Title: Entanglement dynamics from random product states at long times

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Abstract: We study the entanglement dynamics of quantum many-body systems at long times. For upper bounds, we prove the following: (I) For any geometrically local Hamiltonian on a lattice, starting from a random product state the entanglement entropy is bounded away from the maximum entropy at all times with high probability. (II) In a spin-glass model with random all-to-all interactions, starting from any product state the average entanglement entropy is bounded away from the maximum entropy at all times. We also extend these results to any unitary evolution with charge conservation and to the Sachdev-Ye-Kitaev model. For lower bounds, we say that a Hamiltonian is an ``extensive entropy generator" if starting from a random product state the entanglement entropy obeys a volume law at long times with overwhelming probability. We prove that (i) any Hamiltonian whose spectrum has non-degenerate gaps is an extensive entropy generator; (ii) in the space of (geometrically) local Hamiltonians, the non-degenerate gap condition is satisfied almost everywhere. These results imply ``unbounded growth of entanglement" in many-body localized systems.
References: arXiv:2102.07584 & 2104.02053

Zoom Link: https://pitp.zoom.us/j/96027097746?pwd=R2V2ZktJZUV1VkliN3pzM0IzbGR0UT09

Entanglement dynamics from random product states at long times

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About me

I am now a postdoc at the Center for Theoretical Physics, MIT.

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I was a postdoc at the Institute for Quantum Information and Matter, Caltech (2015-2018).

I received a PhD in condensed matter physics from UC Berkeley (2010-2015).

My research interests include condensed matter and quantum information theory.

Entanglement in quantum many-body systems

In the past 20 years, concepts of quantum information theory have been widely used in condensed matter, statistical, and high-energy physics to provide insights beyond those obtained via conventional quantities. In particular, entanglement characterizes or is quantitatively related to critical phenomena, topological order, quantum dynamics, and classical simulability of quantum many-body systems.

We study the entanglement dynamics of quantum many-body systems at long times.

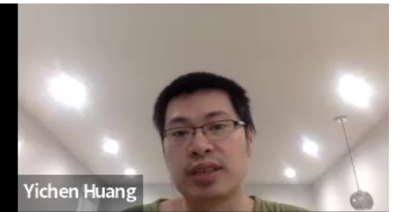
Abstract: upper bound

(I) For any geometrically local Hamiltonian on a lattice, starting from a random product state the entanglement entropy (EE) is bounded away from the maximum entropy at all times with high probability.

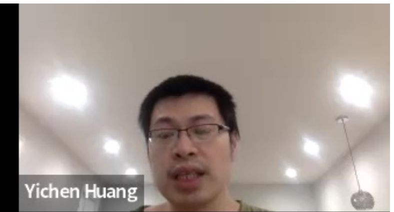
(II) In a spin-glass model with random all-to-all interactions, starting from any product state the average EE is bounded away from the maximum entropy at all times.

We also extend these results to any unitary evolution with charge conservation and to the Sachdev-Ye-Kitaev model.

Our results highlight the difference between the entanglement generated by (chaotic) Hamiltonian dynamics and that of random states.



Abstract: lower bound



In quantum many-body systems, a Hamiltonian is called an "extensive entropy generator" if starting from a random product state the EE obeys a volume law at long times with overwhelming probability. We prove that

(i) any Hamiltonian whose spectrum has non-degenerate gaps is an extensive entropy generator;

(ii) in the space of (geometrically) local Hamiltonians, the non-degenerate gap condition is satisfied almost everywhere.

These results imply "unbounded growth of entanglement" in many-body localized systems [Bardarson et al., PRL 109, 017202 (2012)].

Preliminaries

Let $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ be two functions. One writes

$f(x) = O(g(x))$ if and only if there exist constants $M, x_0 > 0$ such that $f(x) \leq Mg(x)$ for all $x > x_0$;

$f(x) = \Omega(g(x))$ if and only if there exist constants $M, x_0 > 0$ such that $f(x) \geq Mg(x)$ for all $x > x_0$;

$f(x) = \Theta(g(x))$ if and only if there exist constants $M_1, M_2, x_0 > 0$ such that $M_1 g(x) \leq f(x) \leq M_2 g(x)$ for all $x > x_0$;

In particular, "exponentially small in N" is denoted by $e^{-\Omega(N)}$.

Definition (entanglement entropy (EE))

The EE of a bipartite pure state ρ_{AB} is defined as the von Neumann entropy of the reduced density matrix $\rho_A = \mathrm{tr}_B\, \rho_{AB}$:

$$S(\rho_A) := -\mathrm{tr}(\rho_A \ln \rho_A). \quad (1)$$

Deviation from maximum entropy [arXiv:2102.07584]

Definition (Haar-random product state)

Consider a system of N spins. Let $|\Psi\rangle = \bigotimes_{j=1}^{N} |\Psi_j\rangle$ be a Haar-random product state, where each $|\Psi_j\rangle$ is chosen independently and uniformly at random with respect to the Haar measure.

A Haar-random product state is typically a "massive" superposition of energy eigenstates.[1]

The time evolution under a chaotic local Hamiltonian is so complex that heuristically, one might expect that the state at long times behaves like a random state.

Conjecture

For chaotic (not necessarily geometrically) local Hamiltonians, starting from a random product state the EE approaches that of a random state at long times.

[1] Huang & Harrow, arXiv:1907.13392.

Page curve

Theorem (entanglement of random states. conjectured and partially proved by Page (PRL 71, 1291, 1993); proved in (Foong & Kanno, PRL, 1994; Sánchez-Rui, PRE, 1995; Sen, PRL, 1996))

For a bipartite pure state ρ_{AB} chosen uniformly at random with respect to the Haar measure,

$$\mathop{\mathbb{E}}_{\rho_{AB}} S(\rho_A) = \sum_{k=d_B+1}^{d_A d_B} \frac{1}{k} - \frac{d_A - 1}{2d_B} = \ln d_A - \frac{d_A}{2d_B} + \frac{O(1)}{d_A d_B}, \qquad (2)$$

where $d_A \leq d_B$ are the local dimensions of subsystems A, B, respectively.

The distribution of $S(\rho_A)$ is highly concentrated around the mean $\mathbb{E}_{\rho_{AB}} S(\rho_A)$.[2] This is easily seen from the exact formula[3] for $\mathrm{Var}_{\rho_{AB}} S(\rho_A)$.

[2] Hayden et al., Commun. Math. Phys. 265, 95, (2006).

[3] Vivo et al., PRE 93, 052106 (2016); Wei, PRE 96, 022106 (2017).

Geometrically local Hamiltonians

Consider a chain of N qubits governed by a local Hamiltonian $H^{\mathrm{lat}} = \sum_{j=1}^{N} H_j$, where H_j acts on qubits at positions j and $j+1$. (A similar result holds for qudits in higher spatial dimensions.)

We do not assume translational invariance.

Let $\mathbb{E}_{|A|=n}$ denote averaging over all contiguous subsystems of size n. There are N such subsystems.

Theorem

Initialize the system in a Haar-random product state $|\Psi\rangle$. *Let* $\rho_A(t) = \mathrm{tr}_{\bar{A}}(e^{-iH^{\mathrm{lat}}t}|\Psi\rangle\langle\Psi|e^{iH^{\mathrm{lat}}t})$ *be the reduced density matrix of subsystem* A *at time* t. *For* $n > 1$,

$$\Pr_{\Psi}\left(\sup_{t\in\mathbb{R}} \mathop{\mathbb{E}}_{|A|=n} S(\rho_A(t)) = n\ln 2 - \Omega(n/N)\right) \geq 1 - \delta, \tag{3}$$

where $\delta > 0$ *is an arbitrarily small constant.*

Unitary evolution with charge conservation

Consider a system of N qubits without an underlying lattice structure.

Let m, n be positive integers such that mn is a multiple of N. Let $A_1, A_2, \ldots, A_m$ be m possibly overlapping subsystems, each of which has exactly n qubits. Suppose that each qubit in the system is in exactly mn/N out of these m subsystems.

Let $\sigma^z := \sum_{j=1}^{N} \sigma_j^z$ be the total charge operator and $U(t)$ be a unitary time evolution operator such that $[U(t), \sigma^z] = 0$.

For example, $U(t)$ can be the time evolution operator of the SY model[4] or of a quantum circuit with charge conservation.[5]

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[4] Sachdev & Ye, PRL 70, 3339 (1993).

[5] Khemani et al., PRX 8, 031057 (2018); Rakovszky et al., PRX 8, 031058 (2018).

Unitary evolution with charge conservation

Theorem

Initialize the system in a Haar-random product state $|\Psi\rangle$. *Let*

$$\rho_{A_j}(t) = \mathrm{tr}_{\bar{A}_j}(U(t)|\Psi\rangle\langle\Psi|U^\dagger(t)) \tag{4}$$

be the reduced density matrix of subsystem A_j *at time* t. *Then,*

$$\Pr_{\Psi}\left(\sup_{t\in\mathbb{R}} \frac{1}{m}\sum_{j=1}^{m} S(\rho_{A_j}(t)) = n\ln 2 - \Omega(n/N)\right) \geq 1-\delta, \tag{5}$$

where $\delta > 0$ *an arbitrarily small constant.*

Spin-glass model

Consider a system of N qubits. Let

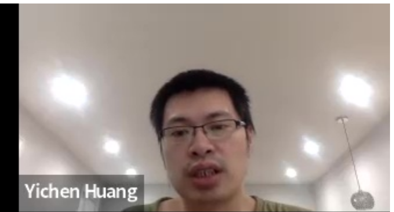
$$\sigma_j^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_j^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_j^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6}$$

be the Pauli matrices for the qubit at site j.

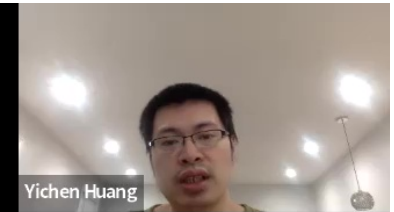
Let $J := \{J_{jklm}\}_{1\le j<k\le N}^{l,m\in\{x,y,z\}}$ be a collection of $9N(N-1)/2$ independent standard normal random variables. The Hamiltonian of the spin-glass model is

$$H_J^{\text{sg}} = \sum_{1\le j<k\le N} \sum_{l,m\in\{x,y,z\}} J_{jklm}\sigma_j^l \sigma_k^m. \tag{7}$$

Let $\mathbb{E}_{|A|=n}$ denote averaging over all subsystems of size n. There are $\binom{N}{n}$ such subsystems.



Spin-glass model



Theorem

Initialize the system in an arbitrary (deterministic) product state $|\psi\rangle = \bigotimes_{j=1}^{N} |\psi_j\rangle$. *Let*

$$\rho_{J,A}(t_J) = \mathrm{tr}_{\bar{A}}(e^{-iH_J^{\mathrm{sg}} t_J} |\psi\rangle\langle\psi| e^{iH_J^{\mathrm{sg}} t_J}) \quad (8)$$

be the reduced density matrix of subsystem A at time t_J. *For* $n > 1$,

$$\mathop{\mathbb{E}}_{J} \sup_{t_J \in \mathbb{R}} \mathop{\mathbb{E}}_{|A|=n} S(\rho_{J,A}(t_J)) = n \ln 2 - \Omega(n^2/N^2). \quad (9)$$

SYK model

Consider a system of N Majorana fermions $\chi_1, \chi_2, \ldots, \chi_N$, where N is an even number.

Let $K := \{K_{jklm}\}_{1\le j<k<l<m\le N}$ be a collection of $\binom{N}{4}$ independent standard normal random variables. The Hamiltonian of the SYK model is[6]

$$H_K^{\mathrm{SYK}} = \sum_{1\le j<k<l<m\le N} K_{jklm}\chi_j\chi_k\chi_l\chi_m, \quad \{\chi_j, \chi_k\} = 2\delta_{jk}. \qquad (10)$$

Let $\mathbb{E}_{|A|=n}$ denote averaging over all subsystems of size n (even). There are $\binom{N}{n}$ such subsystems.

[6] Kiteav 2015; Maldacena & Stanford, PRD 94, 106002 (2016).

Random product states in fermionic systems

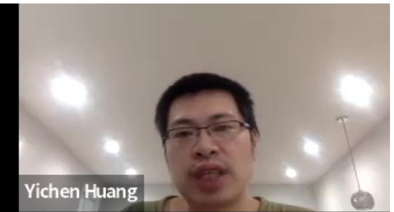
In fermionic systems, defining a Haar-random product state is tricky.

Since the Hamiltonian conserves fermion parity, the Hilbert space is split into an even sector and an odd sector, which do not interact with each other. It is controversial whether to allow the superposition of states from both sectors.

While being compatible with the axioms of quantum mechanics, such a superposition is widely believed to be unphysical. On the other hand, it is not clear how to define a Haar-random product state with definite fermion parity.

The statement of our theorem avoids the controversy and related technical difficulties by introducing the condition (11) instead of claiming $|\psi\rangle$ to be a Haar-random product state.

SYK model



Theorem

Initialize the system in a state $|\psi\rangle$ such that a constant fraction of the expectation values $\{\langle\psi|\chi_j\chi_k\chi_l\chi_m|\psi\rangle\}_{1\le j<k<l<m\le N}$ are non-vanishing, i.e.,

$$|\{(j,k,l,m) : |\langle\psi|\chi_j\chi_k\chi_l\chi_m|\psi\rangle| = \Theta(1)\}| = \Theta(N^4). \tag{11}$$

Let

$$\rho_{K,A}(t_K) = \mathrm{tr}_{\bar{A}}(e^{-iH_K^{\mathrm{SYK}}t_K}|\psi\rangle\langle\psi|e^{iH_K^{\mathrm{SYK}}t_K}) \tag{12}$$

be the reduced density matrix of subsystem A at time t_K. For $n \ge 4$,

$$\mathop{\mathbb{E}}_{K} \sup_{t_K\in\mathbb{R}} \mathop{\mathbb{E}}_{|A|=n} S(\rho_{K,A}(t_K)) = \frac{n\ln 2}{2} - \Omega(n^4/N^4). \tag{13}$$

Extensive entropy [arXiv:2104.02053]

In textbooks, we learned that entropy is an extensive quantity:

> *Initializing a system in a low entropy state, generically (although not always) the entropy will grow with time and eventually become proportional to the system size.*

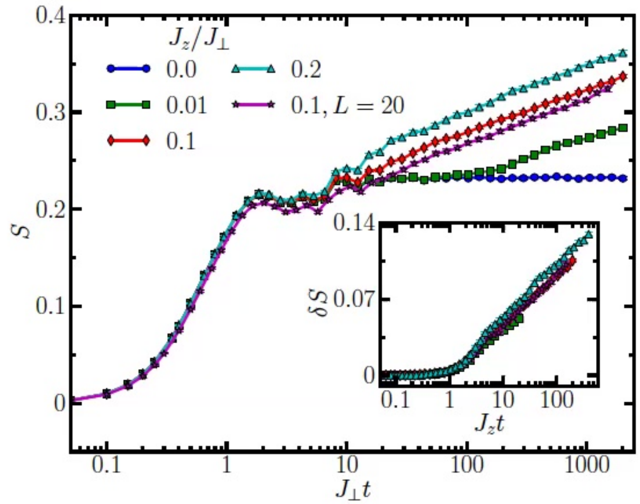
To observe non-trivial entropy dynamics in the unitary evolution of a pure state, we divide the system into two parts A (small) and B (large).

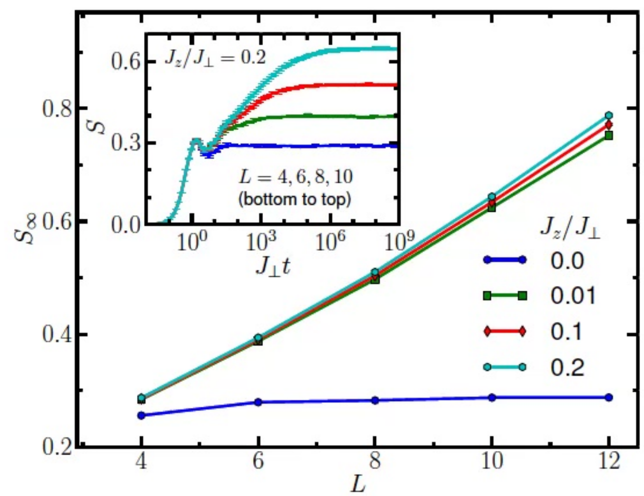
We view B as a bath of A and consider the entropy of A.

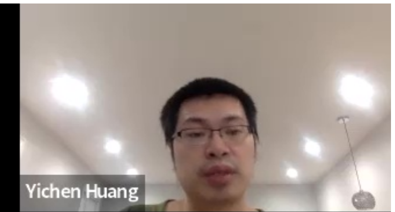
Extensive subsystem entropy is also known as a volume law for entanglement.

The most general unentangled state with zero entropy for all subsystems is a random product state.

Localized systems







Time evolution and saturation of EE from random product states under the Hamiltonian[7]

$$H = J_\perp \sum_i (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + \sum_i h_i \sigma_i^z + J_z \sum_i \sigma_i^z \sigma_{i+1}^z. \quad (14)$$

[7] Bardarson et al., PRL 2012. For heuristic explanations of the numerical results, see Vosk & Altman, PRL 2013; Serbyn et al., PRL 2013; Huse et al., PRB 2014.

Spectrum

Definition (non-degenerate spectrum)

The spectrum of a Hamiltonian is non-degenerate if all eigenvalues are distinct.

Definition (non-degenerate gaps)

The spectrum $\{E_j\}$ of a Hamiltonian has non-degenerate gaps if the differences $\{E_j - E_k\}_{j \neq k}$ are all distinct, i.e., for any $j \neq k$,

$$E_j - E_k = E_{j'} - E_{k'} \implies (j = j') \text{ and } (k = k'). \tag{15}$$

By definition, the non-degenerate gap condition implies that the spectrum is non-degenerate.

Setting and quantum revival

Consider a system of N spins or qudits with local dimension d.

Let m, n be positive integers such that mn is a multiple of the system size N. Let $A_1, A_2, \dots, A_m$ be m possibly overlapping subsystems, each of which consists of exactly $n \le N/2$ spins. Suppose that each spin in the system is in exactly mn/N out of these m subsystems.

Initialize the system in a Haar-random product state $|\Psi\rangle$. Let $\rho_{A_j}(t) := \mathrm{tr}_{\bar{A}_j}(e^{-iHt}|\Psi\rangle\langle\Psi|e^{iHt})$ be the reduced density matrix of subsystem A_j at time t.

Theorem (Quantum recurrence theorem (Bocchieri & Loinger, 1957))

Any finite quantum system will, after a sufficiently long but finite time, return to a state arbitrarily close to the initial state.

For any $t' > 0$, there exists $t > t'$ such that entanglement at time t is arbitrarily close to 0. We can only prove a volume law at *most* $t \in \mathbb{R}$.

Volume law

Theorem

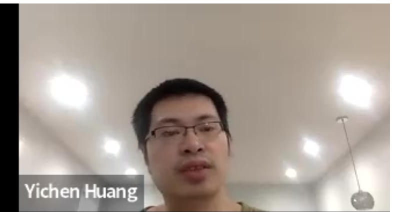
Let $n \leq \frac{(1-\epsilon)N\ln\frac{d+1}{2}}{2\ln d}$, *where* $\epsilon > 0$ *is an arbitrarily small constant. For any Hamiltonian* H *whose spectrum has non-degenerate gaps,*

$$\Pr_{\Psi}\left(\Pr_{t\in\mathbb{R}}\left(\frac{1}{m}\sum_{j=1}^{m}S(\rho_{A_j}(t)) \geq (1-\epsilon)n\sum_{k=2}^{d}\frac{1}{k}\right) = 1 - e^{-\Omega(N)}\right) = 1 - e^{-\Omega(N)}. \quad (16)$$

There is no underlying lattice in the statement of this theorem. In quantum lattice systems,

$$\Pr_{\Psi}\left(\Pr_{t\in\mathbb{R}}\left(\mathop{\mathbb{E}}_{|A|=n} S(\rho_A(t)) = \Omega(n)\right) = 1 - e^{-\Omega(N)}\right) = 1 - e^{-\Omega(N)} \quad (17)$$

for any $n \leq N/2$, where $\mathbb{E}_{|A|=n}$ denotes averaging over all contiguous subsystems of size n (there are N such subsystems).



Non-degenerate gaps

Define an ensemble of Hamiltonians

$$\alpha := (\alpha_j^k|_{1\le j\le N}^{k\in\{x,y,z\}}, \alpha_j^{kl}|_{1\le j\le N-1}^{k,l\in\{x,y,z\}}) \in [-1,1]^{\times(12N-9)}, \tag{18}$$

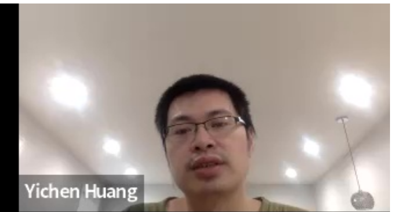
$$H(\alpha) := \sum_{j=1}^{N} \sum_{k\in\{x,y,z\}} \alpha_j^k \sigma_j^k + \sum_{j=1}^{N-1} \sum_{k,l\in\{x,y,z\}} \alpha_j^{kl} \sigma_j^k \sigma_{j+1}^l. \tag{19}$$

Theorem

The set of all α such that the spectrum of $H(\alpha)$ has degenerate gaps is of measure zero.

Similar results can be proved in a similar way for other types of systems including qudit systems with short-range interactions in higher spatial dimensions or even with non-local interactions.

Many-body localization



Corollary

The set of all α *such that the spectrum of the Imbrie model*

$$H^{\mathrm{Imb}}(\alpha) := \sum_{j=1}^{N} (\lambda \alpha_j^x \sigma_j^x + \alpha_j^z \sigma_j^z) + \sum_{j=1}^{N-1} \alpha_j^{zz} \sigma_j^z \sigma_{j+1}^z. \tag{20}$$

has degenerate gaps is of measure zero, where $\lambda > 0$ *is a small constant.*

Corollary

The set of all α *such that the spectrum of the perturbed XX model*

$$H^{XXZ}(\alpha) := \sum_{j=1}^{N} (h_x \alpha_j^x \sigma_j^x + h_z \alpha_j^z \sigma_j^z) + \sum_{j=1}^{N-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + h_{zz} \alpha_j^{zz} \sigma_j^z \sigma_{j+1}^z) \tag{21}$$

has degenerate gaps is of measure zero, where $h_x, h_{zz} > 0$ *are small constants.*

Volume-law coefficient is tight

Consider a chain of N qudits.

Let S_j^z be the z component of the spin operator at position j. Let $(h_1, h_2, \dots, h_N) \in \mathbb{R}^N$ be such that the spectrum of $H^{\text{loc}} := \sum_{j=1}^N h_j S_j^z$ is non-degenerate. Let $H^{\text{mbl}} := H^{\text{loc}} + \Delta H$, where ΔH is an infinitesimal random local perturbation.

The spectrum of H^{mbl} almost surely has non-degenerate gaps.

Theorem

Initialize the system in a Haar-random product state $|\Psi\rangle$. Let $\rho_A(t) := \mathrm{tr}_{\bar{A}}(e^{-iH^{\text{mbl}}t}|\Psi\rangle\langle\Psi|e^{iH^{\text{mbl}}t})$ be the reduced density matrix of subsystem A at time t. For any A,

$$\mathop{\mathbb{E}}_{\Psi}\lim_{\tau\to+\infty}\frac{1}{\tau}\int_0^{\tau} S(\rho_A(t))\,\mathrm{d}t \le |A|\sum_{k=2}^{d}\frac{1}{k}. \tag{22}$$

Summary

We study the entanglement dynamics of quantum many-body systems at long times and prove the following: (I) For any geometrically local Hamiltonian on a lattice, starting from a random product state the EE is bounded away from the maximum entropy at all times with high probability. (II) In a spin-glass model with random all-to-all interactions, starting from any product state the average EE is bounded away from the maximum entropy at all times. We also extend these results to any unitary evolution with charge conservation and to the Sachdev-Ye-Kitaev model.

We say that a Hamiltonian is an "extensive entropy generator" if starting from a random product state the EE obeys a volume law at long times with overwhelming probability. We prove that (i) any Hamiltonian whose spectrum has non-degenerate gaps is an extensive entropy generator; (ii) in the space of (geometrically) local Hamiltonians, the non-degenerate gap condition is satisfied almost everywhere. These results imply "unbounded growth of entanglement" in many-body localized systems [Bardarson et al., PRL 109, 017202 (2012)].