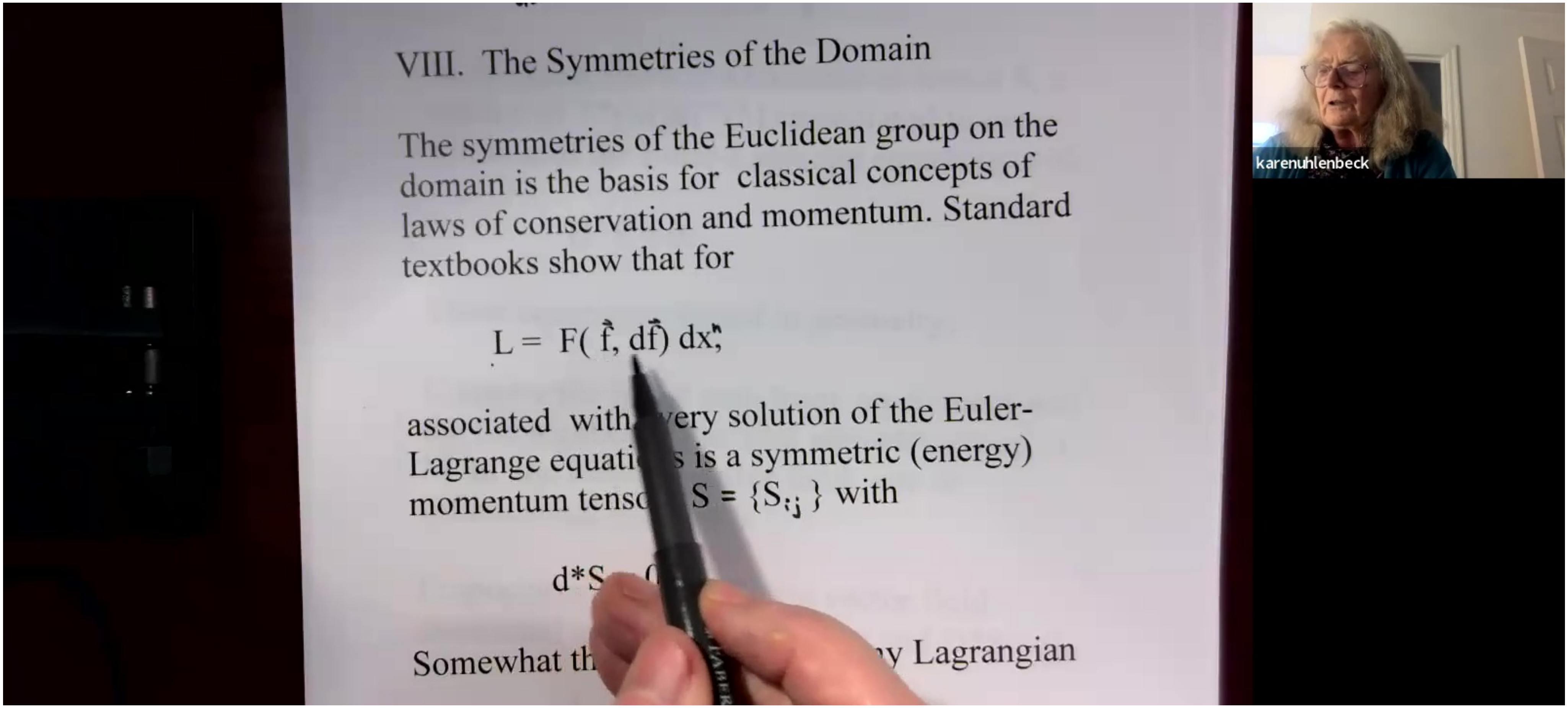
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karenuhlenbeck

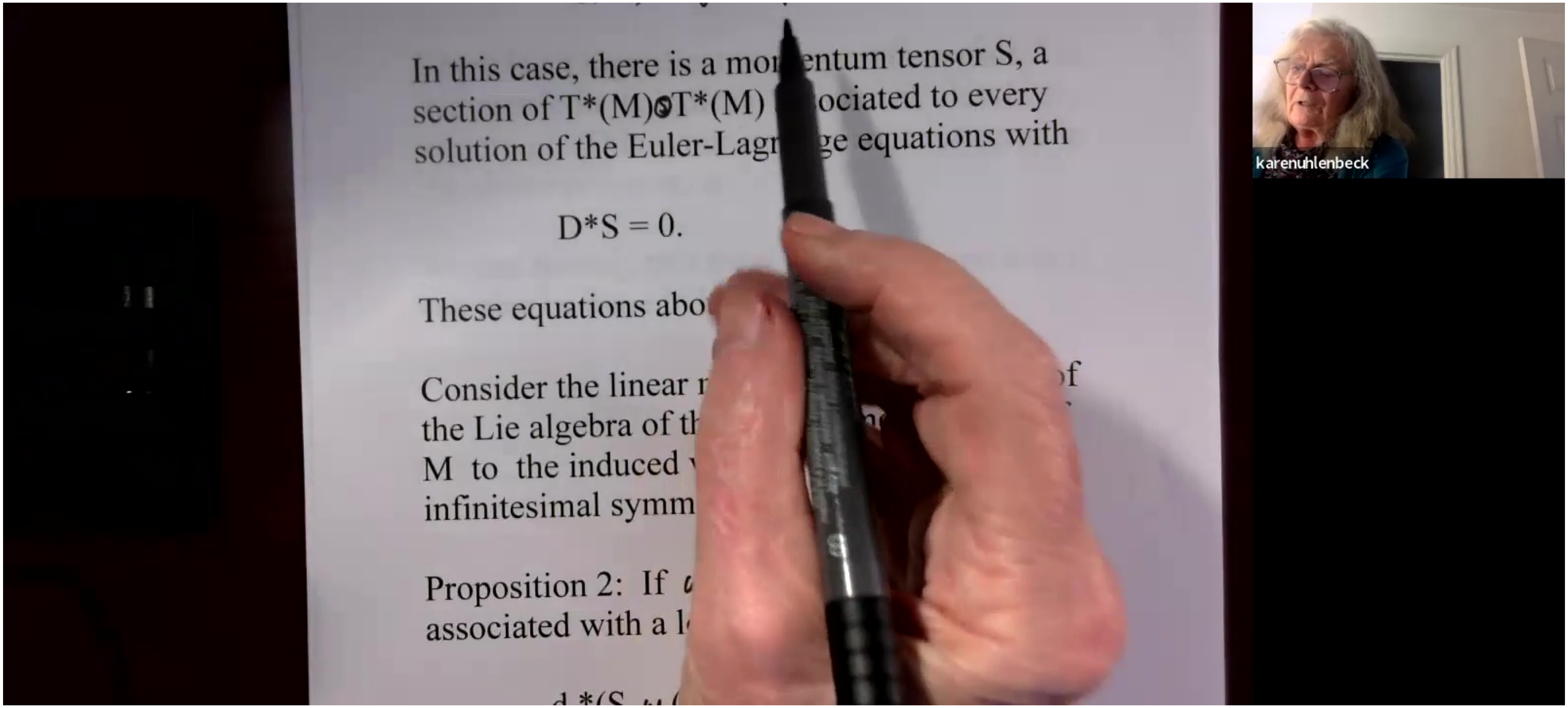
associated with every solution of the Euler-Lagrange equations is a symmetric (energy) momentum tensor $S = \{S_{ij}\}$ with

$$d^*S = 0.$$

Somewhat the same is true for any Lagrangian

$$L = F(f, Q(df))d\mu$$

with domain a Riemannian manifold M. Here the metric on M appears only in the volume form and in the quadratic form


In this case, there is a mo entum tensor S, a
section of T*(M)⊗T*(M) ociated to every
solution of the Euler-Lagr ge equations with
D*S = 0.
These equations abo
Consider the linear f
the Lie algebra of th
M to the induced
infinitesimal symm
Proposition 2: If
associated with a l
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These equations …

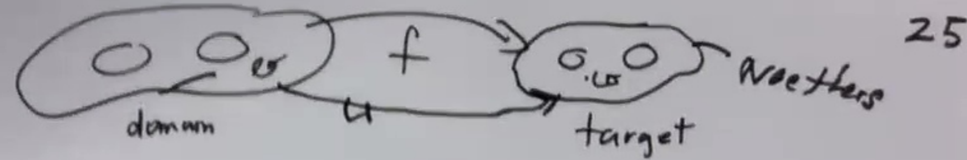
Consider the linear map from an element a of the Lie algebra of the local isometry group of M to the induced vector field $\omega(a)$ or infinitesimal symmetry.

Proposition 2: If $\omega(a)$ is the vector field associated with a local isometry and $D^*S = 0$.

$$d * (S, \omega(a)) = 0.$$

This can be packaged into a global theorem identifying S with a closed n-1 form with values

can also be done in



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$$L = \int F(\vec{f}, d\vec{f})\, dx^n$$

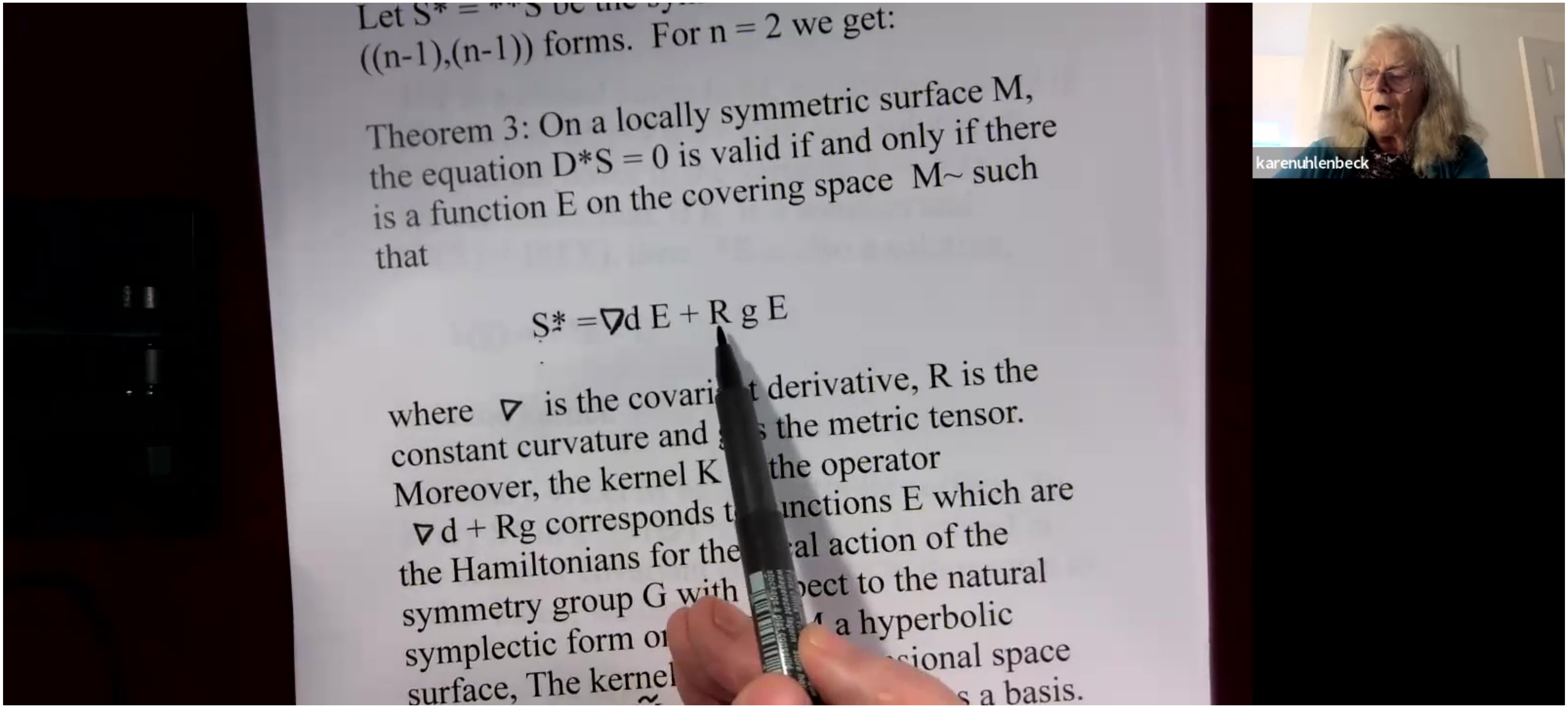
associated with every solution of the Euler-Lagrange equations is a symmetric

Let S* = **S be the symmetric matrix of ((n-1),(n-1)) forms. For n = 2 we get:

Theorem 3: On a locally symmetric surface M, the equation D*S = 0 is valid if and only if there is a function E on the covering space M~ such that

$$S^* = \nabla d\, E + R\, g\, E$$

where ∇ is the covariant derivative, R is the constant curvature and g is the metric tensor. Moreover, the kernel K of the operator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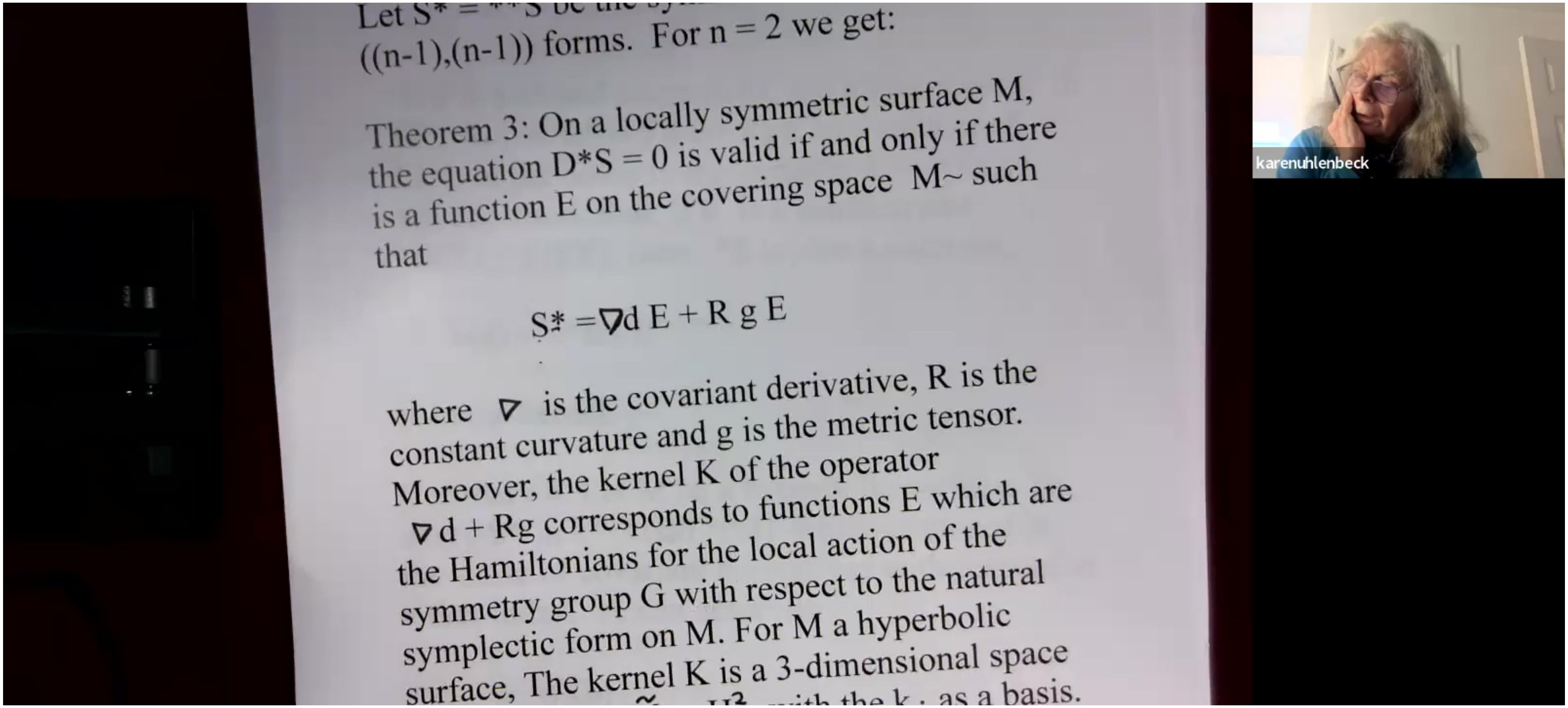
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Moreover, the kernel K of the operator ∇d + Rg corresponds to functions E which are the Hamiltonians for the local action of the symmetry group G with respect to the natural symplectic form on M. For M a hyperbolic surface, The kernel K is a 3-dimensional space
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... a curve in

If γ is a closed curv ... n E =

H^2. Solve S* = ∇ ... x,y).

E(X) in the cover ...

We can check th ... n,

*E(X) = E(γX), ... E

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is in the kernel.

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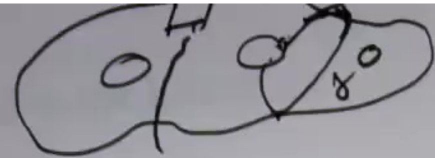
Corollary 4: Let ... osed in

every S* in T*(... respect to



If γ is a closed curve in M, we lift to a curve in H^2. Solve $S^* = \nabla d E + RgE$ as a solution $E = E(X)$ in the cover in the variable $X \longrightarrow (x,y)$. We can check that if E is a solution and $*E(X) = E(\gamma X)$, then $*E$ is also a solution,

$$k(\gamma) = *E - E$$

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Corollary 4: Let M be a hyperbolic surface. To every S^* in $T(M) \otimes T^*(M)$ which is closed in ... derivatives with respect to

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is in the kernel.

Corollary 4: Let M be a hyperbolic surface. To every S^* in $T^*(M) \circledS T^*(M)$ which is closed in the sense of covariant derivatives with respect to either index, we can associate

$$k: \pi_1(M) \longrightarrow K$$
$$k(\gamma_1\gamma_2) = k(\gamma_1) + k(\gamma_2).$$

...used to cons...ct an affine bundle.

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$$\ldots k(\gamma_1) + k(\gamma_2).$$

… to construct an affine bundle.

Noether's theorem. I will give only one example.

A. The Willmore Functional

I was fortunate to find a recent paper by Yann Bernard [B], who not only derives equations from Noether's formula, but also mentions many of the examples I knew about. The Willmore functional does not fall into the class of problems I discussed, as the Lagrangian is of second order. The Lagrangian in the parametric approach is expressed in terms of a second fundamental form of an immersion f: M—> R

$$I(f) = \iint H(f)\, d\mu(f).$$

IX. Found in the Literature

Many theorems in geometric analysis are very general, but examples treating special cases are often the first to be studied. In most, if not all of the applications I am aware of, the identities were first discovered without reference to Noether's theorem. I will give only one example.

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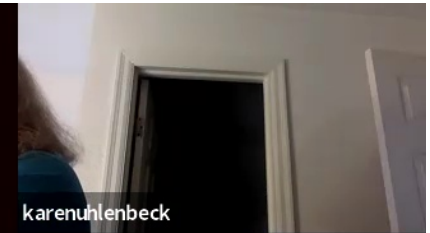

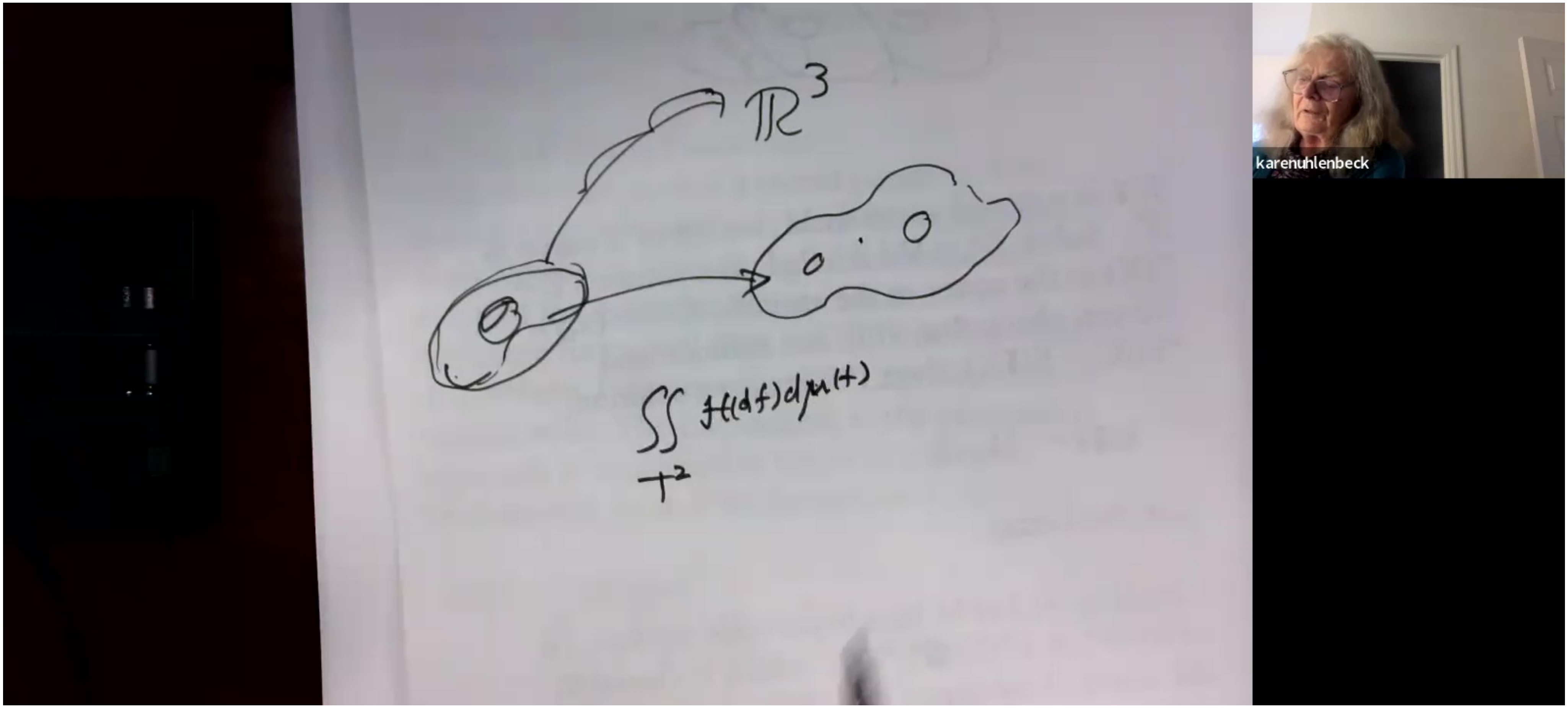
ℝ³
T²
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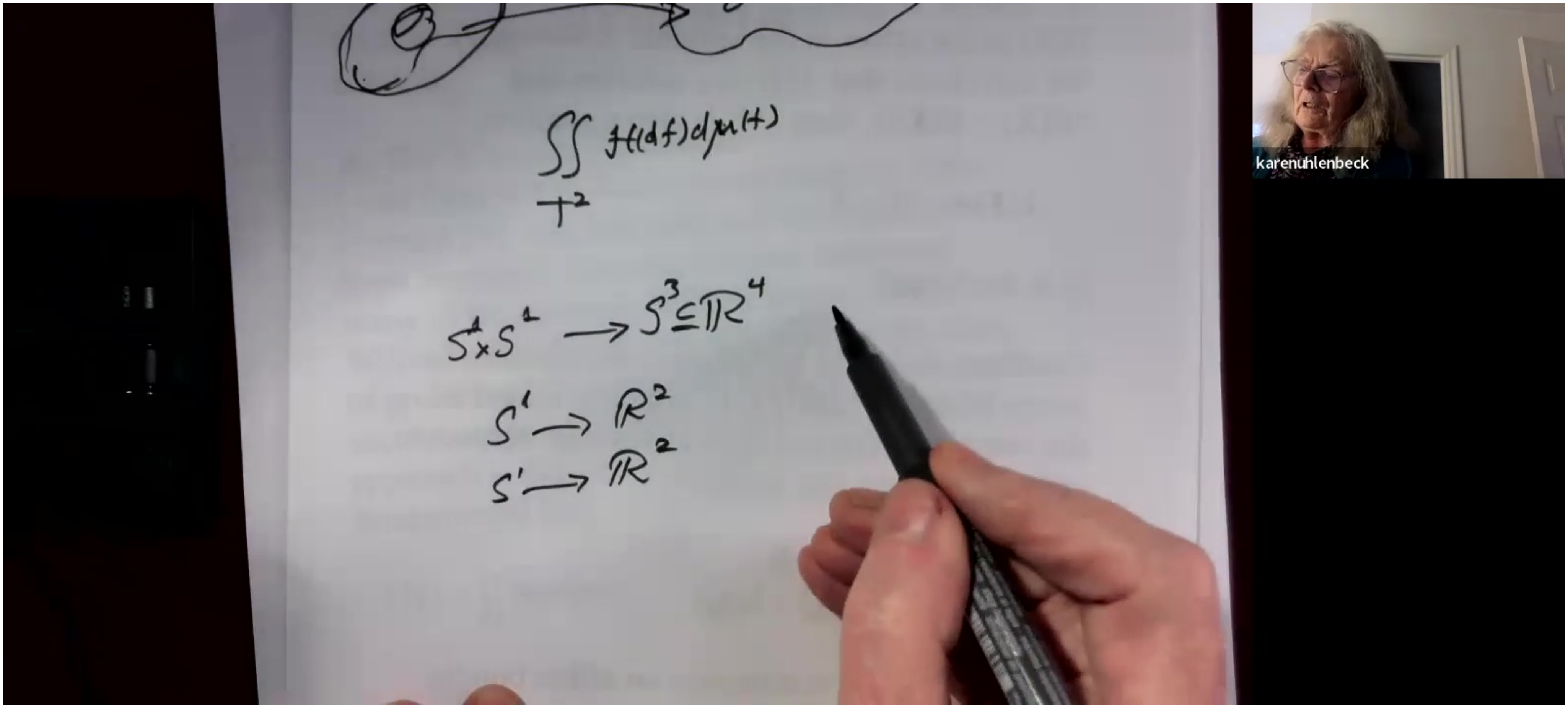
∬ f(df) dμ(t)
T²
S¹×S¹ → S³ ⊆ ℝ⁴
S¹ → ℝ²
S¹ → ℝ²
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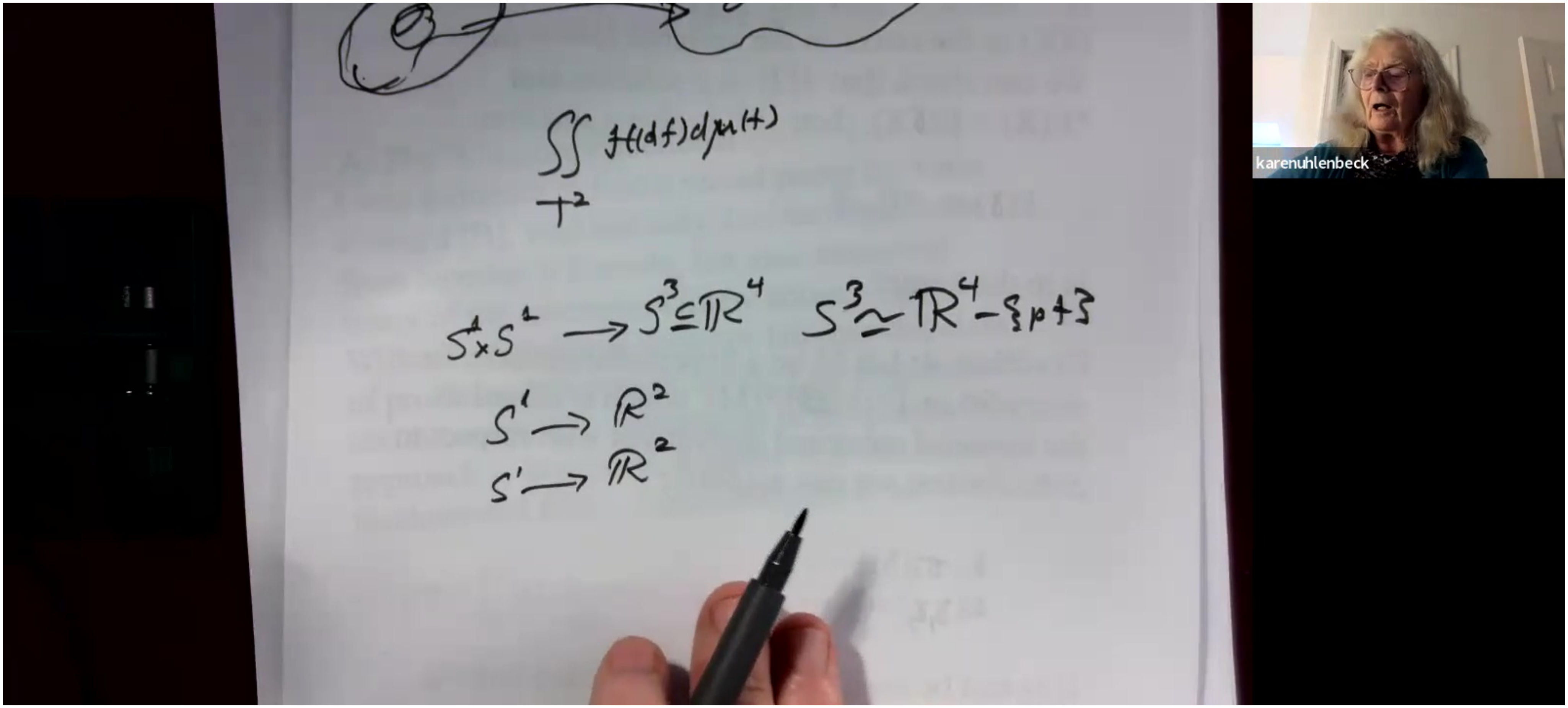
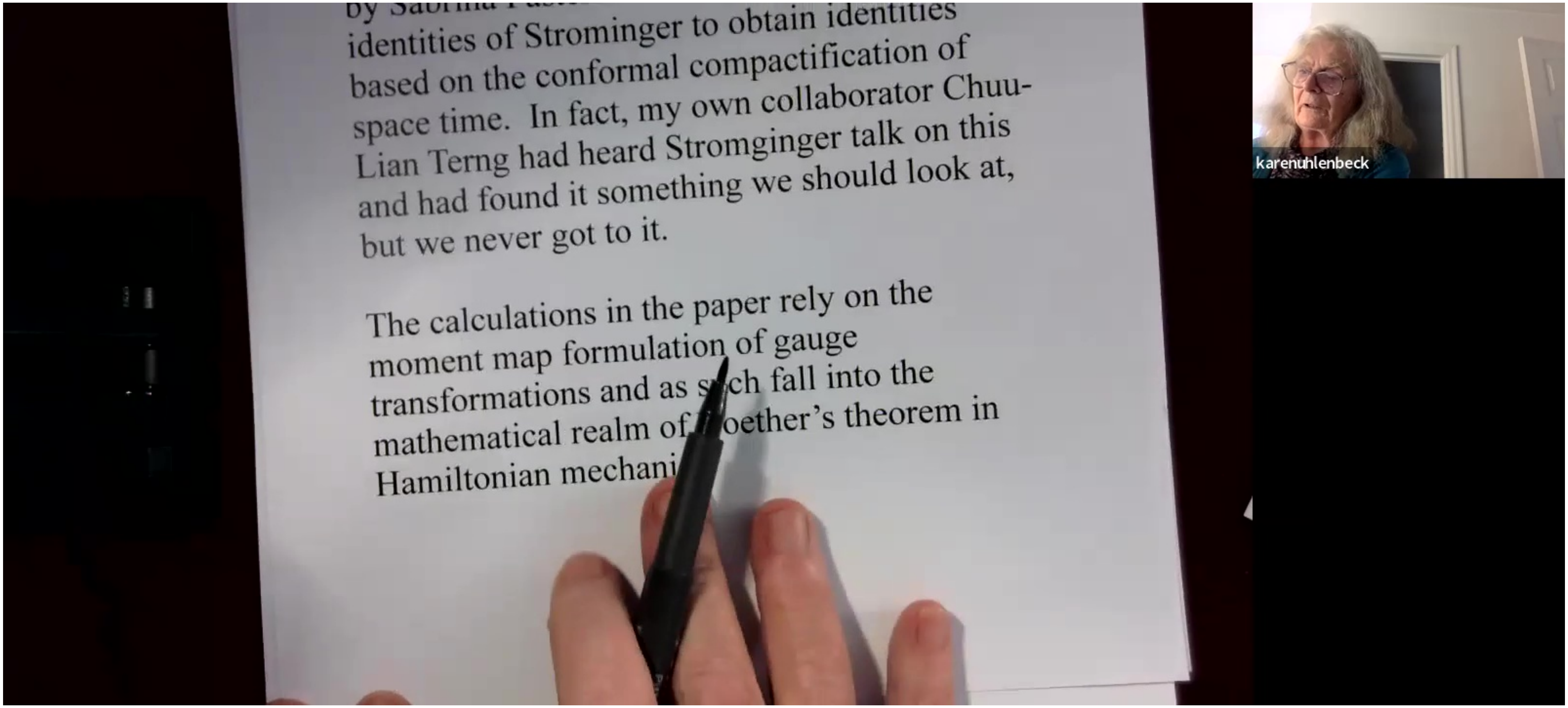

S^1 × S^1 → S^3 ⊆ ℝ^4
S^3 ≃ ℝ^4 − {pt}
S^1 → ℝ^2
S^1 → ℝ^2
karenuhlenbeck

The problem is to find a minimum when M is the torus. (It is Known to be the Clifford torus)

The reparameterizations form a symmetry dependent on two functions of two variables, and the Euclidean group acts on the target. However, the special property of this integral is that it is also invariant under dilations in R^3, implying that I is invariant under the full conformal group of R^3. The calculations needed to carry out Noether's program in this case are considerably more difficult than any described in this talk. The conservation laws obtained from these symmetries had been discovered without appealing to Noether by Riviere and used in a his solution of the Willmore problen (the first solution is due to Marques and Neve



identities of Strominger to obtain identities based on the conformal compactification of space time. In fact, my own collaborator Chuu-Lian Terng had heard Strominger talk on this and had found it something we should look at, but we never got to it.

The calculations in the paper rely on the moment map formulation of gauge transformations and as s[illegible]ch fall into the mathematical realm of [illegible]oether's theorem in Hamiltonian mechani[illegible]

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B. My colleague Antonella Grassi heard an earlier version of this talk, and pointed out to me a very recent paper in physics using Noether's theorem. I hesitate to include it, because the physics constructions need a lot of translation to be understandable to a mathematician. The paper is "A Shorter Path to Celestial Currents" by Sabrina Pasterski. It derives the Ward identities of Strominger to obtain identities based on the conformal compactification of space time. In fact, my own collaborator Chuu-Lian Terng had heard Stromginger talk on this and had found it something we should look at, but we never got to it.

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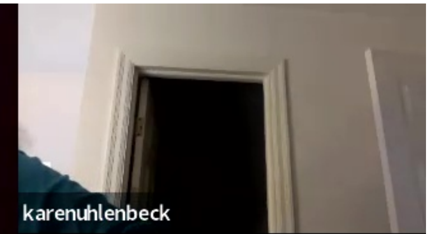

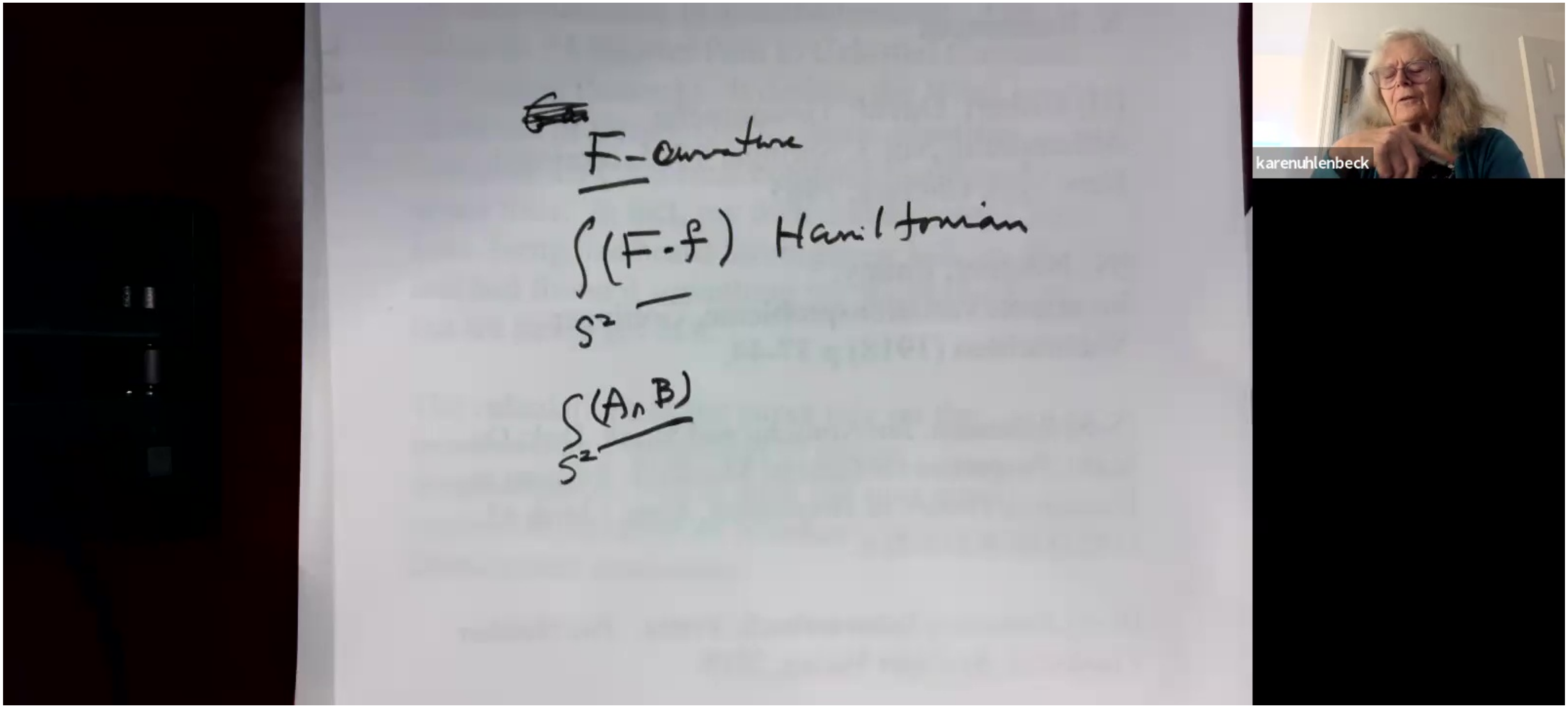
F - curvature
∫ (F·f) Hamiltonian
S²
∫ (A∧B)
S²
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Have I convinced you that the Noether theorems are still in the toolbox of many branches of mathematics? And despite their age, are still worth keeping dusted off on top?