Title: BGG-type relations for transfer matrices of rational spin chains and the shifted Yangians

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Abstract: In this talk, I will discuss: (1) the new BGG-type resolutions of finite dimensional representations of simple Lie algebras that lead to BGG-relations expressing finite-dimensional transfer matrices via infinite-dimensional ones, (2) the factorization of infinite-dimensional ones into the product of two Q-operators, (3) the construction of a large family of rational Lax matrices from antidominantly shifted Yangians. This talk is based on the joint works with R.Frassek, I. Karpov, and V.Pestun.

BGG-type relations for transfer matrices of rational spin chains and the shifted Yangians

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(joint works with R. Frassek, I. Karpov, V. Pestun)

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Main Results

- New BGG-type resolutions for finite-dimensional $\mathfrak{g}$-modules (comparison to the well-known BGG and BGGL resolutions)
- Application to transfer matrices of ABCD-type rational spin chains (expressing finite-dimensional via the infinite-dimensional ones)
- Factorisation of the aforementioned infinite-dimensional transfer matrices into two commuting Baxter Q-operators
- A uniform construction of a large family of degenerate Lax matrices (via the shifted Yangians and the GKLO-type homomorphisms)
- The RTT realization of antidominantly shifted extended Yangians (application to coproduct, integral forms, and integrable systems)

BGG resolution

- $\mathfrak{g}$–simple Lie algebra, Δ–root system, $\{\alpha_i\}_{i=1}^r$–simple roots
- Λ–weight lattice, Λ^+–integral dominant weights
- W–Weyl group, $\ell\colon W \to \mathbb{Z}_{\geq 0}$–length function
- $\rho \in \Lambda$–sum of fundamental weights
- **dot action** $W \curvearrowright \Lambda$: $w \cdot \mu = w(\mu + \rho) - \rho$
- M_μ–Verma module, L_μ–its irreducible quotient
- **Theorem (BGG '75):** For $\lambda \in \Lambda^+$, there is a $\mathfrak{g}$-module resolution

$$0 \to M_{w_0 \cdot \lambda} \to \cdots \to \bigoplus_{w \in W}^{\ell(w)=2} M_{w\cdot\lambda} \to \bigoplus_{w \in W}^{\ell(w)=1} M_{w\cdot\lambda} \to M_\lambda \to L_\lambda \to 0$$

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- Similar resolutions exist for arbitrary Kac-Moody $\mathfrak{g}$ ([Rocha-Caridi&Wallach '82]–symmetrizable, [Kumar '90]–general)

Truncated BGG-type resolution

- S–subset of $\{1,\ldots,r\}$
- Standard parabolic: $B \subset P_S \subset G$ and $\mathfrak{b} \subset \mathfrak{p}_S \subset \mathfrak{g}$
- $W_S \subset W$–Weyl group, $\Delta_S = \Delta \cap \bigoplus_{i\in S} \mathbb{Z}\alpha_i$–root system
- ${}^S W$–shortest representatives of W/W_S
- For $w \in W$ and $\lambda \in \Lambda$ dominant w.r.t. Δ_S, consider:

$$ {}^S M_{w\cdot\lambda} := M_{w\cdot\lambda}/\text{singular vectors of weights } s_{w(\alpha)}(w\cdot\lambda),\ \alpha\in\Delta_S $$

- **Theorem (FKT '21):** For $\lambda \in \Lambda^+$, there is a $\mathfrak{g}$-module resolution

$$ \cdots \to \bigoplus_{w\in {}^S W}^{\ell(w)=2} {}^S M_{w\cdot\lambda} \to \bigoplus_{w\in {}^S W}^{\ell(w)=1} {}^S M_{w\cdot\lambda} \to {}^S M_{1\cdot\lambda} \to L_\lambda \to 0 $$

with each summand admitting a resolution by Vermas:

$$ \cdots \to \bigoplus_{v\in W_S}^{\ell(v)=2} M_{wv\cdot\lambda} \to \bigoplus_{v\in W_S}^{\ell(v)=1} M_{wv\cdot\lambda} \to M_{w\cdot\lambda} \to {}^S M_{w\cdot\lambda} \to 0 $$

BGGL resolution

- W^S–shortest representatives of $W_S \backslash W$
- $w \in W$, $\lambda \in \Lambda^+ \rightsquigarrow$ **parabolic Verma** modules:

$$M^S_{w\cdot\lambda} := M_{w\cdot\lambda}/\text{singular vectors of weights } s_\alpha(w\cdot\lambda),\ \alpha \in \Delta_S$$

- **Theorem (Lepowsky '77):** For $\lambda \in \Lambda^+$, there is a resolution

$$\cdots \to \bigoplus_{w\in W^S}^{\ell(w)=2} M^S_{w\cdot\lambda} \to \bigoplus_{w\in W^S}^{\ell(w)=1} M^S_{w\cdot\lambda} \to M^S_{1\cdot\lambda} \to L_\lambda \to 0$$

with each summand admitting a resolution by Vermas:

$$\cdots \to \bigoplus_{v\in W_S}^{\ell(v)=2} M_{vw\cdot\lambda} \to \bigoplus_{v\in W_S}^{\ell(v)=1} M_{vw\cdot\lambda} \to M_{w\cdot\lambda} \to M^S_{w\cdot\lambda} \to 0$$

- $\{{}^S M_{w\cdot\lambda}\}_{w\in {}^S W}$ all have "similar" characters, $\{M^S_{w\cdot\lambda}\}_{w\in W^S}$–do not

Resolutions via Cousin complexes

- X–topological space, $B \subset A \subset X$–closed subsets, $\mathcal{E}$–sheaf on X
- $H^\bullet_{A/B}(X, \mathcal{E})$–**cohomology groups with relative support**
- $G \curvearrowright X$–smooth alg. var., $\mathcal{D}_X \curvearrowright \mathcal{E}$–coherent $\rightsquigarrow \mathfrak{g} \curvearrowright H^\bullet_{A/B}(X, \mathcal{E})$
- $Z_n \subset Z_{n-1} \subset \cdots \subset Z_0 \subset X$–closed subsets $\rightsquigarrow$ **Cousin complexes**:

$$\mathcal{C}_k: \ H^k_{Z_0/Z_1}(X, \mathcal{E}) \to H^{k+1}_{Z_1/Z_2}(X, \mathcal{E}) \to H^{k+2}_{Z_2/Z_3}(X, \mathcal{E}) \to \cdots$$

- **Theorem (Kempf '78):** If $\mathcal{C}_{l \neq k} = 0$, then $H^\bullet(\mathcal{C}_k) = H^\bullet_{Z_0}(X, \mathcal{E})$
- **Theorem (Brylinski '81):** For $X = G/B$, $\mathcal{E} = \mathcal{L}_\lambda = G \times_B \mathbb{C}_{-\lambda}$, $Z_i = \bigsqcup_{w \in W: \ell(w) \geq i} B^- wB/B$, recover the dual of the BGG resolution
- **Theorem (Murray-Rice '92):** For $X = G/B$, $\mathcal{E} = \mathcal{L}_\lambda$, and $Z_i = \bigsqcup_{w \in W^S: \ell(w) \geq i} P^- wB/B$, recover the dual of the BGGL resolution
- **Theorem (FKP '21):** For $X = G/P$, $\mathcal{E} = R^0\pi_{(G/B \to G/P)*}(\mathcal{L}_\lambda)$, $Z_i = \bigsqcup_{w \in {}^S W: \ell(w) \geq i} B^- wP/P$, recover the dual of the truncated BGG

Type A: recap

- $\mathrm{P} = \sum_{i,j=1}^{n} e_{ij} \otimes e_{ji} \in (\mathrm{End}\,\mathbb{C}^n)^{\otimes 2}$–permutation operator
- $R(z) = \mathrm{Id} + \frac{\mathrm{P}}{z}$–rational R-matrix satisfying *quantum YB equation*
- $Y^{\mathrm{rtt}}(\mathfrak{gl}_n)$: generated by $\{t_{ij}^{(k)}\}_{1\leq i,j\leq n}^{k\geq 1}$ subject to *RTT relation*:

$$R_{12}(z-w)T_1(z)T_2(w) = T_2(w)T_1(z)R_{12}(z-w)$$

with $T(z)_{ij} = \delta_{ij} + \sum_k t_{ij}^{(k)} z^{-k}$

- $\mathrm{ev}\colon Y^{\mathrm{rtt}}(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n),\ t_{ij}^{(k)} \mapsto \delta_{k1}E_{ji} \rightsquigarrow \mathrm{Rep}\,\mathfrak{gl}_n \overset{\mathrm{ev}^*}{\to} \mathrm{Rep}\,Y^{\mathrm{rtt}}(\mathfrak{gl}_n)$
- $\pi_V\colon Y^{\mathrm{rtt}}(\mathfrak{gl}_n) \to \mathrm{End}\,V \rightsquigarrow$ Lax matrix $L_V(x) = \pi_V(T(x)) \rightsquigarrow$

$$\text{transfer matrix} \quad T_V(x) = \mathrm{tr}_V \underbrace{\prod_{1\leq i\leq r} \tau_i^{E_{ii}}}_{\text{twist}} \underbrace{L_V(x)\otimes\cdots\otimes L_V(x)}_{N}$$

- **Thm (Bazhanov-Frassek-Lukowski-Meneghelli-Staudacher'12):**

$$T_{L_\lambda}(x) = \sum_{\sigma\in S_n} (-1)^{\ell(\sigma)} T_{M_{\sigma\cdot\lambda}}(x), \qquad \forall\,\lambda\in\Lambda^+$$

BCD types

Setup

- $\mathfrak{g} = \mathfrak{sp}_{K=2r}, \mathfrak{so}_{K=2r}, \mathfrak{so}_{K=2r+1}$; $i' := K + 1 - i$; $\kappa = \frac{K}{2} \mp 1$ (BD vs C)
- $Q = \sum_{i,j=1}^{K} \pm e_{ij} \otimes e_{i'j'} \in (\mathrm{End}\, \mathbb{C}^K)^{\otimes 2}$ ($\pm = +$ for BD types)
- $R(z) = \mathrm{Id} + \frac{P}{z} - \frac{Q}{z+\kappa}$–Zamolodchikov-Zamolodchikov's R-matrix
- RTT relation $\rightsquigarrow$ extended RTT Yangian $X^{\mathrm{rtt}}(\mathfrak{g})$
- $X^{\mathrm{rtt}}(\mathfrak{g}) \simeq Y^{\mathrm{rtt}}(\mathfrak{g}) \otimes Z(X^{\mathrm{rtt}}(\mathfrak{g}))$

Obstacles

- There is no evaluation homomorphism $\mathrm{ev}\colon X^{\mathrm{rtt}}(\mathfrak{g}) \to U(\mathfrak{g})$
- Very few $\mathfrak{g} \curvearrowright L_\lambda$ can be lifted to $X^{\mathrm{rtt}}(\mathfrak{g}) \curvearrowright L_\lambda$
- Even if $X^{\mathrm{rtt}}(\mathfrak{g}) \curvearrowright L_\lambda$, the Vermas $M_{\sigma.\lambda}$ don't necessarily lift

Our approach

- Consider only multiples of "minuscule" coweights (cf. [KR, CGY])

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Our approach

- Consider only multiples of "minuscule" coweights (cf. [KR, CGY])
- Replace the full BGG resolution with its truncated version

C-type oscillator Lax matrices

- $\frac{r(r+1)}{2}$ pairs of oscillators $\{(\mathbf{a}_{j',i}, \bar{\mathbf{a}}_{i,j'})\}_{1\le i\le j\le r}$, $[\mathbf{a}_{j',i}, \bar{\mathbf{a}}_{k,\ell'}] = \delta_{ik}\delta_{j\ell}$

$$\bar{\mathbf{A}} = \begin{pmatrix} \bar{\mathbf{a}}_{1,r'} & \cdots & \bar{\mathbf{a}}_{1,2'} & 2\bar{\mathbf{a}}_{1,1'} \\ \vdots & \iddots & 2\bar{\mathbf{a}}_{2,2'} & \bar{\mathbf{a}}_{1,2'} \\ \bar{\mathbf{a}}_{r-1,r'} & \iddots & \iddots & \vdots \\ 2\bar{\mathbf{a}}_{r,r'} & \bar{\mathbf{a}}_{r-1,r'} & \cdots & \bar{\mathbf{a}}_{1,r'} \end{pmatrix}, \mathbf{A} = \begin{pmatrix} \mathbf{a}_{r',1} & \cdots & \mathbf{a}_{r',r-1} & \mathbf{a}_{r',r} \\ \vdots & \iddots & \iddots & \mathbf{a}_{r',r-1} \\ \mathbf{a}_{2',1} & \mathbf{a}_{2',2} & \iddots & \vdots \\ \mathbf{a}_{1',1} & \mathbf{a}_{2',1} & \cdots & \mathbf{a}_{r',1} \end{pmatrix}$$

- $\mathcal{L}(x) = \begin{pmatrix} (x+t)\mathrm{I}_r - \bar{\mathbf{A}}\mathbf{A} & -\bar{\mathbf{A}}(2t+r+1-\mathbf{A}\bar{\mathbf{A}}) \\ -\mathbf{A} & (x-t-r-1)\mathrm{I}_r + \mathbf{A}\bar{\mathbf{A}} \end{pmatrix}$ is C_r-type Lax
- $\mathfrak{sp}_{2r} \curvearrowright$ F–Fock module (generated by Fock vacuum $|0\rangle$, $\mathbf{a}_{j',i}|0\rangle = 0$)
- $\mathrm{F} \simeq (M^{\{1,\ldots,r-1\}}_{t\omega_r})^* \simeq ({}^{\{1,\ldots,r-1\}}M_{t\omega_r})^*$
  $t \in \mathbb{Z}_{\ge 0} \rightsquigarrow$ Fock vacuum $|0\rangle$ generates finite-dimensional $L_{t\omega_r}$
- $\mathfrak{sp}_{2r} \curvearrowright L_{t\omega_r}$ thus explicitly extends to $Y^{\mathrm{rtt}}(\mathfrak{sp}_{2r}) \curvearrowright L_{t\omega_r}$ for $t \in \mathbb{Z}_{\ge 0}$
- $\{\pm 1\}^r \leftrightarrow W/W_{\{1,\ldots,r-1\}} \leftrightarrow {}^{\{1,\ldots,r-1\}}W$, hence $\{\pm 1\}^r \ni \vec{\mu} \rightsquigarrow w_{\vec{\mu}}$
- Lax $\{\mathcal{L}_{\vec{\mu}}(x)\}_{\vec{\mu}\in\{\pm 1\}^r}$ from $\mathcal{L}(x)$ via similarity and particle-hole
- Makes F into $\mathfrak{sp}_{2r}$-module $M^{+}_{\vec{\mu},t}$
- $M^{+}_{\vec{\mu},t} \simeq ({}^{\{1,\ldots,r-1\}}M_{w_{\vec{\mu}}\cdot t\omega_r})^*$ for $t \notin \frac{1}{2}\mathbb{Z}$

C-type: transfer matrices

- $\vec{\mu} \in \{\pm 1\}^r, t \in \mathbb{C} \rightsquigarrow$ length N infinite-dimensional transfer matrices

$$T^+_{\vec{\mu},t}(x) = \mathrm{tr}_{\mathsf{F}} \left(\underbrace{\prod_{1 \le i \le r} \tau_i^{\mathcal{F}^{\vec{\mu}}_{ii}}}_{\text{twist}} \underbrace{\mathcal{L}_{\vec{\mu}}(x) \otimes \cdots \otimes \mathcal{L}_{\vec{\mu}}(x)}_{N} \right)$$

where $\mathcal{L}_{\vec{\mu}}(x)_{ij} = x\delta_{ij} + \mathcal{F}^{\vec{\mu}}_{ji}$

- $t \in \mathbb{Z}_{\geq 0} \rightsquigarrow$ finite-dimensional transfer matrices $T_{r,t}(x)$ via $\mathrm{tr}_{L_{t\omega_r}}$
- **Theorem (FKP '21):**

$$T_{r,t}(x) = \sum_{\vec{\mu} \in \{\pm 1\}^r} (-1)^{\sum_{i=1}^r (r-i+1)\delta^-_{\mu_i}} \, T^+_{\vec{\mu},t}(x), \qquad \forall\, t \in \mathbb{Z}_{\geq 0}$$

- Allows to analytically continue $T_{r,t}(x)$ from $t \in \mathbb{Z}_{\geq 0}$ to $t \in \mathbb{C}$

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- Allows to analytically continue $T_{r,t}(x)$ from $t \in \mathbb{Z}_{\geq 0}$ to $t \in \mathbb{C}$
- **Lemma:** $T_{r,t}(x) = (-1)^{\frac{r(r+1)}{2}} T_{r,-r-1-t}(x)$ for any $t \in \mathbb{C}$

D-type transfer matrices: spinorial case

- Type D_r Lax matrix $\mathcal{L}(x)$ is defined likewise with $1 \leq i < j \leq r$ and

$$\bar{\mathbf{A}} = \begin{pmatrix} \bar{a}_{1,r'} & \cdots & \bar{a}_{1,2'} & 0 \\ \vdots & \iddots & 0 & -\bar{a}_{1,2'} \\ \bar{a}_{r-1,r'} & \iddots & \iddots & \vdots \\ 0 & -\bar{a}_{r-1,r'} & \cdots & -\bar{a}_{1,r'} \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a_{r',1} & \cdots & a_{r',r-1} & 0 \\ \vdots & \iddots & \iddots & -a_{r',r-1} \\ a_{2',1} & 0 & \iddots & \vdots \\ 0 & -a_{2',1} & \cdots & -a_{r',1} \end{pmatrix}$$

- $\mathrm{F} \simeq (M_{2t\omega_r}^{\{1,\ldots,r-1\}})^*$; $t \in \frac{1}{2}\mathbb{Z}_{\geq 0} \rightsquigarrow$ finite-dimensional $L_{2t\omega_r}$
- $\mathfrak{so}_{2r} \curvearrowright L_{2t\omega_r}$ explicitly extends to $Y(\mathfrak{so}_{2r}) \curvearrowright L_{2t\omega_r}$ for $t \in \frac{1}{2}\mathbb{Z}_{\geq 0}$
- Lax $\{\mathcal{L}_{\vec{\mu}}(x)\}_{\vec{\mu} \in \{\pm 1\}^r}$ from $\mathcal{L}(x)$ via similarity and particle-hole
- $T^{+}_{\vec{\mu},t}(x)$–infinite-dimensional transfer matrices
- $T^{\pm}_{t}(x)$–finite-dimensional ones for $L_{2t\omega_r}$ and $L_{2t\omega_{r-1}}$
- **Theorem (FKP '21)**=[Ferrando-Frassek-Kazakov '20, Conjecture]:

$$T^{\pm}_{t}(x) = \sum_{\vec{\mu} \in \{\pm 1\}^r}^{\mu_1 \cdots \mu_r = \pm 1} (-1)^{\sum_{i=1}^{r}(r-i)\delta^{-}_{\mu_i}}\, T^{+}_{\vec{\mu},t}(x), \qquad \forall\, t \in \frac{1}{2}\mathbb{Z}_{\geq 0}$$

- **Lemma:** $T^{\pm}_{t}(x) = (-1)^{\frac{r(r-1)}{2}}\, T^{\pm(-1)^r}_{-r+1-t}(x)$ for any $t \in \mathbb{C}$

BD-type transfer matrices: first fundamental case

- ([Frassek, '20], [FT, '21]) Explicit $SO(\mathrm{K})$-type Lax matrices $\mathcal{L}(x)$
- $\mathcal{L}(x)$ is quadratic in x and depends on $\mathrm{K}-2$ pairs of oscillators
- $\mathrm{F} \simeq (M^{\{2,\ldots,r\}}_{t\omega_1})^* \simeq ({}^{\{2,\ldots,r\}}M_{t\omega_1})^*$
  $t \in \mathbb{Z}_{\geq 0} \rightsquigarrow |0\rangle$ generates finite-dimensional $L_{t\omega_1}$
- $\{1,\ldots,r,r',\ldots,1'\} \leftrightarrow W/W_{\{2,\ldots,r\}} \leftrightarrow {}^{\{2,\ldots,r\}}W$
- $\mathcal{L}(x)$ + similarity and particle-hole $\rightsquigarrow$ Lax $\{\mathcal{L}_k(x), \mathcal{L}_{k'}(x)\}_{k=1}^r$
- Transfer matrices: $T^+_{k,t}(x)$, $T^+_{k',t}(x)$, and finite-dimensional $T_{1,t}(x)$
- **Theorem (FKP '21)**=[Ferrando-Frassek-Kazakov '20, Conjecture]:

$$T_{1,t}(x) = \sum_{k=1}^{r}(-1)^{k-1}\, T^+_{k,t}(x) + \sum_{k=1}^{r}(-1)^{k-1+\mathrm{K}}\, T^+_{k',t}(x), \quad \forall\, t \in \mathbb{Z}_{\geq 0}$$

Part III

- **Fusion and Factorisation into Q-operators**

Type A: recap

- For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Lambda^+$, we have:

$$T_{M^*_\lambda}(x) \sim Q_1(x+\ell_1)Q_2(x+\ell_2)\cdots Q_n(x+\ell_n), \quad \ell_i = \lambda_i - i + 1$$

with Q-operators $Q_i(x)$ arising from degenerate Lax matrices $L_i(x)$:

  - "degenerate" as they contain x only in the (i,i)-entry
  - $L_i(x)_{kl} = \delta_{kl}$ for $k, l \neq i$
  - "normalized limits" of the nondegenerate Lax matrices realizing M^*_λ

- Combining with the BGG relation, we get ([BFLMS '12]):

$$T_{L_\lambda}(x) = \prod_{1\leq i<j\leq n} \left(\tau_j^{-1} - \tau_i^{-1}\right)^{-1} \cdot \det \left\| \tau_i^{\ell_j} Q_i(x+\ell_j) \right\|_{1\leq i,j\leq n}, \ \lambda \in \Lambda^+$$

- $\{Q_i(x)\}_{i=1}^n \rightsquigarrow \{Q_I(x)\}_{I\subset\{1,\ldots,n\}}$ satisfying QQ-relations:

$$Q_{I\sqcup i\sqcup j}(x+\tfrac{1}{2})Q_I(x-\tfrac{1}{2}) = \frac{\tau_j}{\tau_j - \tau_i} Q_{I\sqcup i}(x-\tfrac{1}{2})Q_{I\sqcup j}(x+\tfrac{1}{2}) - \frac{\tau_i}{\tau_j - \tau_i} Q_{I\sqcup j}(x-\tfrac{1}{2})Q_{I\sqcup i}(x+\tfrac{1}{2}), \qquad i,j \notin I$$

Fusion in C-type

- Relabel $\mathbf{A} \rightsquigarrow \mathbf{A}_1, \bar{\mathbf{A}} \rightsquigarrow \bar{\mathbf{A}}_1$, encoding $\{\mathbf{a}_{j',i}, \bar{\mathbf{a}}_{i,j'}\}_{1\le i\le j\le r}$
- Obtain $\mathbf{A}_2, \bar{\mathbf{A}}_2$, encoding $\{\bar{\mathbf{a}}_{j',i}, \mathbf{a}_{i,j'}\}_{1\le i\le j\le r}$, from $-\bar{\mathbf{A}}_1, \mathbf{A}_1$ via

$$\mathrm{P.H.}: \quad \bar{\mathbf{a}}_{i,j'} \mapsto -\mathbf{a}_{i,j'}, \quad \mathbf{a}_{j',i} \mapsto \bar{\mathbf{a}}_{j',i}$$

- $L_{(+,\ldots,+)}(x) := \begin{pmatrix} x\mathrm{I}_r - \bar{\mathbf{A}}_1\mathbf{A}_1 & \bar{\mathbf{A}}_1 \\ -\mathbf{A}_1 & \mathrm{I}_r \end{pmatrix}$ is degenerate C_r-type Lax as:

$$L_{(+,\ldots,+)}(x) = \lim_{t\to\infty} \left\{\mathcal{L}(x-t)\cdot \mathrm{diag}\Big(1,\cdots,1;-1/2t,\cdots,-1/2t\Big)\right\}$$

- "Opposite" degenerate C_r-type Lax matrix:

$$L_{(-,\ldots,-)}(x) = \begin{pmatrix} \mathrm{I}_r & \mathrm{J}_r\bar{\mathbf{A}}_2\mathrm{J}_r \\ \mathrm{J}_r\mathbf{A}_2\mathrm{J}_r & x\mathrm{I}_r + \mathrm{J}_r\mathbf{A}_2\bar{\mathbf{A}}_2\mathrm{J}_r \end{pmatrix}, \ \mathrm{J}_r = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \ldots & 0 \end{pmatrix}$$

- **Key Observation:** $\mathcal{L}(x)$ is a *fusion* of these two degenerate Lax:

$$L_{(+,\ldots,+)}(x+t)L_{(-,\ldots,-)}(x-t-r-1) = \mathbf{S}\mathcal{L}(x)\mathrm{G}\mathbf{S}^{-1} \quad \text{with}$$

$$\mathbf{S} = \exp\left[\sum_{1\le i\le j\le r} (1+\delta_{ij})\,\bar{\mathbf{a}}_{ij'}\mathbf{a}_{ij'}\right], \quad \mathrm{G} = \begin{pmatrix} \mathrm{I}_r & \bar{\mathbf{A}}_2 \\ 0 & \mathrm{I}_r \end{pmatrix}$$

Two-term factorisation: C-type

- $D_{(+,\dots,+)} := \prod_{i\leq j\leq r} (\tau_i\tau_j)^{-\bar{\mathbf{a}}_{ij'}\mathbf{a}_{j'i}}$, $D_{(-,\dots,-)} := \prod_{i\leq j\leq r} (\tau_i\tau_j)^{-\bar{\mathbf{a}}_{j'i}\mathbf{a}_{ij'}}$
- Define Q-operators $Q_{(+,\dots,+)}(x), Q_{(-,\dots,-)}(x) \in \mathrm{End}(\mathbb{C}^{2r})^{\otimes N}$ via:

$$Q_{(\pm,\dots,\pm)}(x) = \widehat{\mathrm{tr}}_{D_{(\pm,\dots,\pm)}}\Big(\underbrace{L_{(\pm,\dots,\pm)}(x)\otimes\cdots\otimes L_{(\pm,\dots,\pm)}(x)}_{N}\Big)$$

with $\widehat{\mathrm{tr}}_Y(X) := \mathrm{tr}_F(YX)/\mathrm{tr}_F(Y)$

- Then: $T^+_{(+1,\dots,+1),t}(x) \sim Q_{(+,\dots,+)}(x+t)Q_{(-,\dots,-)}(x-t-r-1)$
- Likewise: $T^+_{\vec{\mu},t}(x) \sim Q_{\vec{\mu}}(x+t)Q_{-\vec{\mu}}(x-t-r-1) \quad \forall\, \vec{\mu}\in\{\pm 1\}^r$

Two-term factorisation: BD-types

- For the spinor D-type case, this goes through word-by-word $\rightsquigarrow$

$$T_t^{\pm}(x) = \sum_{\vec{\mu}\in\{\pm 1\}^r}^{\mu_1\cdots\mu_r=\pm 1} (-1)^{\mathrm{sgn}\,\vec{\mu}} \mathrm{ch}^{+}_{\vec{\mu},t} \cdot Q_{\vec{\mu}}(x+t)Q_{-\vec{\mu}}(x-t-r+1)$$

- For the first fundamental case in $SO(\mathsf{K})$-types, the fusion looks as:

$$L_1(x-1+(2t+\mathsf{K})/4)L_{1'}(x-(2t+\mathsf{K})/4) = \mathbf{S}\mathcal{L}(x)\mathsf{G}\mathbf{S}^{-1}$$

with $L_1(x), L_{1'}(x)$–two "opposite" degenerate $SO(\mathsf{K})$-type Lax with

$$L_1(x) = \lim_{t\to\infty} \left\{ \mathcal{L}\,(x+1-(2t+\mathsf{K})/4)\cdot \mathrm{diag}(1; -t^{-1}, \cdots, -t^{-1}; t^{-2}) \right\}$$

Part IV

- **Shifted Drinfeld Yangians and GKLO-type homomorphisms**

Yangians $Y(\mathfrak{g})$

- $Y(\mathfrak{g})$ are Hopf algebras quantizing $U(\mathfrak{g}[z])$ and $\mathbb{C}\Big[G_1[[z^{-1}]]\Big]$
- Yangians $Y(\mathfrak{g})$ admit three realizations:
  - Original J-realization
    (finitely many generators, explicit Hopf algebra structure)
  - New Drinfeld (=loop) realization
    (best for representation theory, Hopf algebra structure - harder)
  - RTT realization
    (explicit Hopf algebra structure, good for representation theory; motivated and best suitable for integrable systems)
- Drinfeld-Gavarini duality provides a Hopf subalgebra $Y'_\hbar(\mathfrak{g}) \subset Y_\hbar(\mathfrak{g})$ quantizing a commutative algebra $\mathbb{C}\Big[G_1[[z^{-1}]]\Big]$:

$$Y'_\hbar(\mathfrak{g}) = \Big\{ a \in Y_\hbar(\mathfrak{g}) \,\Big|\, (\mathrm{Id} - \epsilon)^{\otimes n} \circ \Delta^{(n-1)}(a) \,\vdots\, \hbar^n \quad \forall\, n \geq 1 \Big\}$$

$\Delta^{(n-1)} \colon Y_\hbar(\mathfrak{g}) \to Y_\hbar(\mathfrak{g})^{\otimes n}$–iterated coproduct, $\epsilon \colon Y_\hbar(\mathfrak{g}) \to \mathbb{C}$–counit

Shifted Yangians

- μ–coweight of $\mathfrak{g} \rightsquigarrow Y_\mu(\mathfrak{g})$ generated by

$$\{H_i^{(k)}\}_{1\le i\le r}^{k\ge -(\mu,\alpha_i)} \cup \{F_i^{(k)}\}_{1\le i\le r}^{k\ge 1} \cup \{E_i^{(k)}\}_{1\le i\le r}^{k\ge 1}$$

with $H_i^{(-(\mu,\alpha_i))}=1$ and the usual Yangian defining relations:

$$[E_i^{(k)}, F_j^{(\ell)}] = \delta_{ij} H_i^{(k+\ell-1)}$$

$$[H_i^{(k+1)}, E_j^{(\ell)}] - [H_i^{(k)}, E_j^{(\ell+1)}] = \frac{(\alpha_i,\alpha_j)}{2}(H_i^{(k)}E_j^{(\ell)} + E_j^{(\ell)}H_i^{(k)})$$

$$[H_i^{(k+1)}, F_j^{(\ell)}] - [H_i^{(k)}, F_j^{(\ell+1)}] = -\frac{(\alpha_i,\alpha_j)}{2}(H_i^{(k)}F_j^{(\ell)} + F_j^{(\ell)}H_i^{(k)})$$

$$[E_i^{(k+1)}, E_j^{(\ell)}] - [E_i^{(k)}, E_j^{(\ell+1)}] = \frac{(\alpha_i,\alpha_j)}{2}(E_i^{(k)}E_j^{(\ell)} + E_j^{(\ell)}E_i^{(k)})$$

$$[F_i^{(k+1)}, F_j^{(\ell)}] - [F_i^{(k)}, F_j^{(\ell+1)}] = -\frac{(\alpha_i,\alpha_j)}{2}(F_i^{(k)}F_j^{(\ell)} + F_j^{(\ell)}F_i^{(k)})$$

and Serre relations

Classical limit and Coproduct

- Define the infinite type scheme $\mathcal{W}_\mu \subset G((z^{-1}))$:

$$\pi\colon U[z]\backslash U((z^{-1}))T_1[[z^{-1}]]z^\mu U_-((z^{-1}))/U_-[z] \xrightarrow{\sim} U_1[[z^{-1}]]T_1[[z^{-1}]]z^\mu U_{-,1}[[z^{-1}]] =: \mathcal{W}_\mu$$

- Multiplication maps

$$m_{\mu_1,\mu_2}\colon \mathcal{W}_{\mu_1} \times \mathcal{W}_{\mu_2} \longrightarrow \mathcal{W}_{\mu_1+\mu_2}, \quad (g_1, g_2) \mapsto \pi(g_1 \cdot g_2)$$

- **Theorem (Finkelberg-Kamnitzer-Pham-Rybnikov-Weekes '16):** $Y_\mu(\mathfrak{g})$ quantizes $\mathcal{W}_\mu$:

$$\mathrm{gr}\, Y_\mu(\mathfrak{g}) \simeq \mathbb{C}[\mathcal{W}_\mu]$$

- **Theorem (Main Construction of [FKPRW]):** For any $\mathfrak{g}$, μ_1, μ_2:

$$\Delta_{\mu_1,\mu_2}\colon Y_{\mu_1+\mu_2}(\mathfrak{g}) \longrightarrow Y_{\mu_1}(\mathfrak{g}) \otimes Y_{\mu_2}(\mathfrak{g})$$

homomorphisms, coassociative for $\mu_1, \mu_2, \mu_3 \leq 0$, and $\Delta_{0,0} = \Delta$

- **Theorem (FPT&FT '20-21)**=[FKPRW, Conjecture]:

$$m^*_{\mu_1,\mu_2} = \mathrm{gr}\,\Delta_{\mu_1,\mu_2}\colon \mathbb{C}[\mathcal{W}_{\mu_1+\mu_2}] \longrightarrow \mathbb{C}[\mathcal{W}_{\mu_1}] \otimes \mathbb{C}[\mathcal{W}_{\mu_2}]$$

for classical $\mathfrak{g}$, i.e. Δ_{μ_1,μ_2} quantize m_{μ_1,μ_2}

Truncated shifted Yangians

- Images of $Y_\mu(\mathfrak{g})$ under the GKLO-type homomorphisms $\rightsquigarrow$

$$\text{Truncated Shifted Yangians} \qquad Y_\mu^\lambda(\mathfrak{g}) = \Phi_\mu^\lambda(Y_\mu(\mathfrak{g}))$$

- **Theorem (Braverman-Finkelberg-Nakajima '16):** For $\mu \geq 0$, $Y_\mu^\lambda(\mathfrak{g})$ quantizes the slice in the thick affine Grassmannian:

$$\mathrm{Gr}_\mu^{\overline{\lambda}} = \overline{G[t]t^\lambda} \cap G_1[[t^{-1}]]t^{w_0\mu} \subset G((t^{-1}))/G[t]$$

- **Theorem (Brundan-Kleshchev '06):** There is an isomorphism

$$Y_{-\pi}(\mathfrak{gl}_n)/(\text{explicit Cartan elements}) \simeq U(\mathfrak{gl}_N, e_\pi)$$

the latter being the finite W-algebra quantizing the Slodowy slice

$$\mathcal{S}_{e_\pi} = e_\pi + \mathfrak{c}_{\mathfrak{gl}_N}(f_\pi) \subset \mathfrak{gl}_N, \quad \text{where} \quad (e_\pi, h_\pi, f_\pi) \simeq \mathfrak{sl}_2$$

$\pi = (p_1 \leq \ldots \leq p_n)$, $N = p_1 + \ldots + p_n$, $e_\pi \in \mathfrak{gl}_N$–type π nilpotent

GKLO-type homomorphisms

- $\lambda = \omega_{i_1} + \ldots + \omega_{i_N}$; $\{z_k\}_{k=1}^N \in \mathbb{C} \rightsquigarrow Z_i(z) := \prod_{1\leq k\leq N}^{i_k=i}(z-z_k)$
- Assume: $\lambda - \mu = \sum a_i \alpha_i^\vee$ with $a_i \in \mathbb{N}$, $\alpha_i^\vee$-simple coroots
- $\tilde{\mathcal{A}} := \mathbb{C}\langle w_{i,k}\,,\ \mathrm{u}_{i,k}^{\pm 1}\,,\ (w_{i,k} - w_{i,\ell} + m\mathrm{d}_i)^{-1}\rangle_{1\leq i\leq r, m\in\mathbb{Z}}^{1\leq k\neq \ell\leq a_i}$ with $[\mathrm{u}_{i,k}, w_{j,\ell}] = \mathrm{d}_i\delta_{ij}\delta_{k\ell}\mathrm{u}_{i,k}$, $[w_{i,k}, w_{j,\ell}] = 0 = [\mathrm{u}_{i,k}, \mathrm{u}_{j,\ell}]$, $\mathrm{d}_i = \frac{(\alpha_i,\alpha_i)}{2}$
- $W_i(z) := \prod_{k=1}^{a_i}(z - w_{i,k})$, $W_{i,\ell}(z) := \prod_{1\leq k\leq a_i}^{k\neq\ell}(z-w_{i,k})$
- GKLO-type algebra homomorphism $\Phi_\mu^\lambda\colon Y_\mu(\mathfrak{g}) \to \tilde{\mathcal{A}}$ with

$$E_i(z) \mapsto \frac{1}{\mathrm{d}_i}\sum_{k=1}^{a_i} \frac{\prod_{j\to i}\prod_{p=1}^{-a_{ji}} W_j(w_{i,k}-\frac{1}{2}(\alpha_i,\alpha_j)-p\mathrm{d}_j)}{(z-w_{i,k})W_{i,k}(w_{i,k})}\mathrm{u}_{i,k}^{-1},$$

$$F_i(z) \mapsto -\sum_{k=1}^{a_i} \frac{Z_i(w_{i,k}+\mathrm{d}_i)\prod_{j\leftarrow i}\prod_{p=1}^{-a_{ji}} W_j(w_{i,k}+\mathrm{d}_i-\frac{1}{2}(\alpha_i,\alpha_j)-p\mathrm{d}_j)}{(z-w_{i,k}-\mathrm{d}_i)W_{i,k}(w_{i,k})}\mathrm{u}_{i,k},$$

$$H_i(z) \mapsto \frac{Z_i(z)\prod_{j-i}\prod_{p=1}^{-a_{ji}} W_j(z-\frac{1}{2}(\alpha_i,\alpha_j)-p\mathrm{d}_j)}{W_i(z)W_i(z-\mathrm{d}_i)}, \quad a_{ij} := \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$$

Shifted RTT Yangians: A type

- $Y^{\mathrm{rtt}}(\mathfrak{gl}_n) \rightsquigarrow$ Gauss decomposition $T(z) = F(z)H(z)E(z)$
- $Y^{\mathrm{rtt}}(\mathfrak{gl}_n) \simeq Y^{\mathrm{rtt}}(\mathfrak{sl}_n) \otimes Z(Y^{\mathrm{rtt}}(\mathfrak{gl}_n))$; center: $\mathrm{qdet}\, T(z)$
- **Theorem (Brundan-Kleshchev '05):** The assignment

$$E_i(z) \mapsto e_{i,i+1}(z), \quad F_i(z) \mapsto f_{i+1,i}(z), \quad D_i(z) \mapsto h_i(z)$$

gives rise to $Y(\mathfrak{gl}_n) \xrightarrow{\sim} Y^{\mathrm{rtt}}(\mathfrak{gl}_n)$ and $Y(\mathfrak{sl}_n) \xrightarrow{\sim} Y^{\mathrm{rtt}}(\mathfrak{sl}_n)$

- μ-dominant $\rightsquigarrow$ shifted RTT Yangian $Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)$:
  - generated by $t_{ij}^{(k)}$ with $1 \leq i, j \leq n$ and $k \in \mathbb{Z}$
  - RTT relation: $R(z-w)T_1(z)T_2(w) = T_2(w)T_1(z)R(z-w)$
  - Gauss decomposition $T(z) = F(z)H(z)E(z)$ is subject to

$$F(z) = o(1), \quad E(z) = o(1), \quad H(z) = z^{-\mu}(\mathrm{Id}_n + o(1))$$

- **Theorem (FPT '20):** For $\mu \geq 0$, there is an algebra isomorphism

$$\Upsilon_{-\mu}\colon\ Y_{-\mu}(\mathfrak{gl}_n) \xrightarrow{\sim} Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)$$

provided by the same assignment as in the above unshifted case

Applications

- Coassociative coproduct homomorphisms

$$Y^{\mathrm{rtt}}_{-\mu_1-\mu_2}(\mathfrak{gl}_n) \longrightarrow Y^{\mathrm{rtt}}_{-\mu_1}(\mathfrak{gl}_n) \otimes Y^{\mathrm{rtt}}_{-\mu_2}(\mathfrak{gl}_n), \quad T(z) \mapsto T(z) \otimes T(z)$$

recovering the type A construction of [FKPRW]:

$$\Delta_{\nu_1,\nu_2} \colon Y_{\nu_1+\nu_2}(\mathfrak{sl}_n) \longrightarrow Y_{\nu_1}(\mathfrak{sl}_n) \otimes Y_{\nu_2}(\mathfrak{sl}_n)$$

- Bethe subalgebras $B(C) \subset Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)$ generated by coefficients of

$$\mathrm{tr}_{(\mathbb{C}^n)^{\otimes k}} \Big(A_k C_1 \cdots C_k T_1(u) T_2(u-1) \cdots T_k(u-k+1) \Big)$$

where $C \in \mathrm{End}\, \mathbb{C}^n$, A_k is the antisymmetrizer, $k \in \{1, \ldots, n\}$

- $B(C) \rightsquigarrow$ commutative subalgebras of quantized Coulomb branches (providing a new interesting family of quantum integrable systems)
- In the trigonometric setup, recover a natural $\mathbb{C}[q, q^{-1}]$ integral form

Type A degenerate Lax matrices

- $\lambda, \mu, \{z_k\} \rightsquigarrow \Lambda^+$-valued divisor on $\mathbb{CP}^1$: $D = \sum_{k=1}^{N} \omega_{i_k}[z_k] + \mu[\infty]$
- Lax $T_D(z) =$ the image of $T(z)$ under $Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n) \xrightarrow{\sim} Y_{-\mu}(\mathfrak{gl}_n) \to \tilde{\mathcal{A}}$
- If D contains $\omega_{i_k}[z_k]$ and $D' = D + \omega_{i_k}([\infty] - [z_k])$, then:

$$T_{D'}(z) = \lim_{t\to\infty} \left\{ T_D(z) \cdot (-z_k)^{\omega_{i_k}} \right\}$$

- **Theorem (FPT):** $T_D(z)$ is polynomial in z
- Example ($n = 2$, $\lambda + \mu = 2\omega = \alpha^\vee$, $w \leftrightarrow p$, $\mathrm{u} \leftrightarrow e^{-q}$):
  - for $\lambda = 0, \mu = 2\omega$, get local Lax matrix for the Toda chain

$$T_D(z) = \begin{pmatrix} z - p & -e^q \\ e^{-q} & 0 \end{pmatrix}$$

  - for $\lambda = \omega, \mu = \omega$, get local Lax matrix for the DST chain

$$T_D(z) = \begin{pmatrix} z - p & -(p - z_1)e^q \\ e^{-q} & 1 \end{pmatrix}$$

  - for $\lambda = 2\omega, \mu = 0$, get local Lax matrix for the Heisenberg magnet

$$T_D(z) = \begin{pmatrix} z - p & -(p - z_1)(p - z_2)e^q \\ e^{-q} & z + p + 1 - z_1 - z_2 \end{pmatrix}$$

RTT realization of BCD antidominantly shifted Yangians

- $\mathfrak{g} = \mathfrak{so}_N, \mathfrak{sp}_N \rightsquigarrow X^{\mathrm{rtt}}(\mathfrak{g}) \rightsquigarrow$ Gauss decomp. $T(z) = F(z)H(z)E(z)$
- $X^{\mathrm{rtt}}(\mathfrak{g}) \simeq Y^{\mathrm{rtt}}(\mathfrak{g}) \otimes Z(X^{\mathrm{rtt}}(\mathfrak{g}))$, center is polynomial in $\mathfrak{z}_k$:

$$T(z)T'(z-\kappa) = T'(z-\kappa)T(z) = \mathfrak{z}(z)\mathrm{Id}_N$$

- **Theorem (Jing-Liu-Molev '17)**: The assignment

$$D_j(z) \mapsto h_j(z),\ E_i(z) \mapsto e_{i,i+1}(z),\ F_i(z) \mapsto f_{i+1,i}(z), i < r, j \le r+1$$

$$E_r(z) \mapsto \begin{cases} e_{r-1,r+1}(z) \\ e_{r,r+1}(z) \\ \frac{1}{2}e_{r,r+1}(z) \end{cases}, \quad F_r(z) \mapsto \begin{cases} f_{r+1,r-1}(z) & \text{if } \mathfrak{g} = \mathfrak{so}_{2r} \\ f_{r+1,r}(z) & \text{if } \mathfrak{g} = \mathfrak{so}_{2r+1} \\ f_{r+1,r}(z) & \text{if } \mathfrak{g} = \mathfrak{sp}_{2r} \end{cases}$$

gives rise to $X(\mathfrak{g}) \xrightarrow{\sim} X^{\mathrm{rtt}}(\mathfrak{g})$ and $Y(\mathfrak{g}) \xrightarrow{\sim} Y^{\mathrm{rtt}}(\mathfrak{g})$

- μ-dominant $\rightsquigarrow$ shifted extended RTT Yangian $X^{\mathrm{rtt}}_{-\mu}(\mathfrak{g})$
- **Theorem (FT '21):** For $\mu \geq 0$, there is an algebra isomorphism

$$\Upsilon_{-\mu}:\ X_{-\mu}(\mathfrak{g}) \xrightarrow{\sim} X^{\mathrm{rtt}}_{-\mu}(\mathfrak{g})$$

provided by the same assignment as in the above unshifted case

Type BCD degenerate Lax matrices

- Λ^+-valued divisor D on $\mathbb{CP}^1 \rightsquigarrow$ Lax matrix $T_D(z)$
- **Theorem (FT):** $T_D(z)$ is polynomial in z
- *Technical ingredient:* shuffle realization of Φ^λ_μ
- For $D = \omega_1[x]$ in type C_r, get:

$$T_D(z) = (z - x - 1)\mathrm{Id}_{2r} + (\bar{\mathbf{a}}_1, \ldots, \bar{\mathbf{a}}_r, \mathbf{a}_r, \ldots, \mathbf{a}_1)^t \cdot (-\mathbf{a}_1, \ldots, -\mathbf{a}_r, \bar{\mathbf{a}}_r, \ldots, \bar{\mathbf{a}}_1)$$

- For $D = \omega_r[x] + \omega_r[\infty]$ in type C_r, get Lax matrices from Part III:

$$T_D(z) = \begin{pmatrix} (z+x)\mathrm{Id}_r - \bar{\mathbf{A}}\mathbf{A} & \bar{\mathbf{A}} \\ -\mathbf{A} & \mathrm{Id}_r \end{pmatrix}$$

- For $D = \begin{cases} \omega_{r-1}[x] + \omega_r[\infty], & r\mathrm{-odd} \\ \omega_r[x] + \omega_r[\infty], & r\mathrm{-even} \end{cases}$ in type D_r, get Lax matrices:

$$T_D(z) = \begin{pmatrix} (z+x)\mathrm{Id}_r - \bar{\mathbf{A}}\mathbf{A} & \bar{\mathbf{A}} \\ -\mathbf{A} & \mathrm{Id}_r \end{pmatrix}$$

The End

thank you!

SLAVA UKRAINE!

GEROYAM SLAVA!

Type BCD degenerate Lax matrices

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- These Lax matrices appeared in [Isaev-Karakhanyan-Kirschner '16], [Frassek '20], [Karakhanyan-Kirschener '20], [FT '21]