Title: Quantum Information 2021/2022

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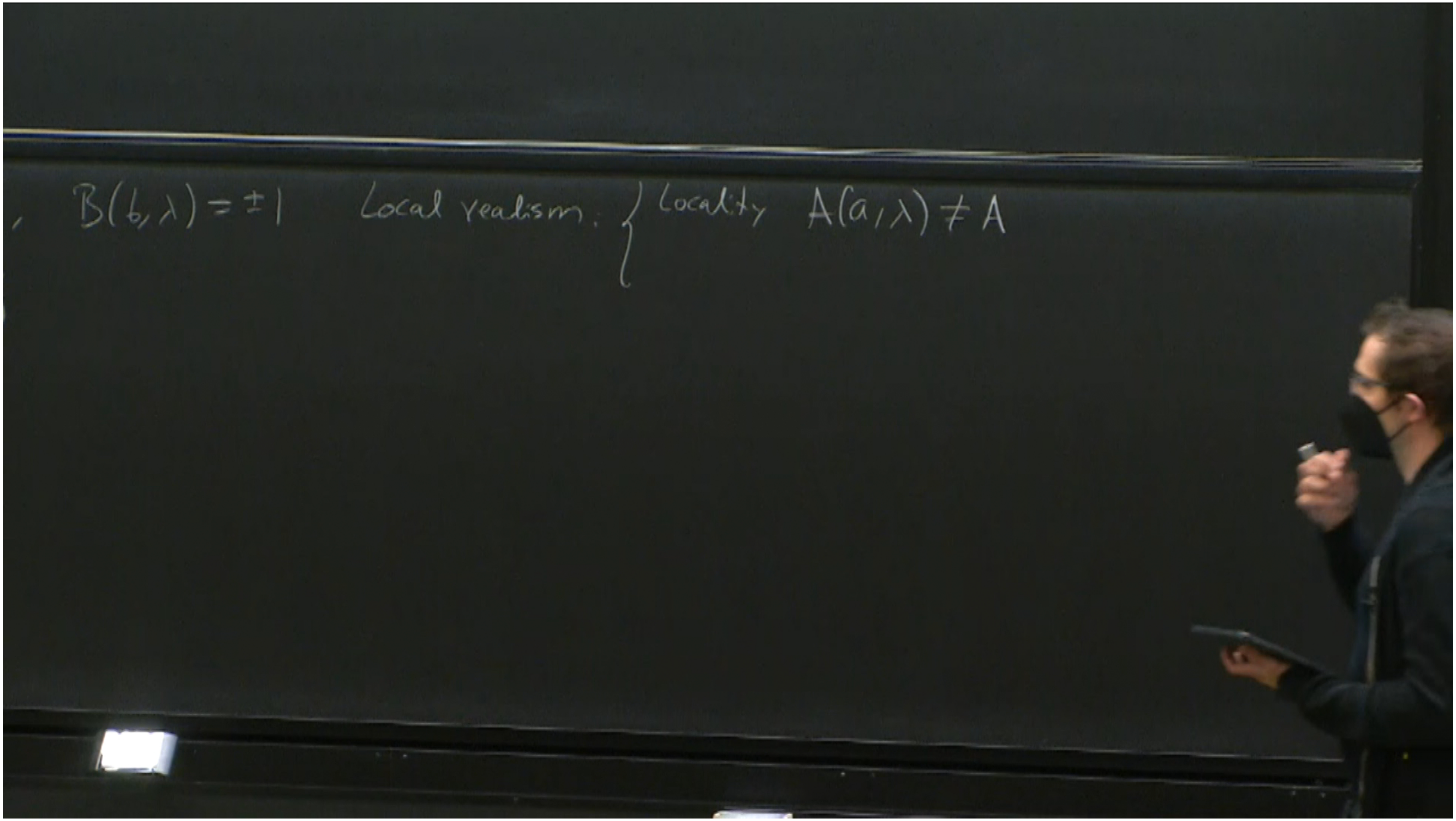
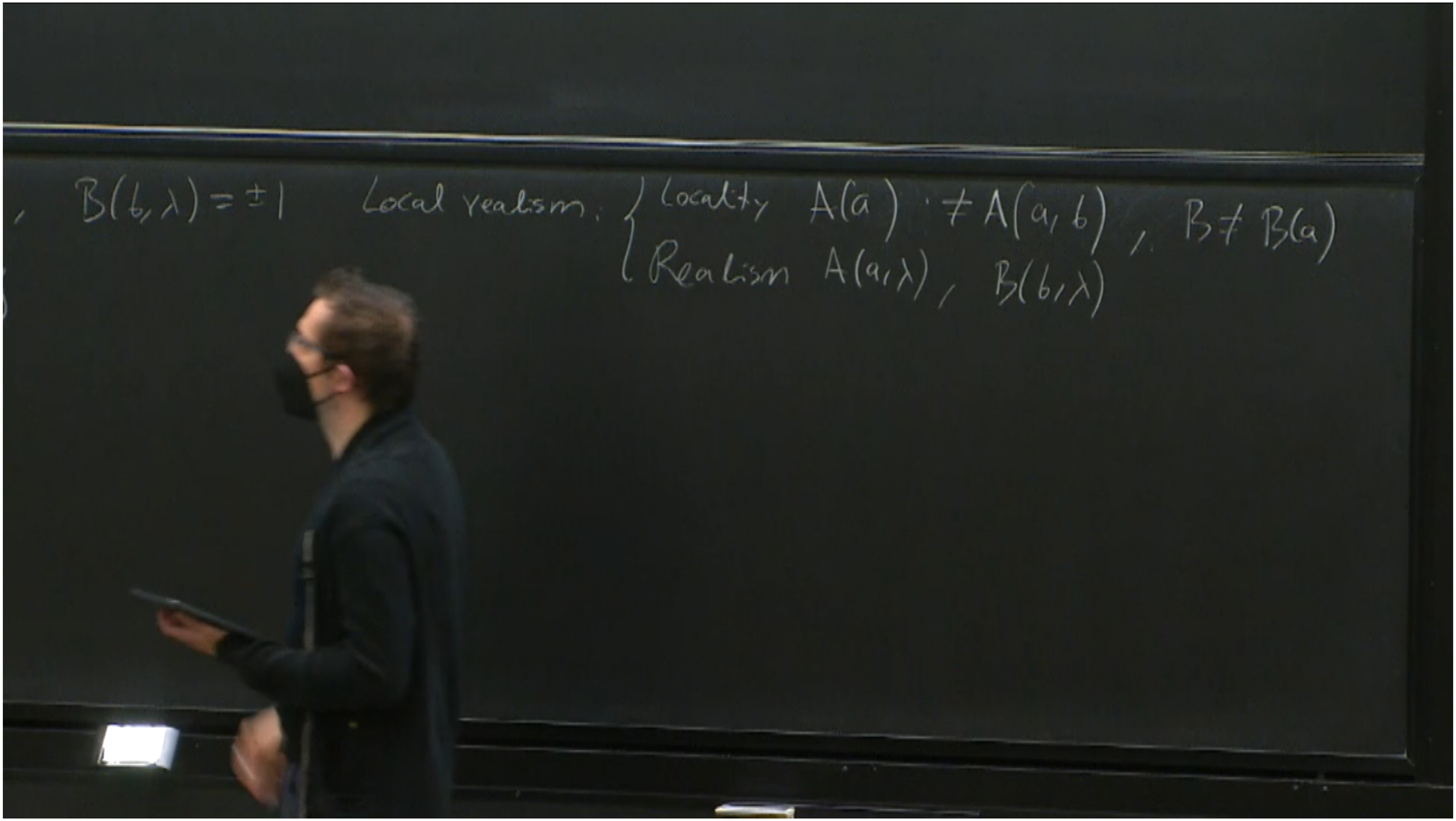
$$C(a,b) = \int d\lambda\, \Delta(\lambda)\, A(a,\lambda)\, B(b,\lambda) \qquad A(a,\lambda) = \pm 1, \quad B(b,\lambda) = \pm 1$$

Local realism: $\{$

Consider two different configurations for A and B : $\{a, a'\}$, $\{b, b'\}$


B(b, λ) = ±1
Local realism: { Locality A(a, λ) ≠ A



$B(b,\lambda) = \pm 1$
Local realism:
Locality $A(a) \neq A(a,b)$, $B \neq B(a)$
Realism $A(a,\lambda)$, $B(b,\lambda)$

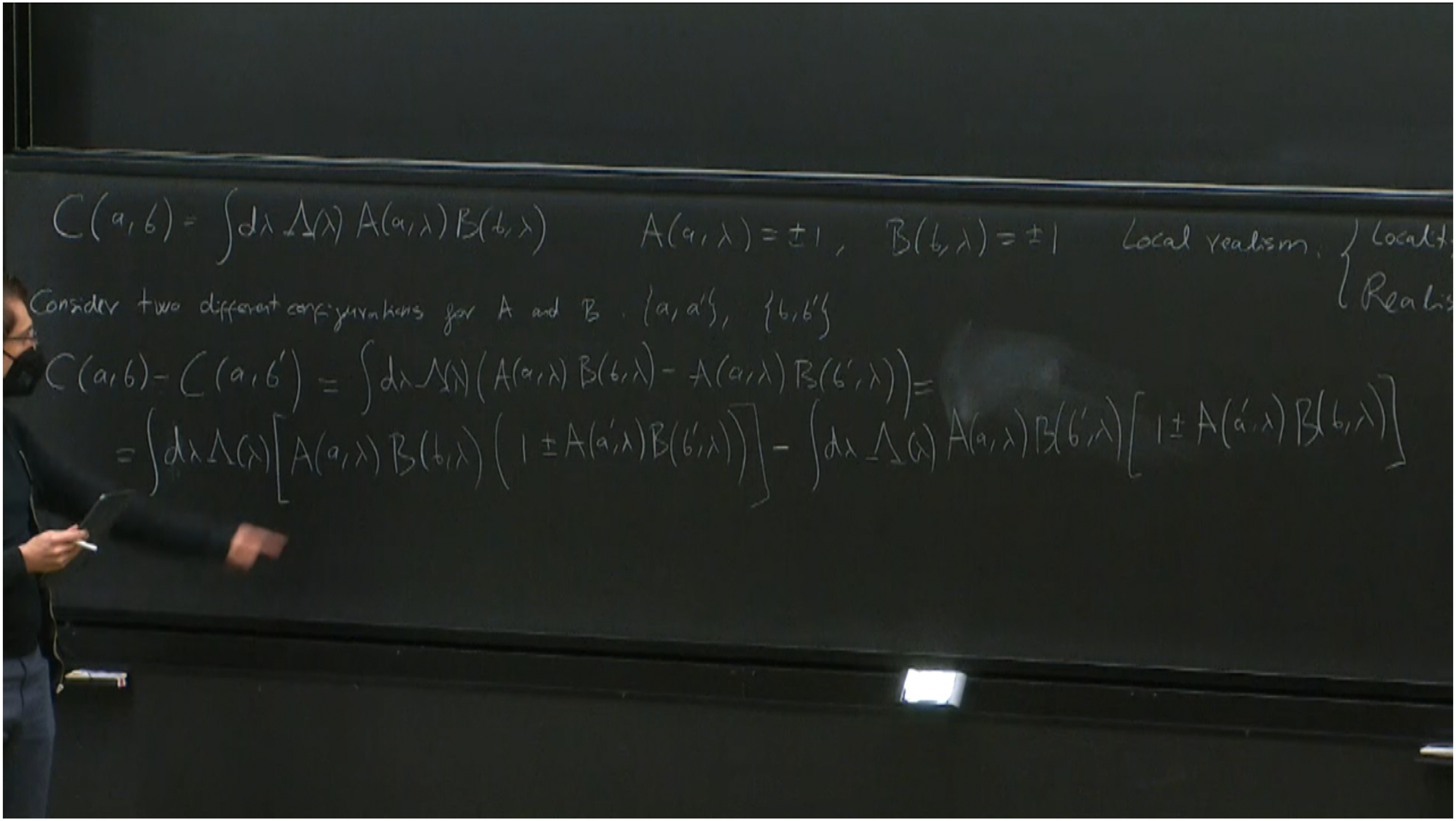
$$C(a,b) = \int d\lambda\, \Lambda(\lambda)\, A(a,\lambda)\, B(b,\lambda) \qquad A(a,\lambda) = \pm 1, \quad B(b,\lambda) = \pm 1$$

Lo

Consider two different configurations for A and B : $\{a, a'\}$, $\{b, b'\}$

$$C(a,b) - C(a,b') = \int d\lambda\, \Lambda(\lambda) \left(A(a,\lambda) B(b,\lambda) - A(a,\lambda) B(b',\lambda) \right) =$$

$$= \int d\lambda\, \Lambda(\lambda) \left[A(a,\lambda) B(b,\lambda) \left(1 \pm A(a',\lambda) B(b',\lambda) \right) \right] - \int d\lambda\, \Lambda(\lambda)\, A(a,\lambda)\, B(b'$$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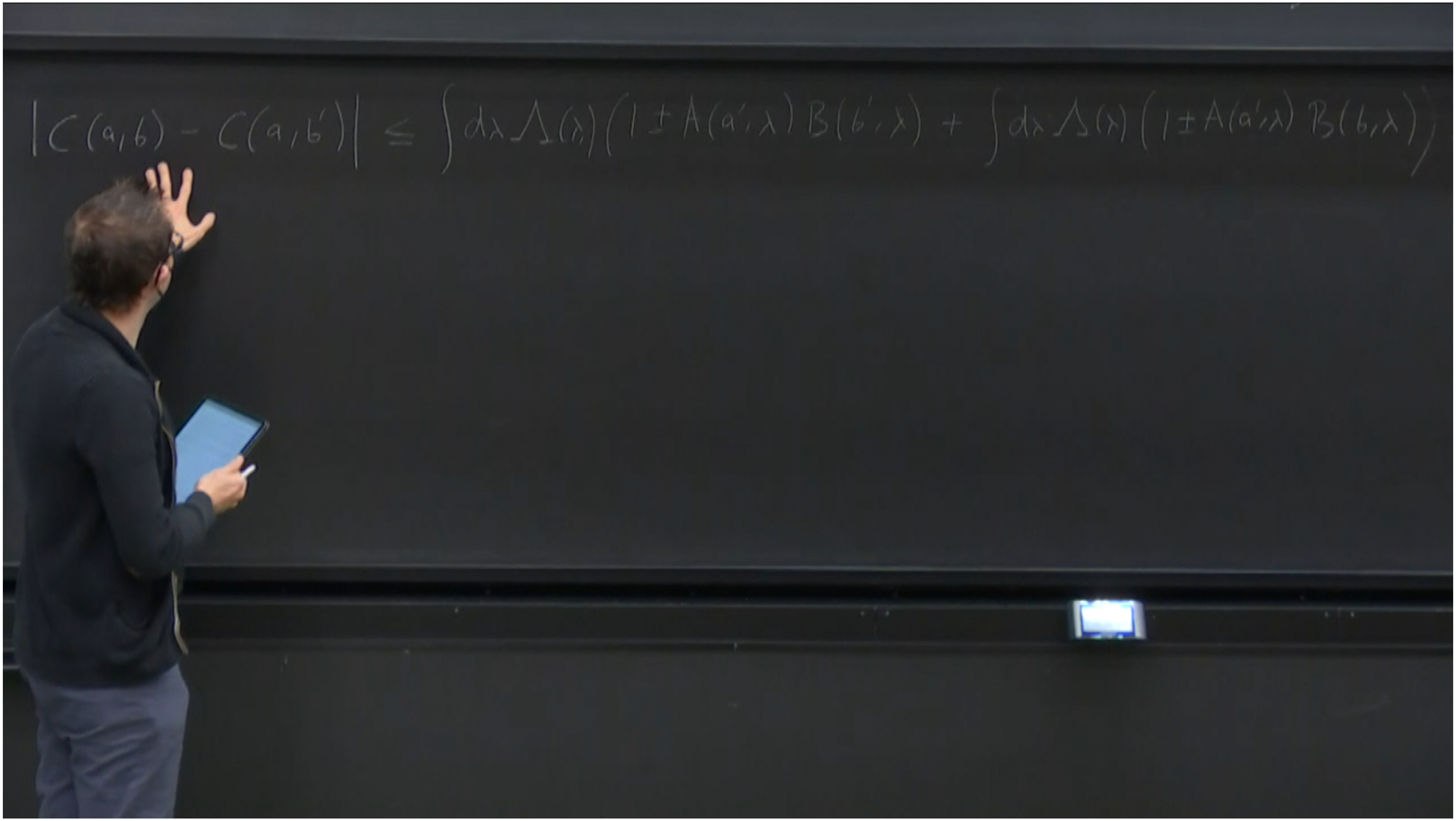
Since $|A(a,\lambda)| \leq 1 \qquad |B(b,\lambda)| \leq 1$


$|C(a,b) - C(a,b')| \leq \int d\lambda\, \Lambda(\lambda)\left(1 \pm A(a',\lambda)\, B(b',\lambda)\right) + \int d\lambda\, \Lambda(\lambda)\left(1 \pm A(a',\lambda)\, B(b,\lambda)\right)$

$$\left|C(a,b) - C(a,b')\right| \leq \int d\lambda\, \Lambda(\lambda)\left(1 \pm A(a',\lambda)\, B(b',\lambda)\right) + \int d\lambda\, \Lambda(\lambda)\,(\ldots$$

$$\boxed{\left|C(a,b) - C(a,b')\right| + \left|C(a',b') + C(a',b)\right| \leq 2}$$

$\|=1$

$$\int d\lambda \, \Lambda(\lambda) = 1$$

$$\int d\lambda \, \Lambda(\lambda) \left(1 \pm A(a',\lambda) \, B(b,\lambda) \right) = 2 \pm \left(C(a',b') + C(a',b) \right)$$

Bell Inequality

Consider a system of two two-level quantum systems

Let's consider a bipartite state $|\Psi_-\rangle = \frac{1}{\sqrt{2}} \left(|0_A 1_B\rangle - |1_A 0_B\rangle \right)$

Since $|A(a,\lambda)| \leq 1 \quad |B(b,\lambda)| \leq 1$

$$|C(a,b) - C(a,b')| \leq \int d\lambda\, \Lambda(\lambda)\left(1 \pm A(a',\lambda)\, B(b',\lambda)\right) + \int d\lambda\, \Lambda(\lambda)\left(1 \pm A(a',\lambda\right.$$

$$\boxed{|C(a,b) - C(a,b')| + |C(a',b') + C(a',b)| \leq 2}$$

Bell Inequality

Let's consider

$$A(a) = \hat{\sigma}_{z_A} \otimes \mathbb{1}_B$$

$$B(b) = \frac{-1}{\sqrt{2}}\, \mathbb{1}_A \otimes \left(\hat{\sigma}_{z_B} + \hat{\sigma}_{x_A}\right)$$

Since $|A(a,\lambda)| \leq 1$ $\quad |B(b,\lambda)| \leq 1$

$$|C(a,b) - C(a,b')| \leq \int d\lambda\, \Lambda(\lambda) \left(1 \pm A(a',\lambda) B(b',\lambda)\right) + \int d\lambda\, \Lambda(\lambda) \left(1 \pm A(a',\lambda \ldots\right.$$

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$\ldots,\lambda)| \leq 1$

$$\int d\lambda\, \Delta(\lambda) = 1$$

$$+ \int d\lambda\, \Delta(\lambda)\left(1 \pm A(a',\lambda)\, B(b,\lambda)\right) = 2 \pm \left(C(a',b') + C(a',b)\right)$$

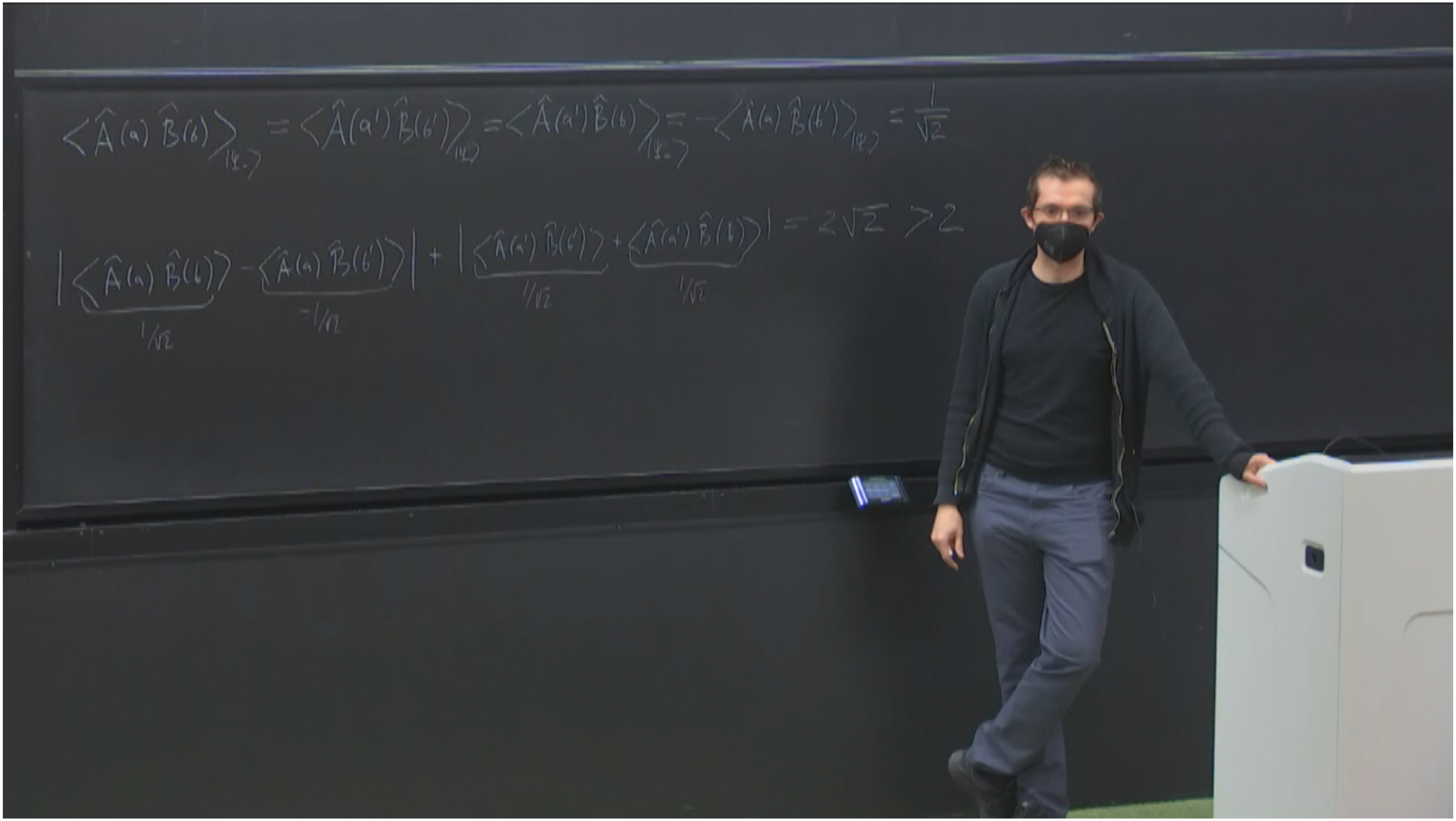
Bell Inequality

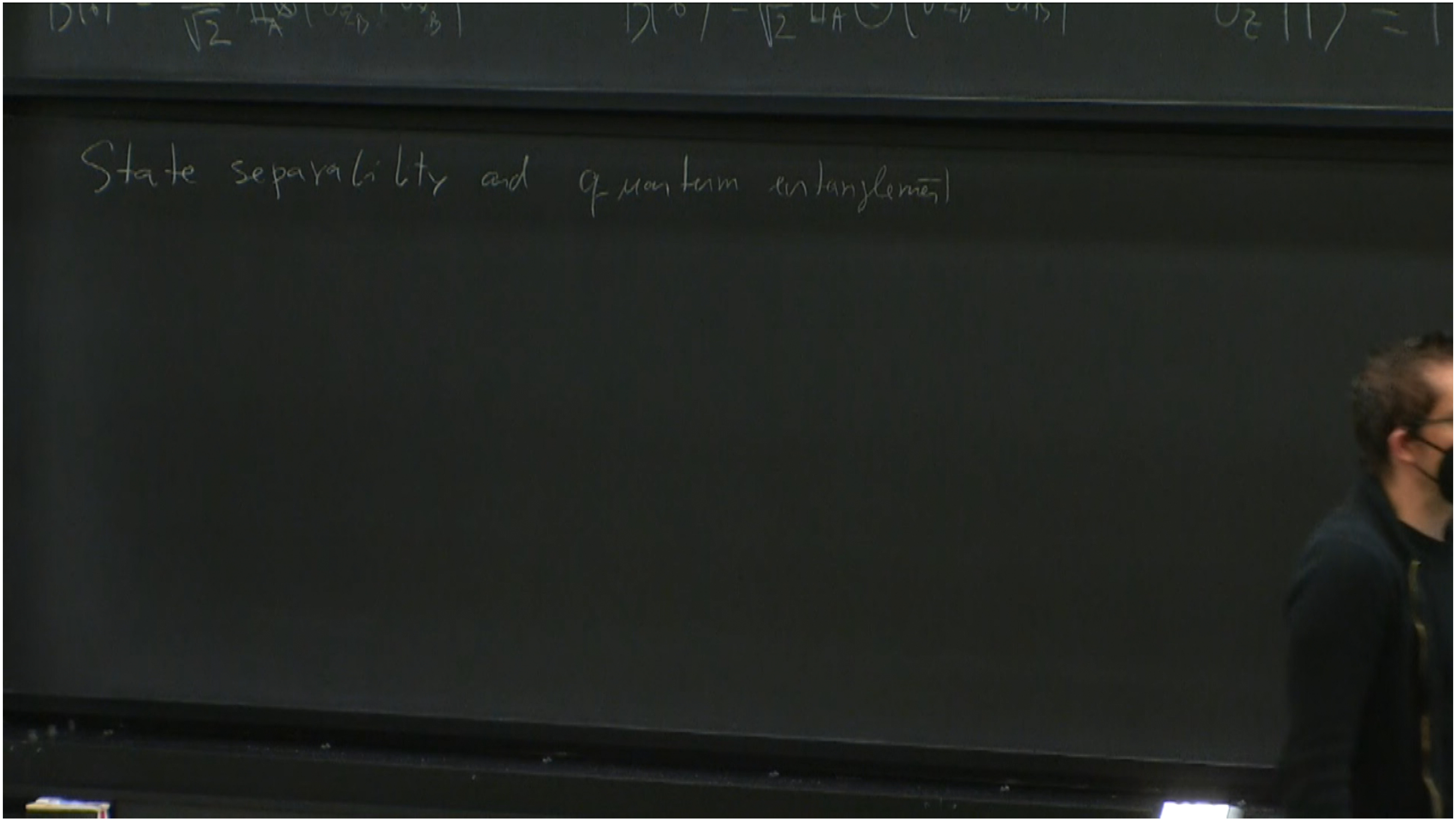
Consider a system of two two-level quantum systems

Let's consider a Bipartite state $|\Psi_-\rangle = \frac{1}{\sqrt{2}}\left(|0_A 1_B\rangle - |1_A 0_B\rangle\right)$

$$\hat{\sigma}_z|0\rangle = -|0\rangle \qquad \hat{\sigma}_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \qquad |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hat{\sigma}_z|1\rangle = |1\rangle \qquad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$


$\langle \hat{A}(a)\hat{B}(b)\rangle_{|\Psi_-\rangle} = \langle \hat{A}(a')\hat{B}(b')\rangle_{|\Psi_-\rangle} = \langle \hat{A}(a')\hat{B}(b)\rangle_{|\Psi_-\rangle} = -\langle \hat{A}(a)\hat{B}(b')\rangle_{|\Psi_-\rangle} = \frac{1}{\sqrt{2}}$
$\left|\underbrace{\langle \hat{A}(a)\hat{B}(b)\rangle}_{1/\sqrt{2}} - \underbrace{\langle \hat{A}(a)\hat{B}(b')\rangle}_{-1/\sqrt{2}}\right| + \left|\underbrace{\langle \hat{A}(a')\hat{B}(b')\rangle}_{1/\sqrt{2}} + \underbrace{\langle \hat{A}(a')\hat{B}(b)\rangle}_{1/\sqrt{2}}\right| = 2\sqrt{2} > 2$


State separability and quantum entanglement

$$\hat{B}(b) = \frac{1}{\sqrt{2}} \mathbb{1}_A \otimes (\hat{\sigma}_{z_B} + \hat{\sigma}_{x_B}) \qquad \hat{B}(b') = \frac{1}{\sqrt{2}} \mathbb{1}_A \otimes (\hat{\sigma}_{z_B} - \hat{\sigma}_{x_B}) \qquad \hat{\sigma}_z |1\rangle =$$

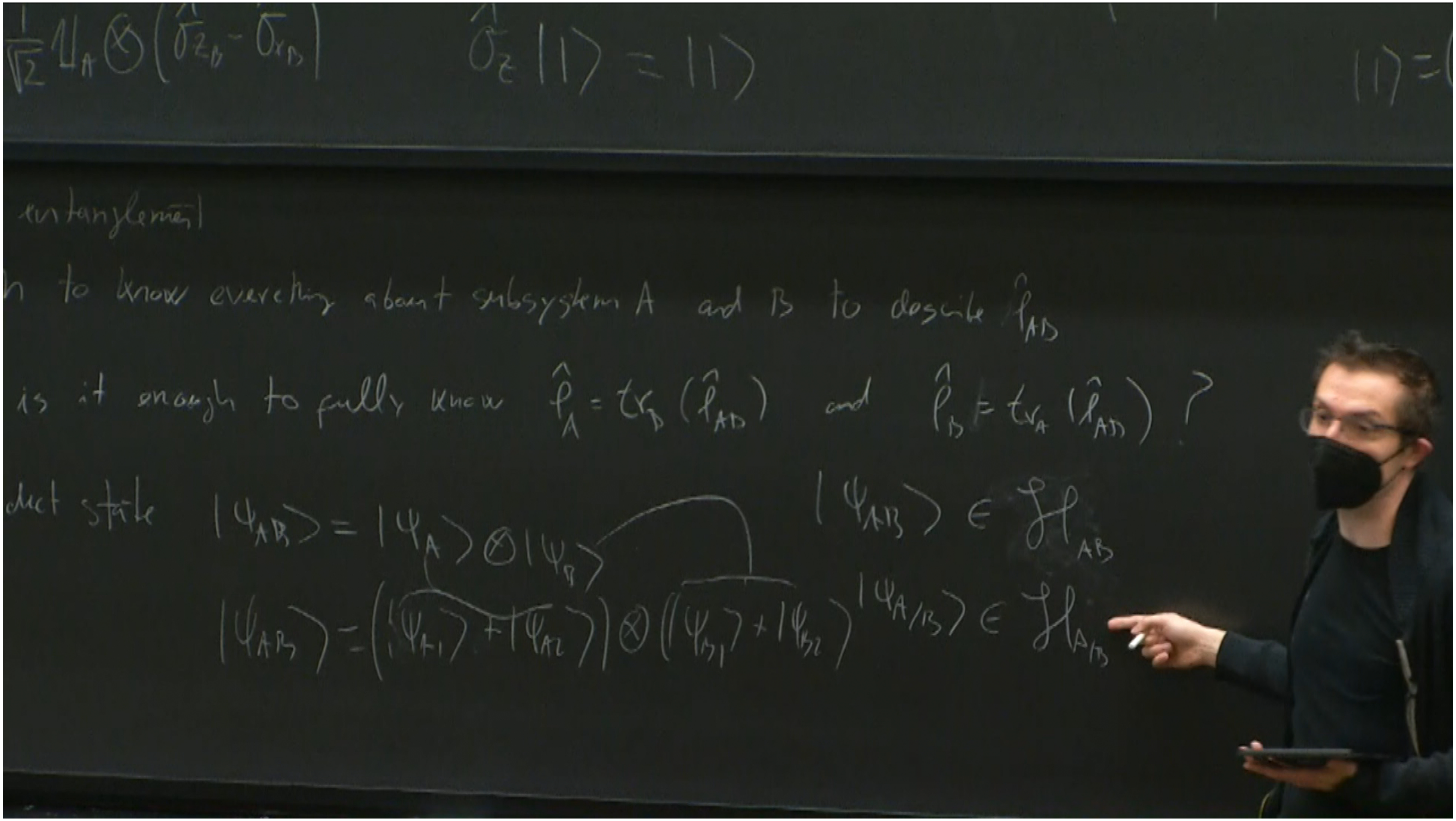
State separability and quantum entanglement

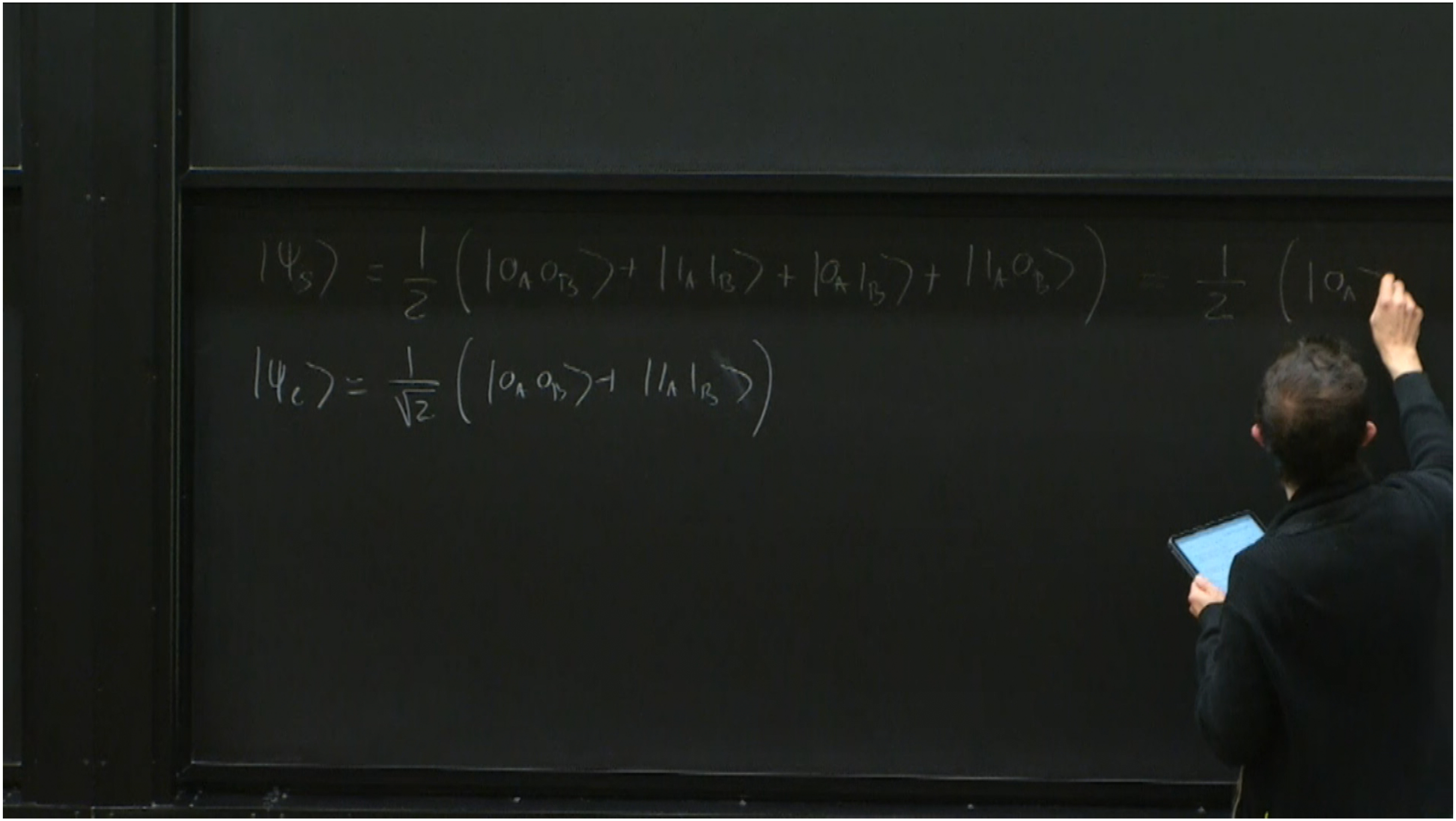
for a bipartite state $\hat{\rho}_{AB}$, is it enough to know everything about subsystem A

for a pure bipartite state $|\psi_{AB}\rangle$ is it enough to fully know $\hat{\rho}_A = \mathrm{tr}_B ($

It's true if the pure state is a product state $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$

$$|\psi_{AB}\rangle = (|\psi_{A_1}\rangle + |\psi_{A_2}\rangle) \otimes ((|$$


$\frac{1}{\sqrt{2}} \mathbb{1}_A \otimes (\hat{\sigma}_{z_B} - \hat{\sigma}_{x_B})$
$\hat{\sigma}_z |1\rangle = |1\rangle$
$|1\rangle = ($
entanglement
n to know everything about subsystem A and B to describe $\hat{\rho}_{AB}$
is it enough to fully know $\hat{\rho}_A = tr_B(\hat{\rho}_{AB})$ and $\hat{\rho}_B = tr_A(\hat{\rho}_{AB})$?
duct state $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$
$|\psi_{AB}\rangle \in \mathcal{H}_{AB}$
$|\psi_{AB}\rangle = (|\psi_{A1}\rangle + |\psi_{A2}\rangle) \otimes (|\psi_{B1}\rangle + |\psi_{B2}\rangle)$
$|\psi_{A/B}\rangle \in \mathcal{H}_{A/B}$


$|\psi_s\rangle = \frac{1}{2}\left(|0_A 0_B\rangle + |1_A 1_B\rangle + |0_A 1_B\rangle + |1_A 0_B\rangle\right) = \frac{1}{2}\left(|0_A\rangle\right.$
$|\psi_c\rangle = \frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle + |1_A 1_B\rangle\right)$

$$= \frac{1}{2} \underbrace{\left(|0_A\rangle + |1_A\rangle \right)}_{|\psi_s^A\rangle} \otimes \underbrace{\left(|0_B\rangle + |1_B\rangle \right)}_{|\psi_s^B\rangle}$$ product state

$$|\psi_S\rangle = \frac{1}{2}\Big(|0_A 0_B\rangle + |1_A 1_B\rangle + |0_A 1_B\rangle + |1_A 0_B\rangle\Big) = \frac{1}{2}\underbrace{\Big(|0_A\rangle + |1_A\rangle\Big)}_{|\psi_S^A\rangle} \otimes$$

$$|\psi_e\rangle = \frac{1}{\sqrt{2}}\Big(|0_A 0_B\rangle + |1_A 1_B\rangle\Big) \neq \Big(\alpha_A|0_A\rangle + \beta_A|1_A\rangle\Big) \otimes \Big(\alpha_B|0_B\rangle + \beta_B|1_B\rangle\Big)$$

$$|\psi_s\rangle = \frac{1}{2}\left(|0_A 0_B\rangle + |1_A 1_B\rangle + |0_A 1_B\rangle + |1_A 0_B\rangle\right) = \frac{1}{2}\underbrace{\left(|0_A\rangle + |1_A\rangle\right)}_{|\psi_s^A\rangle} \otimes$$

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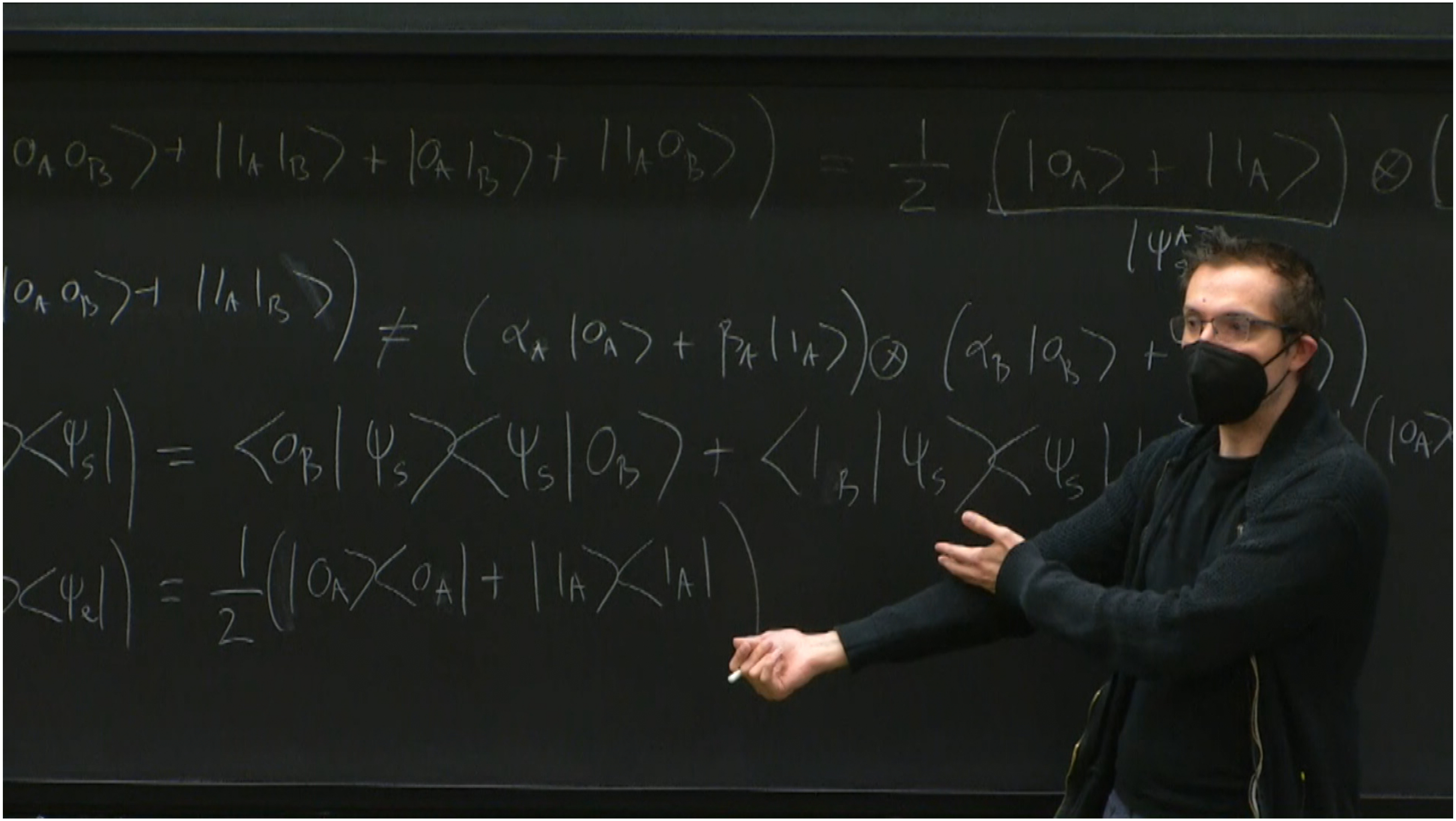
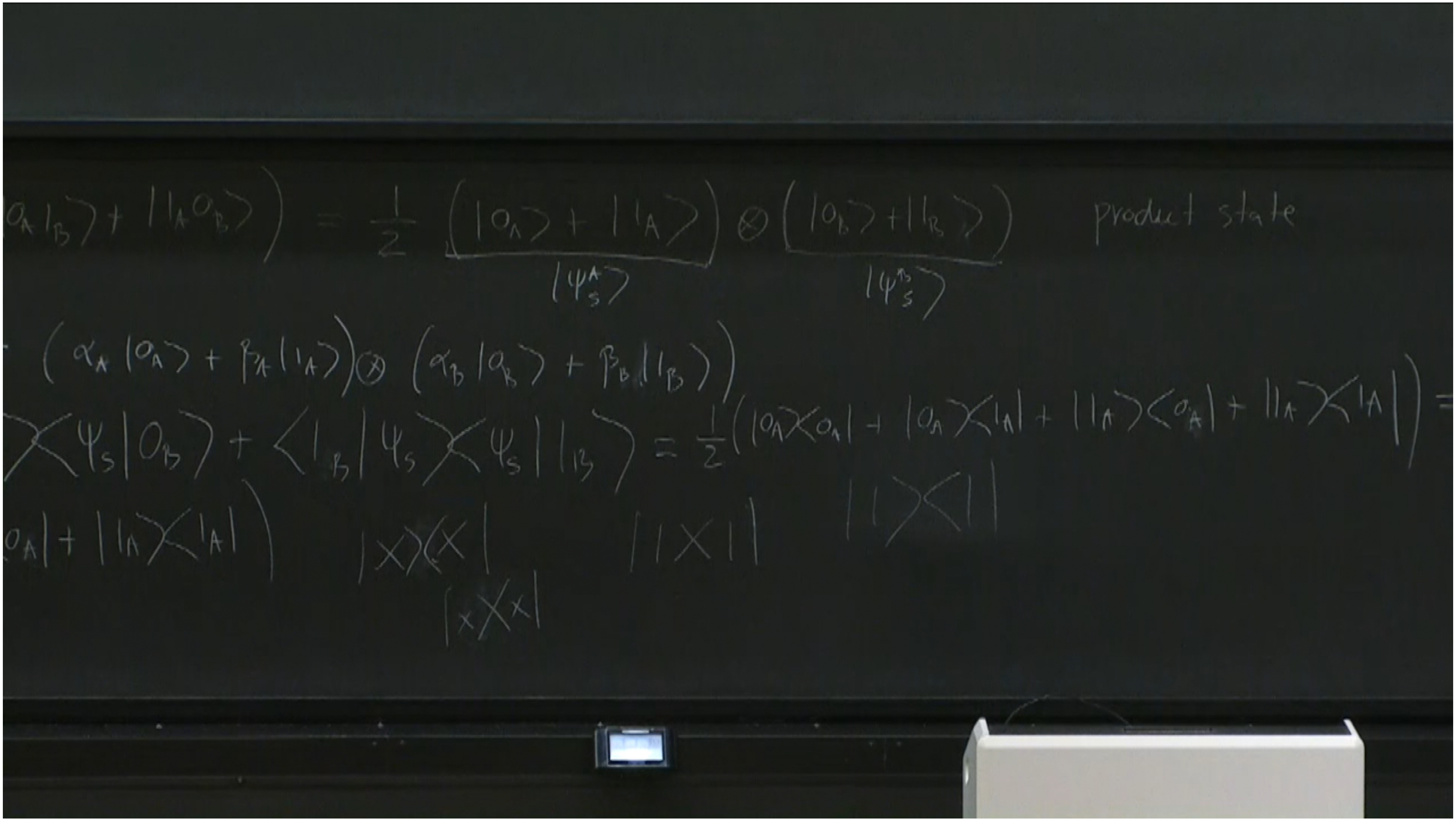
$$\hat{\rho}_s^A = Tr_B\left(|\psi_s\rangle\langle\psi_s|\right) = \langle 0_B|\psi_s\rangle\langle\psi_s|0_B\rangle + \langle 1_B|\psi_s\rangle\langle\psi_s|1_B\rangle$$

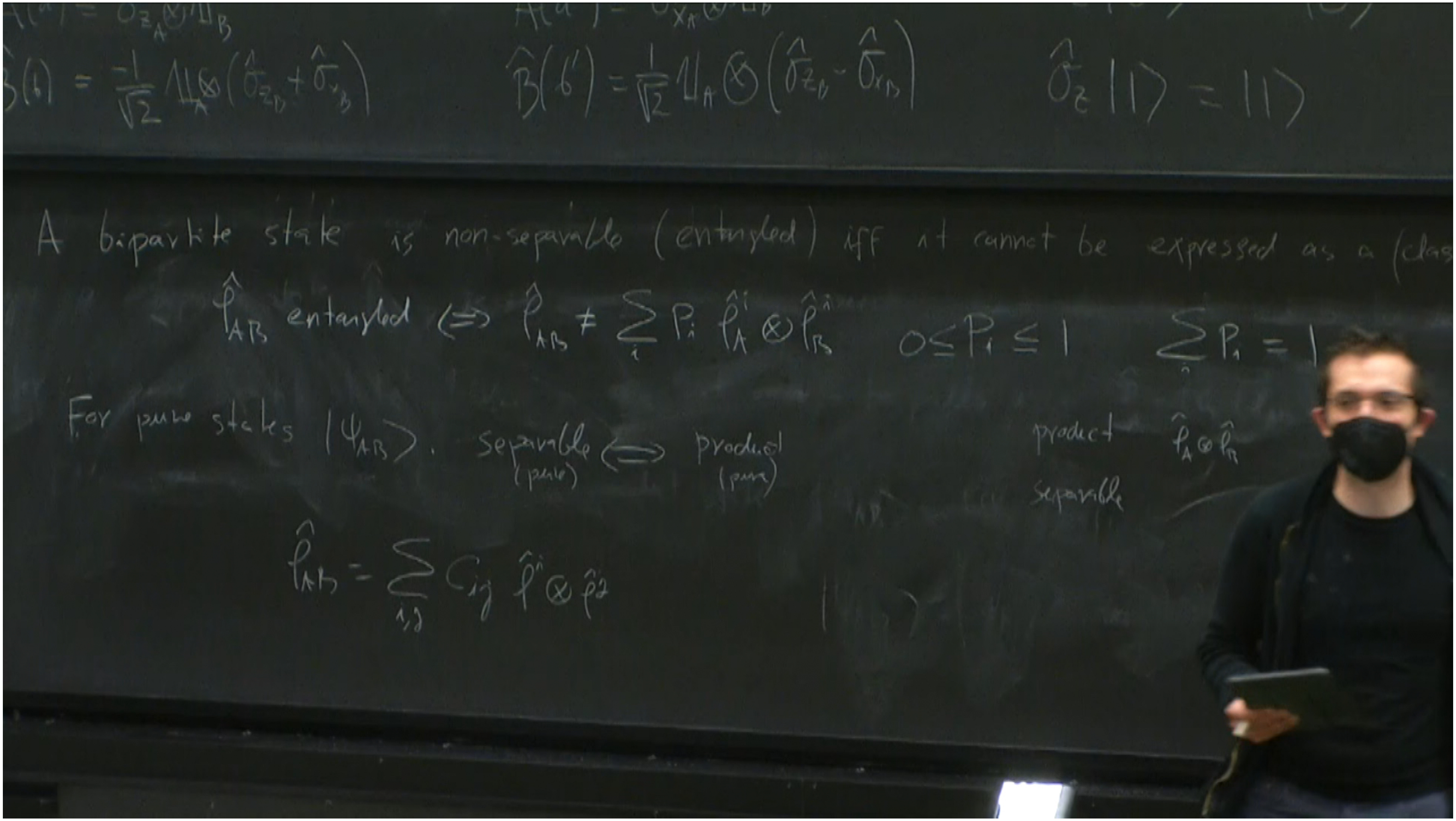
$$\rangle\Big) \otimes \underbrace{\Big(|0_B\rangle + |1_B\rangle\Big)}_{|\psi_s^B\rangle} \quad \text{product state}$$

$$\rangle\Big)\Big)$$

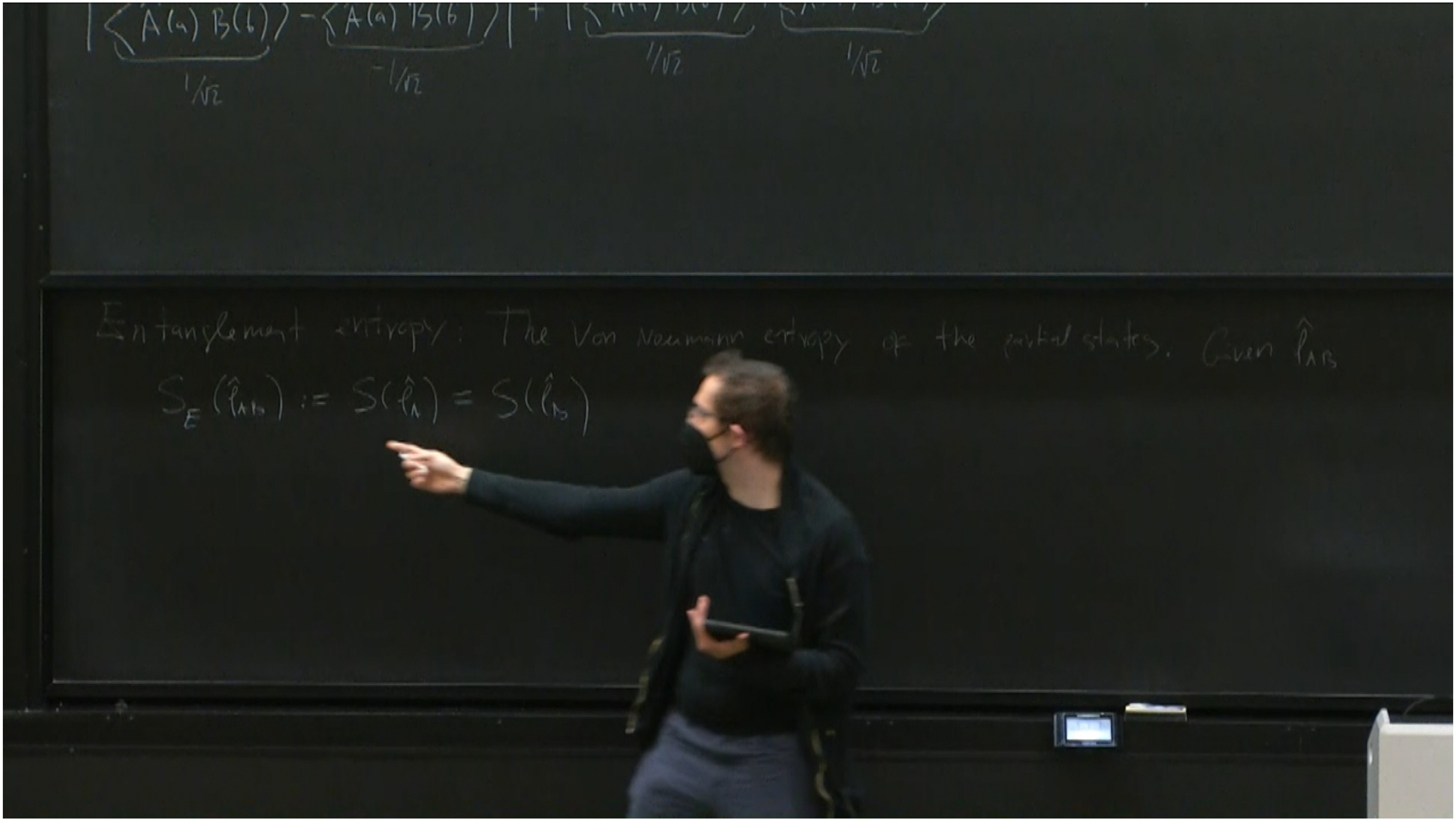
$$\frac{1}{2}\Big(|0_A\rangle\langle 0_A| + |0_A\rangle\langle 1_A| + |1_A\rangle\langle 0_A| + |1_A\rangle\langle 1_A|\Big) = |\psi_s^A\rangle\langle\psi_s^A|$$


$|0_A 0_B\rangle + |1_A 1_B\rangle + |0_A 1_B\rangle + |1_A 0_B\rangle \Big) = \frac{1}{2} \big(|0_A\rangle + |1_A\rangle\big) \otimes \big($
$|\psi^A$
$|0_A 0_B\rangle + |1_A 1_B\rangle \Big) \neq \big(\alpha_A |0_A\rangle + \beta_A |1_A\rangle\big) \otimes \big(\alpha_B |0_B\rangle +$
$\rangle\langle\psi_s|\Big) = \langle 0_B|\psi_s\rangle\langle\psi_s|0_B\rangle + \langle 1_B|\psi_s\rangle\langle\psi_s|$
$(|0_A\rangle$
$\rangle\langle\psi_e|\Big) = \frac{1}{2}\big(|0_A\rangle\langle 0_A| + |1_A\rangle\langle 1_A|\big)$


0_A 1_B⟩ + |1_A 0_B⟩) = 1/2 (|0_A⟩ + |1_A⟩) ⊗ (|0_B⟩ + |1_B⟩) product state
|ψ_s^A⟩
|ψ_s^B⟩
(α_A |0_A⟩ + β_A |1_A⟩) ⊗ (α_B |0_B⟩ + β_B |1_B⟩)
⟩⟨ψ_s|0_B⟩ + ⟨1_B|ψ_s⟩⟨ψ_s|1_B⟩ = 1/2 (|0_A⟩⟨0_A| + |0_A⟩⟨1_A| + |1_A⟩⟨0_A| + |1_A⟩⟨1_A|) =
0_A| + |1_A⟩⟨1_A|)
|x⟩⟨x|
|1⟩⟨1|
|1⟩⟨1|
|x⟩⟨x|



$\hat{B}(b) = \frac{-1}{\sqrt{2}} \mathbb{1}_A \otimes (\hat{\sigma}_{z_B} + \hat{\sigma}_{x_B})$
$\hat{B}(b') = \frac{1}{\sqrt{2}} \mathbb{1}_A \otimes (\hat{\sigma}_{z_B} - \hat{\sigma}_{x_B})$
$\hat{\sigma}_z |1\rangle = |1\rangle$
A bipartite state is non-separable (entangled) iff it cannot be expressed as a (clas
$\hat{\rho}_{AB}$ entangled $\iff \hat{\rho}_{AB} \neq \sum_i P_i \, \hat{\rho}_A^i \otimes \hat{\rho}_B^i$
$0 \leq P_i \leq 1$
$\sum_i P_i = 1$
For pure states $|\psi_{AB}\rangle$: separable (pure) $\iff$ product (pure)
product $\hat{\rho}_A \otimes \hat{\rho}_B$
separable
$\hat{\rho}_{AB} = \sum_{i,j} c_{ij} \, \hat{\rho}^i \otimes \hat{\rho}^j$

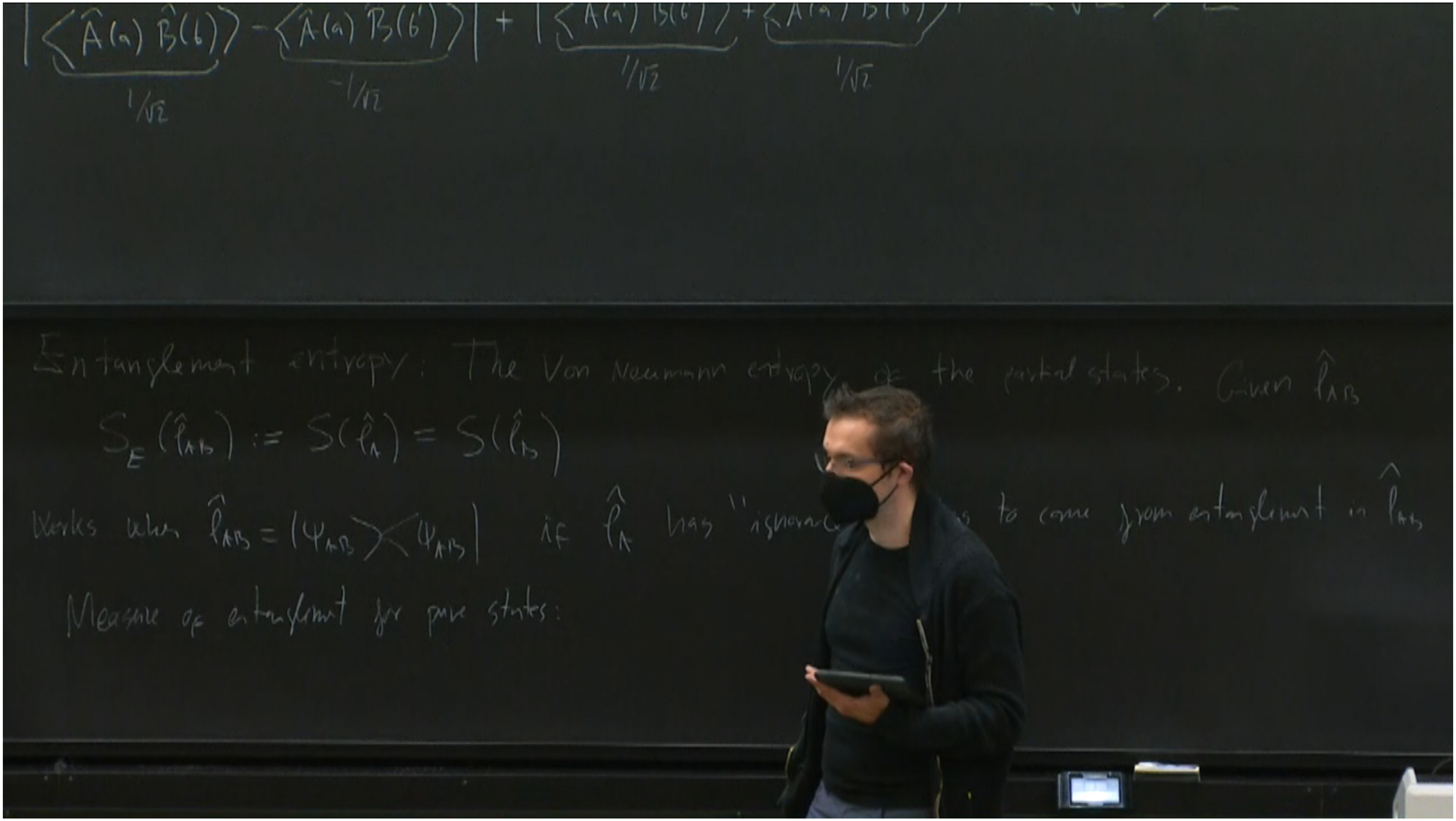

Entanglement entropy: The Von Neumann entropy of the partial states. Given $\hat{\rho}_{AB}$
$S_E(\hat{\rho}_{AB}) := S(\hat{\rho}_A) = S(\hat{\rho}_B)$

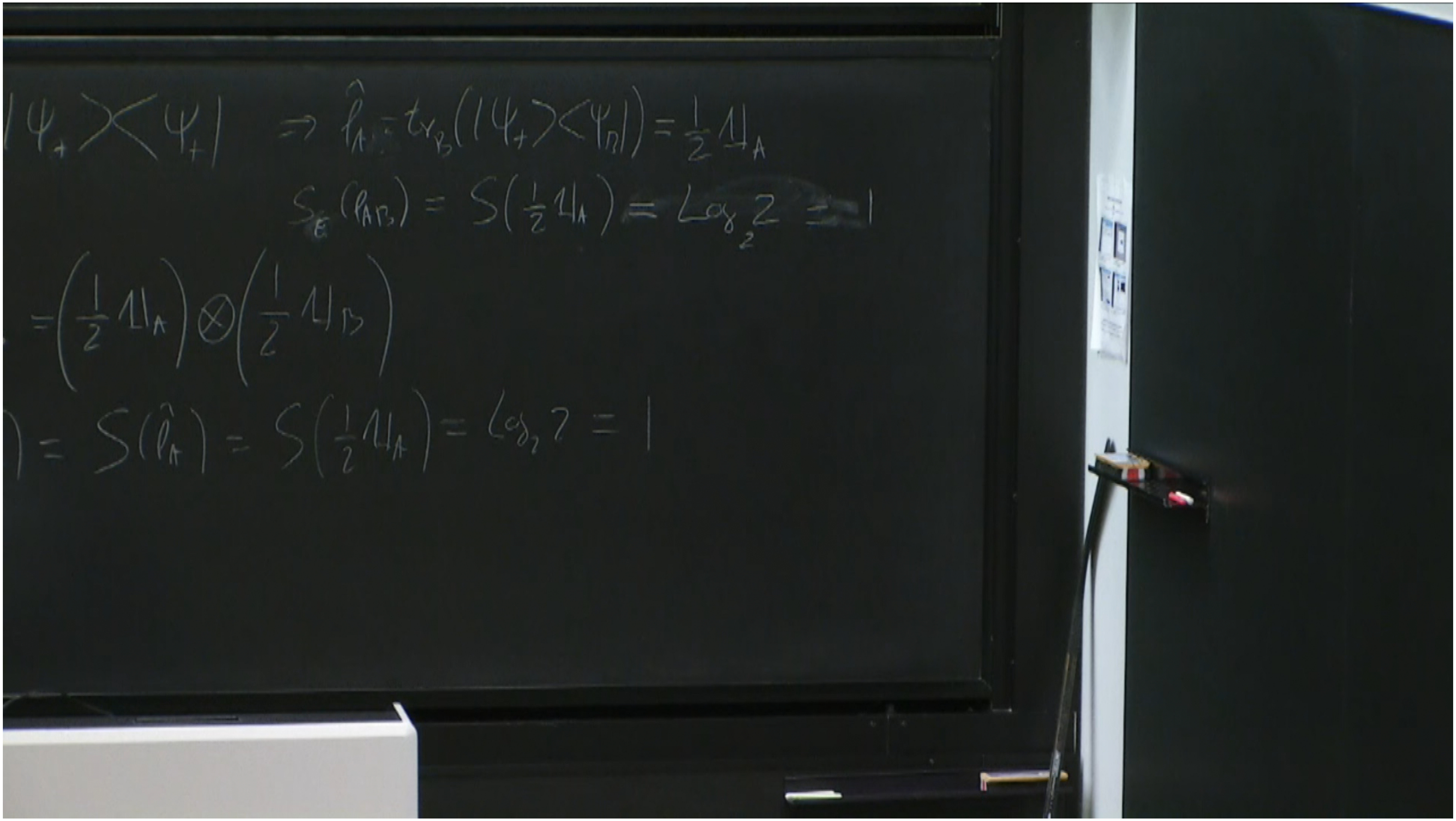
$$\left|\underbrace{\langle \hat{A}(a)\hat{B}(b)\rangle}_{1/\sqrt{2}} - \underbrace{\langle \hat{A}(a)\hat{B}(b')\rangle}_{-1/\sqrt{2}}\right| + \left|\underbrace{\langle \hat{A}(a')\hat{B}(b')\rangle}_{1/\sqrt{2}} + \underbrace{\langle \hat{A}(a')\hat{B}(b)\rangle}_{1/\sqrt{2}}\right|$$

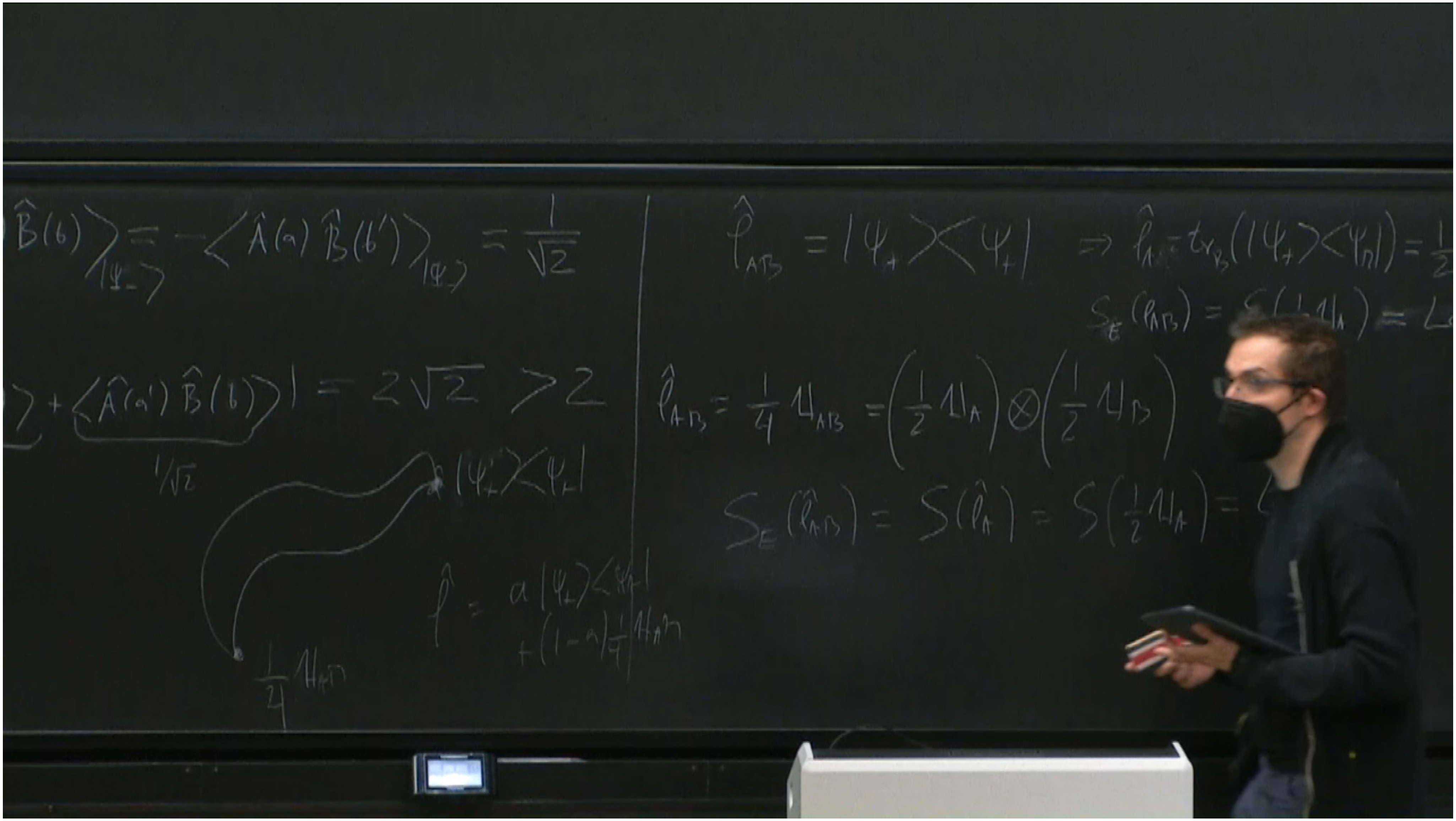
Entanglement entropy: The Von Neumann entropy of the partial states. Given $\hat{\rho}_{AB}$

$$S_E(\hat{\rho}_{AB}) := S(\hat{\rho}_A) = S(\hat{\rho}_B)$$

Works when $\hat{\rho}_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$ if $\hat{\rho}_A$ has "ignorance" it has to come from entanglement in $\hat{\rho}_{AB}$


$\left|\underbrace{\langle \hat{A}(a)\hat{B}(b)\rangle}_{1/\sqrt{2}} - \underbrace{\langle \hat{A}(a)\hat{B}(b')\rangle}_{-1/\sqrt{2}}\right| + \left|\underbrace{\langle \hat{A}(a')\hat{B}(b')\rangle}_{1/\sqrt{2}} + \underbrace{\langle \hat{A}(a')\hat{B}(b)\rangle}_{1/\sqrt{2}}\right|$
Entanglement entropy: The Von Neumann entropy of the partial states. Given $\hat{\rho}_{AB}$
$S_E(\hat{\rho}_{AB}) := S(\hat{\rho}_A) = S(\hat{\rho}_B)$
Works when $\hat{\rho}_{AB} = |\Psi_{AB}\rangle\langle\Psi_{AB}|$ if $\hat{\rho}_A$ has "ignorance" to come from entanglement in $\hat{\rho}_{AB}$
Measure of entanglement for pure states:

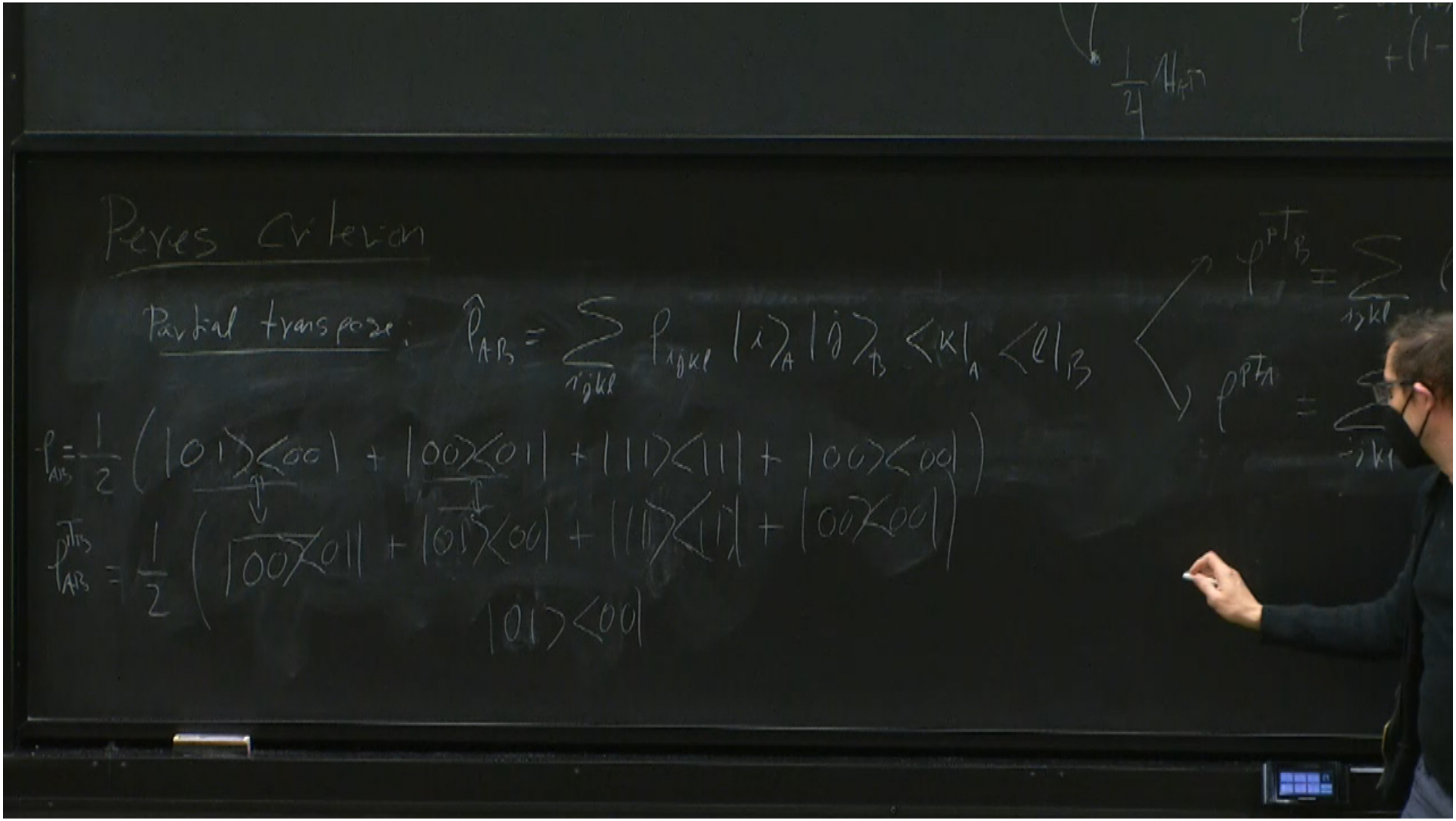

$|\psi_+\rangle\langle\psi_+| \Rightarrow \hat{\rho}_A = tr_B(|\psi_+\rangle\langle\psi_B|) = \frac{1}{2}\mathbb{1}_A$
$S_E(\rho_{AB}) = S(\frac{1}{2}\mathbb{1}_A) = Log_2 2 = 1$
$= (\frac{1}{2}\mathbb{1}_A) \otimes (\frac{1}{2}\mathbb{1}_B)$
$) = S(\hat{\rho}_A) = S(\frac{1}{2}\mathbb{1}_A) = Log_2 2 = 1$

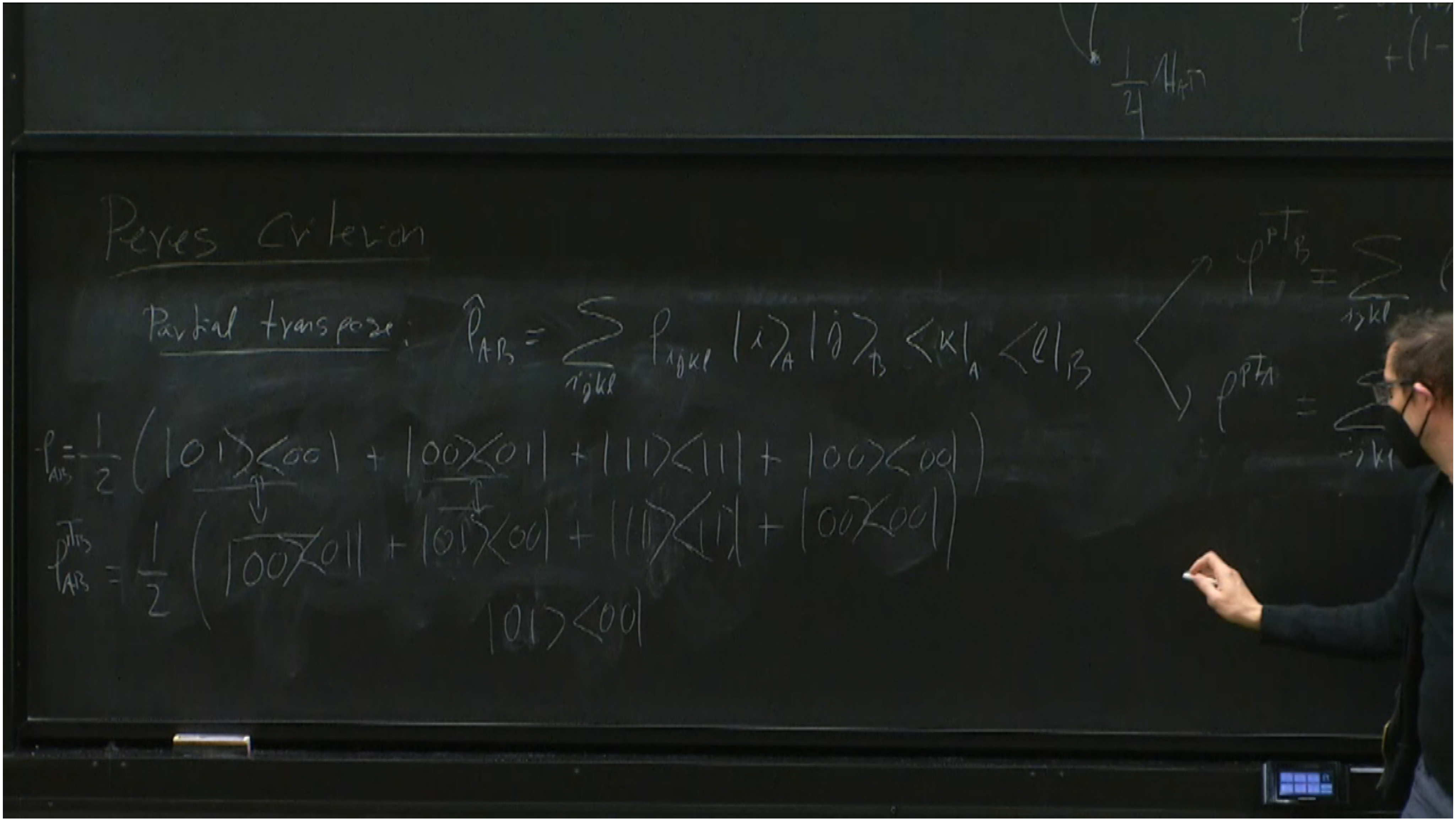

$\hat\rho_{AB} = |\Psi_+\rangle\langle\Psi_+|$
$\hat\rho_{AB} = \frac{1}{4}\mathbb{1}_{AB} = \left(\frac{1}{2}\mathbb{1}_A\right)\otimes\left(\frac{1}{2}\mathbb{1}_B\right)$
$S_E(\hat\rho_{AB}) = S(\hat\rho_A) = S\left(\frac{1}{2}\mathbb{1}_A\right) =$
$= 2\sqrt{2} > 2$

$\hat{B}(b) = \frac{-1}{\sqrt{2}} \mathbb{1}_A \otimes (\hat{\sigma}_{z_B} + \hat{\sigma}_{x_B})$ $\hat{B}(b') = \frac{1}{\sqrt{2}} \mathbb{1}_A \otimes (\hat{\sigma}_{z_B} - \hat{\sigma}_{x_B})$ $\hat{\sigma}_z |1\rangle = |1$

Entanglement measures $\mathcal{E}(\hat{\rho}_{AB}) \in \mathbb{R}$

- Must be maximum for maximally entangled states
- Must be zero for separable states
- Must be non-zero for non-separable states
- Must not increase under Local operations + classical communication (LOCC)


Peres Criterion
Partial transpose:
$\hat{\rho}_{AB} = \sum_{ijkl} \rho_{ijkl} |i\rangle_A |j\rangle_B \langle k|_A \langle l|_B$
$\rho^{pT_B} = \sum_{ijkl}$
$\rho^{pT_A} = \sum_{ijkl}$
$\rho_{AB} = \frac{1}{2}\left(|01\rangle\langle 00| + |00\rangle\langle 01| + |11\rangle\langle 11| + |00\rangle\langle 00| \right)$
$\rho_{AB}^{T_B} = \frac{1}{2}\left(|00\rangle\langle 01| + |01\rangle\langle 00| + |11\rangle\langle 11| + |00\rangle\langle 00| \right)$
$|01\rangle\langle 00|$


Peres Criterion
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$\rho_{AB} = \frac{1}{2}\left(|01\rangle\langle 00| + |00\rangle\langle 01| + |11\rangle\langle 11| + |00\rangle\langle 00|\right)$
$\rho_{AB}^{T_B} = \frac{1}{2}\left(|00\rangle\langle 01| + |01\rangle\langle 00| + |11\rangle\langle 11| + |00\rangle\langle 00|\right)$
$|01\rangle\langle 00|$

$\ldots |i\rangle_A |l\rangle_B \langle k|_A \langle j|_B$

$\ldots |k\rangle_A |j\rangle_B \langle i|_A \langle l|_B |$

$$\hat{\rho}^{pT_A} = \sum p_n \, \hat{\rho}_A^{n\,T} \otimes \hat{\rho}_B^{n}$$

if $\hat{\rho}_{AB}$ is separable $\Rightarrow$ $\hat{\rho}_{AB}^{pT}$ is a density operator

If $\hat{\rho}_{AB}^{pT}$ is Not a positive (semidefinite) operator then $\hat{\rho}_{AB}$ is Non-separable

if $\dim(\mathcal{H}_A) \times \dim(\mathcal{H}_B)$ is 2×2 or 2

then $\hat{\rho}_{AB}^{pT}$ NON positive $\Leftrightarrow$ $\hat{\rho}_{AB}$ entangled

$$
\begin{array}{c|cccc}
 & |00\rangle & |01\rangle & |10\rangle & |11\rangle \\
\hline
|00\rangle & 1 & & & \\
|01\rangle & & 0 & 1 & \\
|10\rangle & & 1 & 0 & \\
|11\rangle & & & & 1
\end{array}
$$

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$