

Title: Geometry and topology for physicists 2021/2022 - Lecture 14

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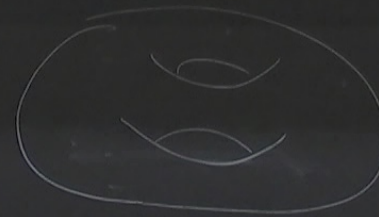
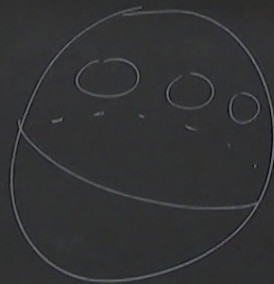
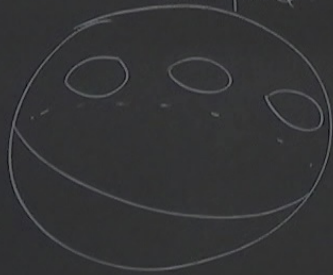
Collection: Geometry and Topology for Physicists 2021/2022

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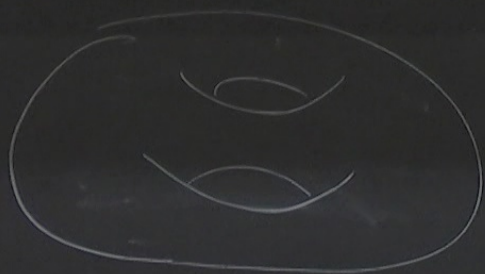
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$S^2 \setminus n$ slits cut

Full surface



Glue those along boundary circles



Example

Consider

$$F(w, z) = \frac{w^3}{3} - z^2 w - 1$$

For each z , the set of sol's to $F(w, z)$ is normally 3 points

Sometimes, z is such that
 $F(w, z)$ has ≤ 3 roots.
This happens when 2 roots
come together.



Sometimes, z is such that
 $F(w, z)$ has < 3 roots.

This happens when 2 roots
come together.

This happens when
we can find solⁿs to

$$F(w, z) = 0$$

$$\frac{\partial}{\partial w} F(w, z) = 0$$

Those z where this eqⁿ

Those z where this eqⁿ can be solved are branch points

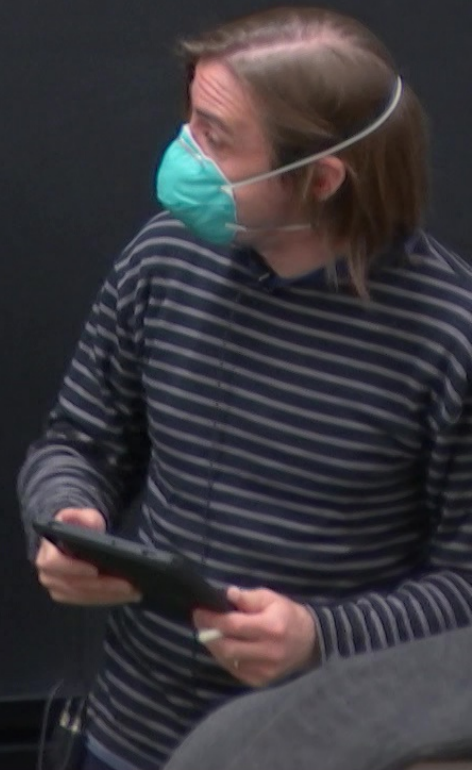
$$F(w, z) = \frac{w^3}{3} - z^2 w - 1 = 0$$

$$\frac{\partial F}{\partial w} = w^2 - z^2 = 0$$

$$z = \pm w$$

$$\text{And, } \frac{\pm z^3}{3} - z^3 = 1$$

$$\text{so, } z^3 = \pm \frac{3}{2} \quad (z^6 = \frac{9}{4})$$



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$$F(w, z) = \frac{w^3}{3} - z^2 w - 1 = 0$$

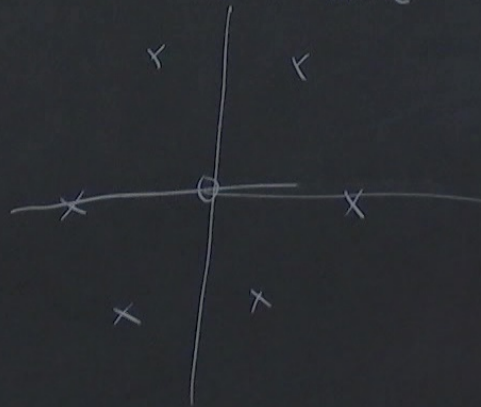
$$\frac{\partial F}{\partial w} = w^2 - z^2 = 0$$

$$z = \pm w$$

$$\text{And, } \frac{\pm z^3}{3} - z^3 = 1$$

$$\text{so, } z^3 = \pm \frac{3}{2} \quad \left(z^6 = \frac{9}{4} \right)$$

There are 6 branch points:

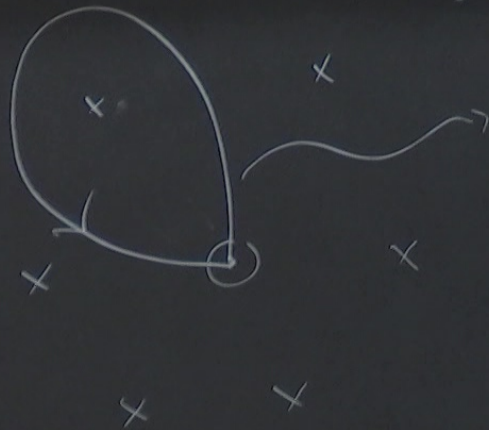


At each branch point,

$$\frac{\partial^2}{\partial w^2} F \neq 0$$

Only 2 roots come together.

$z=0$ 3 roots, cube roots of 3
 $\omega_1, \omega_2, \omega_3$



sends $(\omega_1, \omega_2, \omega_3)$

$\rightarrow (\omega_2, \omega_1, \omega_3)$

2 roots switch, one
stays same.

together.

$\otimes$ $\xrightarrow{z=0}$ Look near here

$$F(w, z) = (w - b)(w - a + z)(w - a) - \circ$$

→ Look near here

$$F(w, z) = (w - b)(w - a + z)(w - a) - O(z)$$

ks like $(w^2 - z) \times (w - b)$

$w = \sqrt{z}$ $w = -\sqrt{z}$
↙ ↗
 $w = b$ these two switch

↑ irrelevant

Most general behaviour:

n polynomial

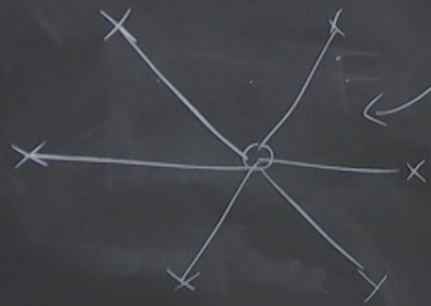
k groups look like

$$w^{n_1} = z$$

$$w^{n_2} = z \dots$$

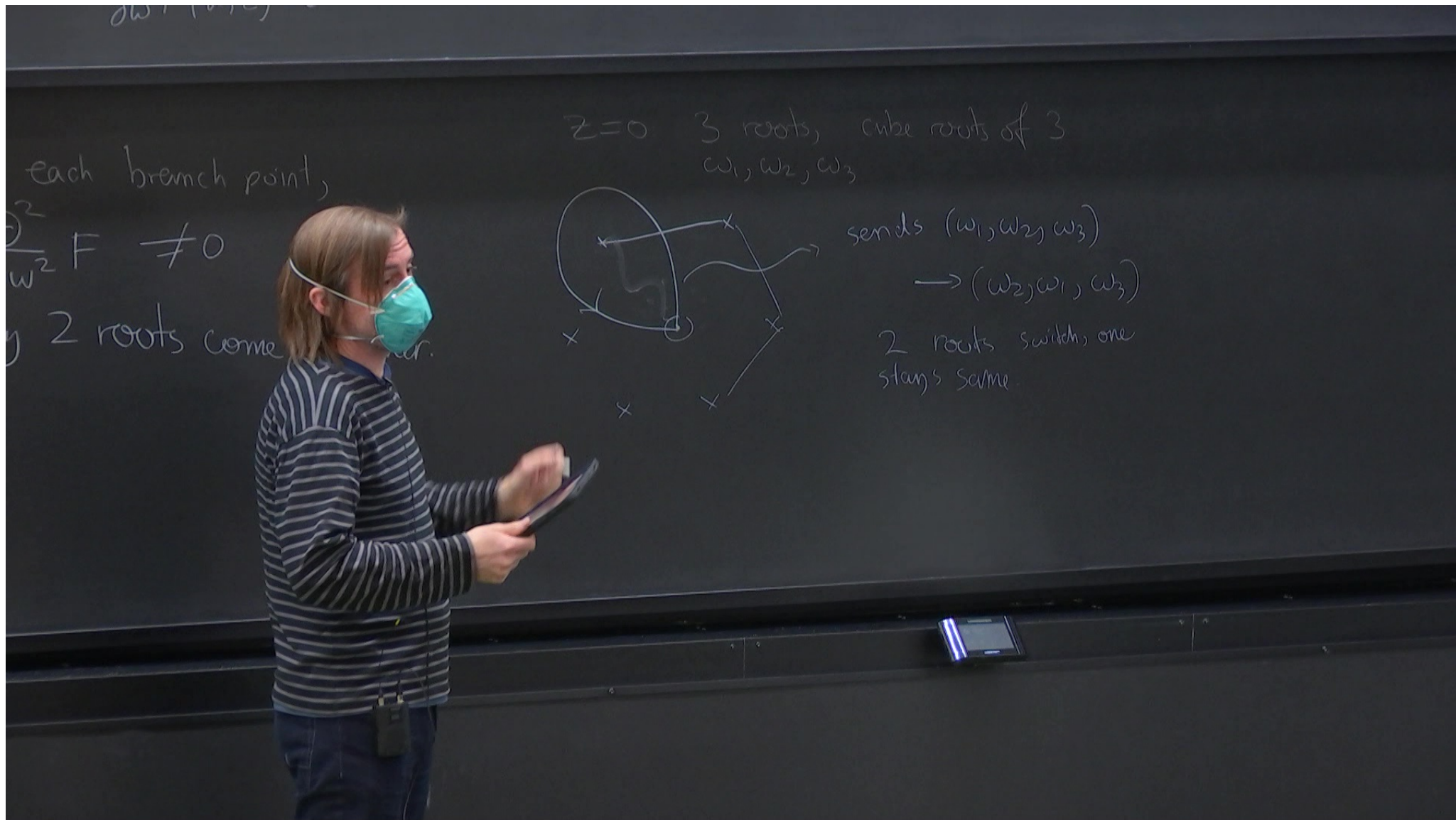
(-b)
↑ irrelevant

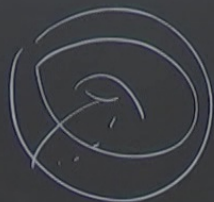
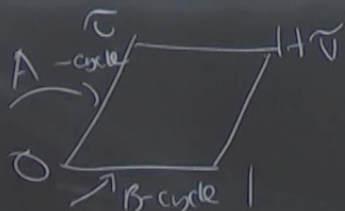
In this example, it is more complicated to build surface.



branch cuts.

When we remove branch cuts,
include ∞ , get 3 12-gons
glued together according to how roots
switch.





$$u, u \sim u+1$$

$$u \sim u + \tau$$

$$H^1(E_\tau) = \mathbb{C}^2$$

a basis is $du, d\bar{u}$.

$$\tau = \frac{\int_{A\text{-cycle}} du}{\int_{B\text{-cycle}} du}$$

E_τ could be parametrized in terms of

$$w^2 = F(z)$$

$$F(z) = (z-p_1) \cdots (z-p_4)$$

What are dx and dy in terms of z ?

Local model:

near

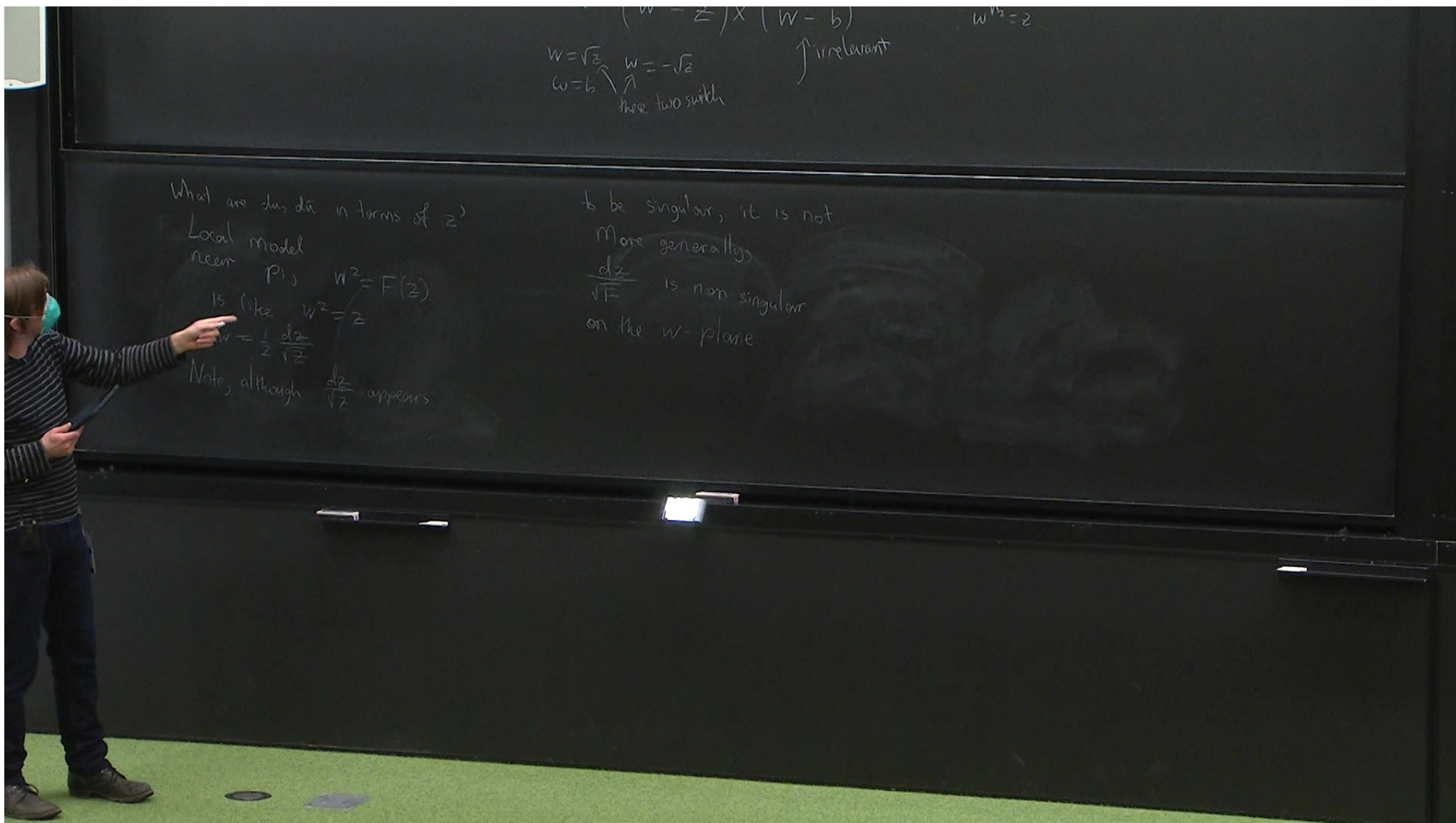
p_1 ,

$$w^2 = F(z)$$

is like $w^2 = z$

$$dw = \frac{1}{2} \frac{dz}{\sqrt{z}}$$

Note, although $\frac{dz}{\sqrt{z}}$ appears



to be singular, it is not.

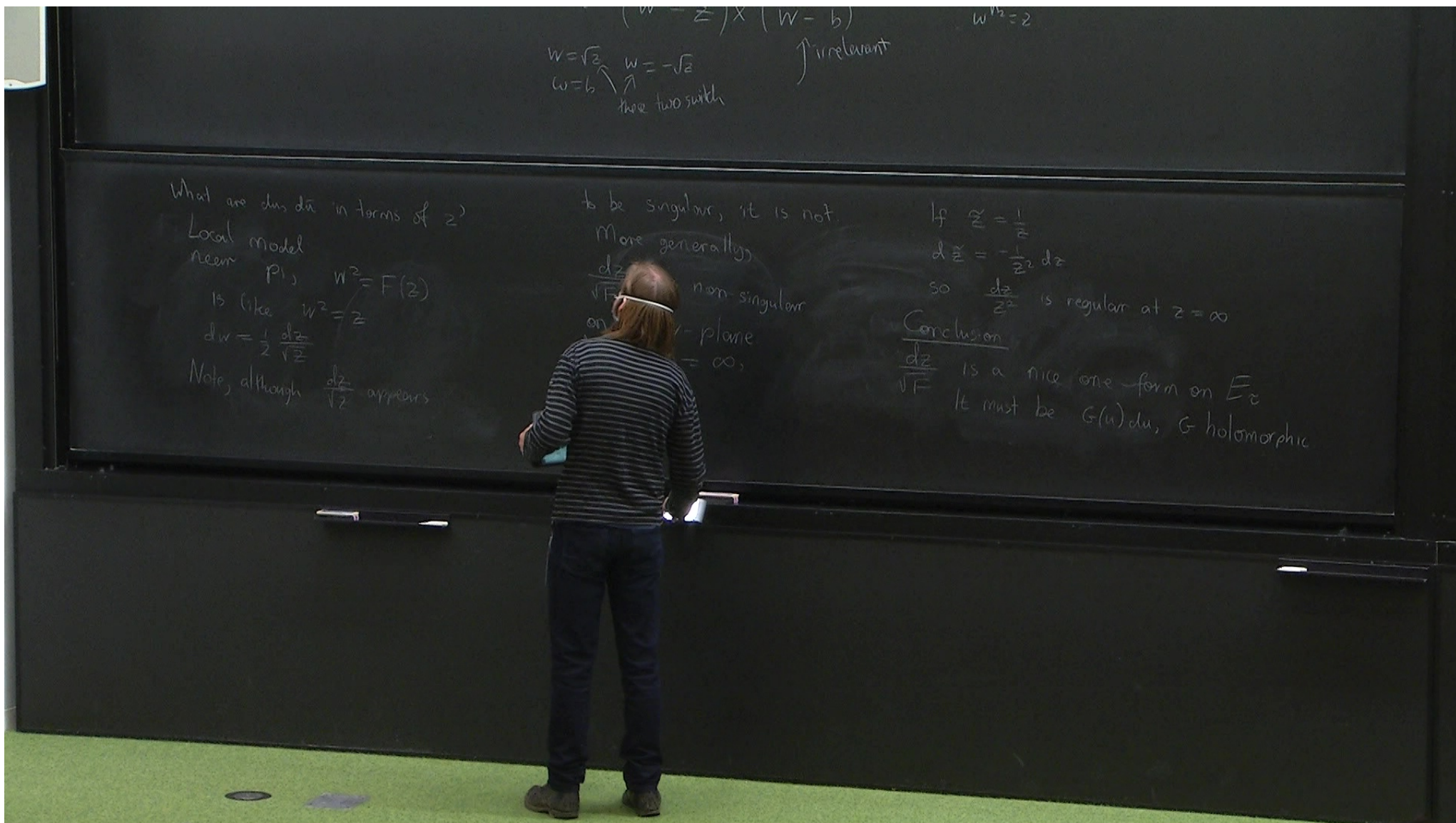
More generally,

$\frac{dz}{\sqrt{F}}$ is non-singular

on the w -plane.

Near $z = \infty$,

$$\frac{dz}{\sqrt{F}} \sim \frac{dz}{z^2}$$



$dw = \frac{1}{z} \frac{dz}{\sqrt{z}}$
 Note, although $\frac{dz}{\sqrt{z}}$ appears

in the w -plane.
 Near $z = \infty$,
 $\frac{dz}{\sqrt{z}} \sim \frac{dz}{z^2}$

$\frac{dz}{\sqrt{z}}$ is a nice one-form on E_τ
 It must be $G(u)du$, G holomorphic

Further, G is doubly periodic,
 it must be bounded,
 by maximum principle it is constant

$$\frac{dz}{\sqrt{z}} = C du$$

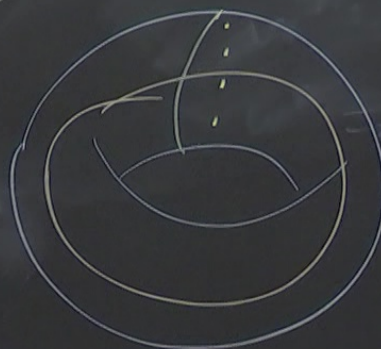
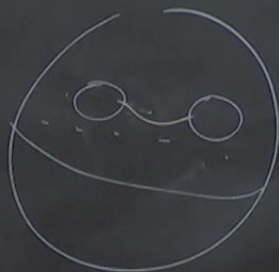
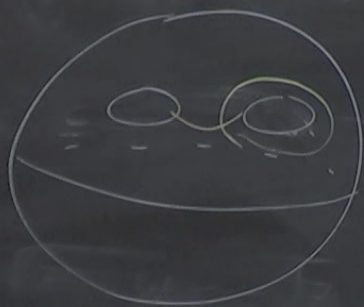
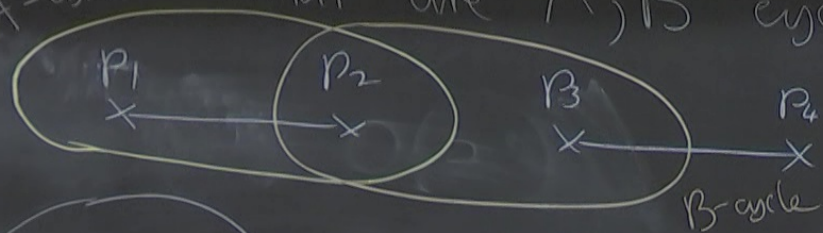
What are A

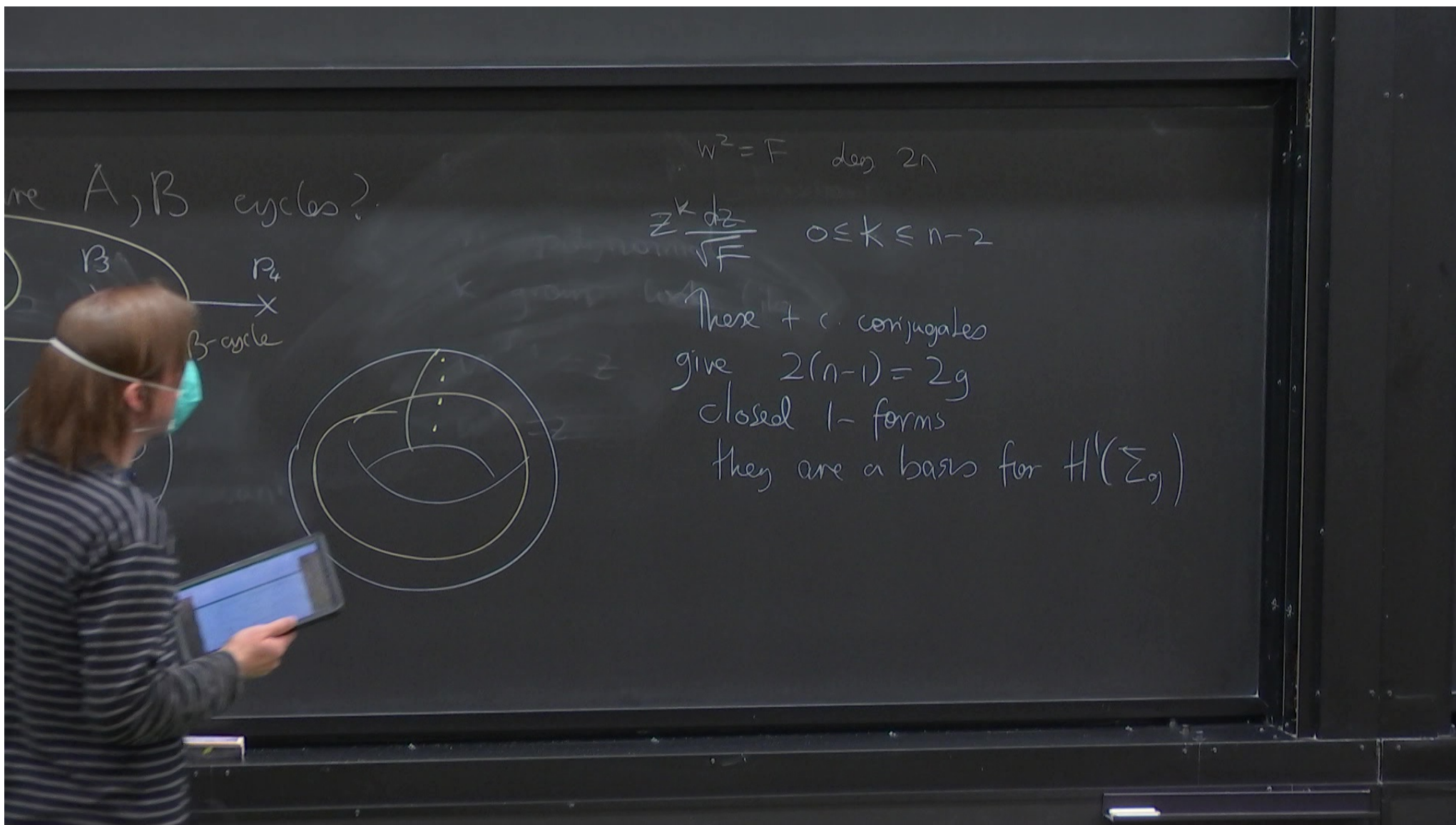


Further, G is doubly periodic,
it must be bounded,
by maximum principle it is constant.

$$\frac{dz}{\sqrt{F}} = C du$$

A-cycle What are A, B cycles?

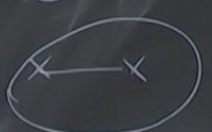




$$\frac{\partial}{\partial w} F(w, z) = 0$$

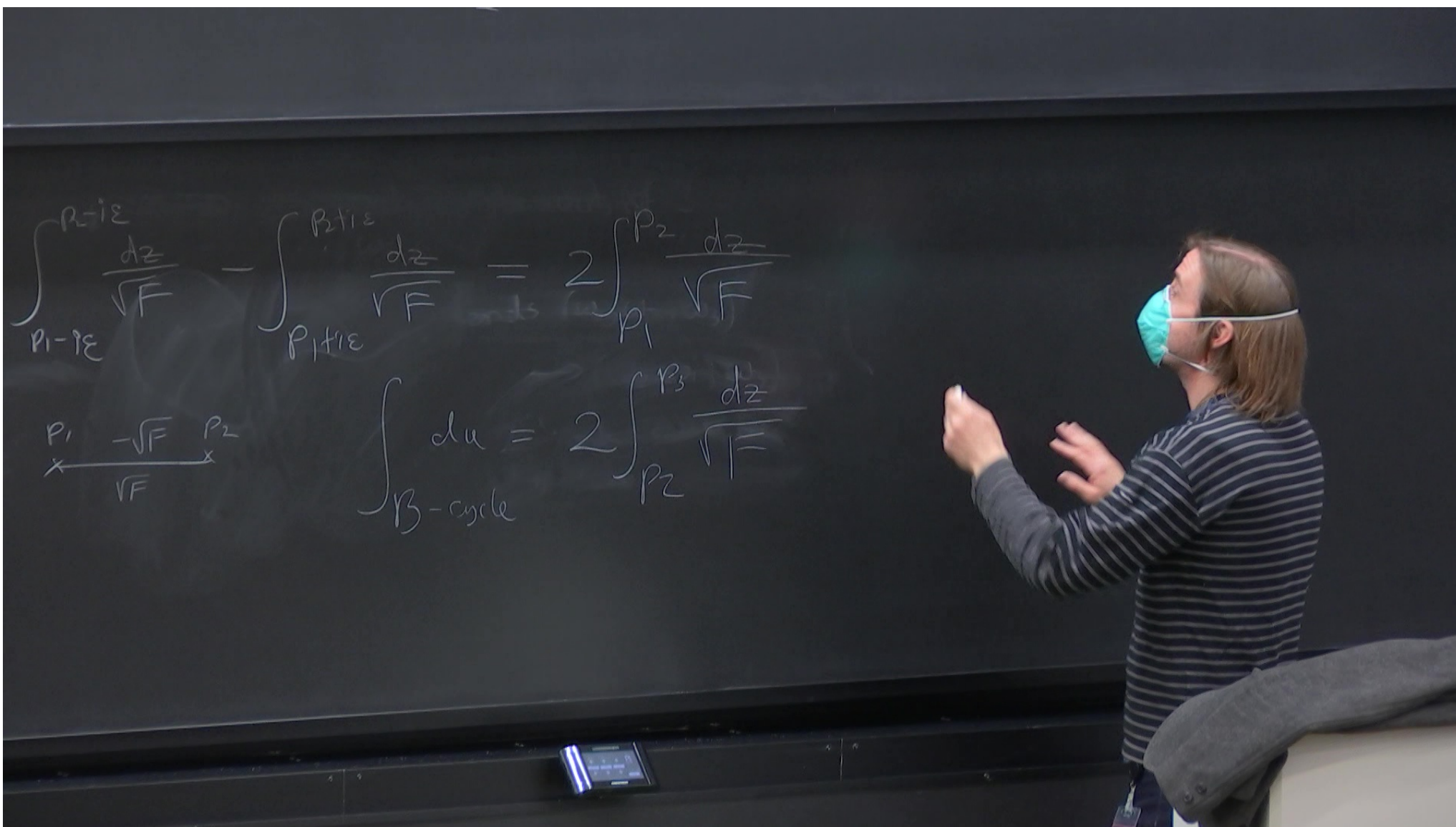
$$\int_{A\text{-cycle}} du = \int \frac{dz}{\sqrt{F}} = \int_{p_1 - i\varepsilon}^{p_2 - i\varepsilon} \frac{dz}{\sqrt{F}} - \int_{p_1 + i\varepsilon}^{p_2 + i\varepsilon} \frac{dz}{\sqrt{F}}$$

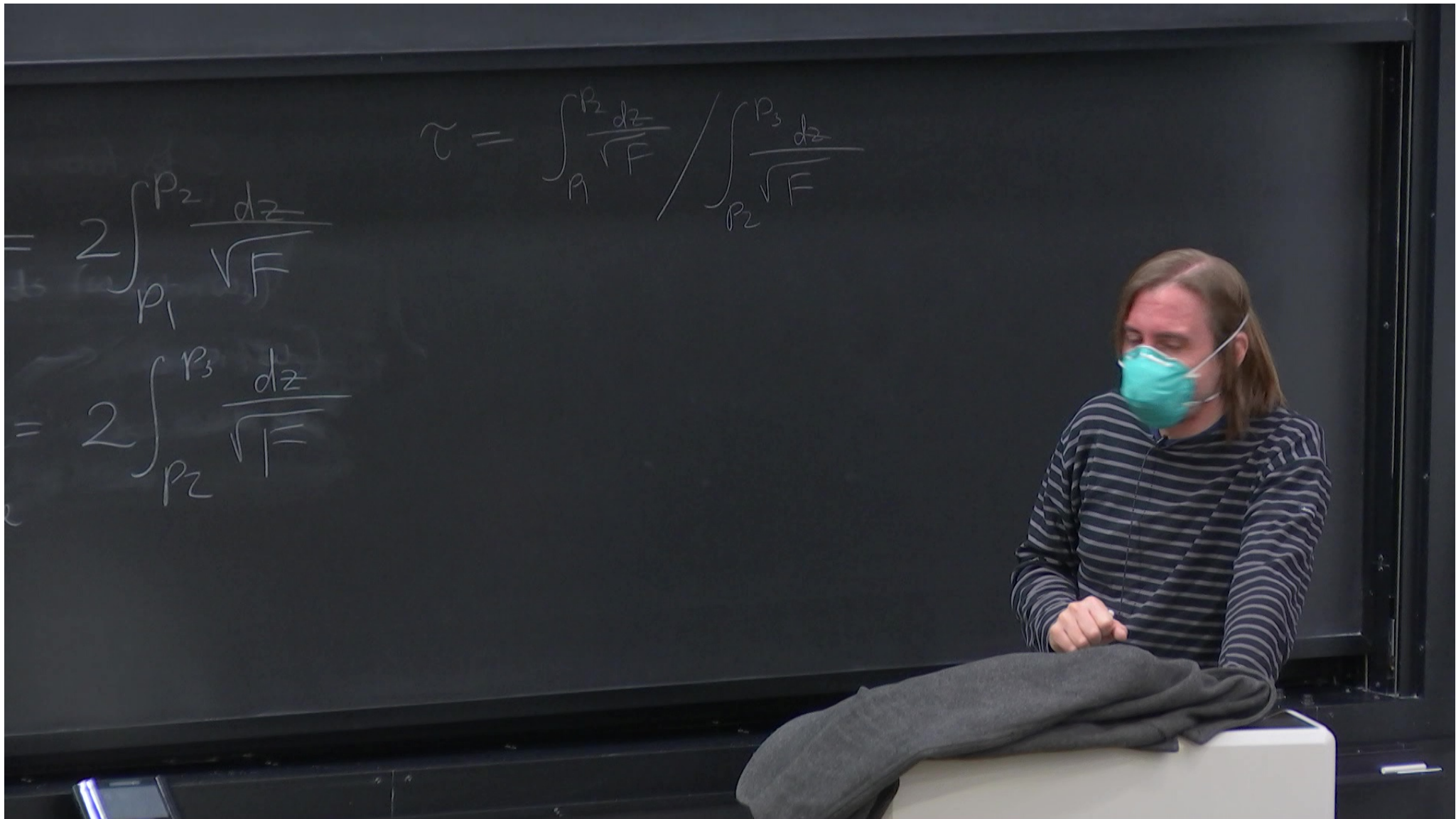
Diagram illustrating the integration path in the complex plane:



Below the diagram, the text $p_1, p_2 \in \mathbb{R}$ is written.

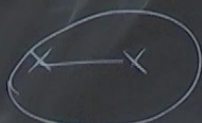
Below the main equation, a diagram shows a horizontal line segment from p_1 to p_2 with two small semi-circular arcs above and below it, forming a closed loop. The points p_1 and p_2 are marked with 'x'. The upper arc is labeled $-\sqrt{F}$ and the lower arc is labeled $\sqrt{F}$.





$$\partial_w F(w, z) = 0$$

$$\int_{A\text{-cycle}} du = \int \frac{dz}{\sqrt{F}} = \int_{p_1 - i\varepsilon}^{p_2 - i\varepsilon} \frac{dz}{\sqrt{F}} - \int_{p_1 + i\varepsilon}^{p_2 + i\varepsilon} \frac{dz}{\sqrt{F}} =$$

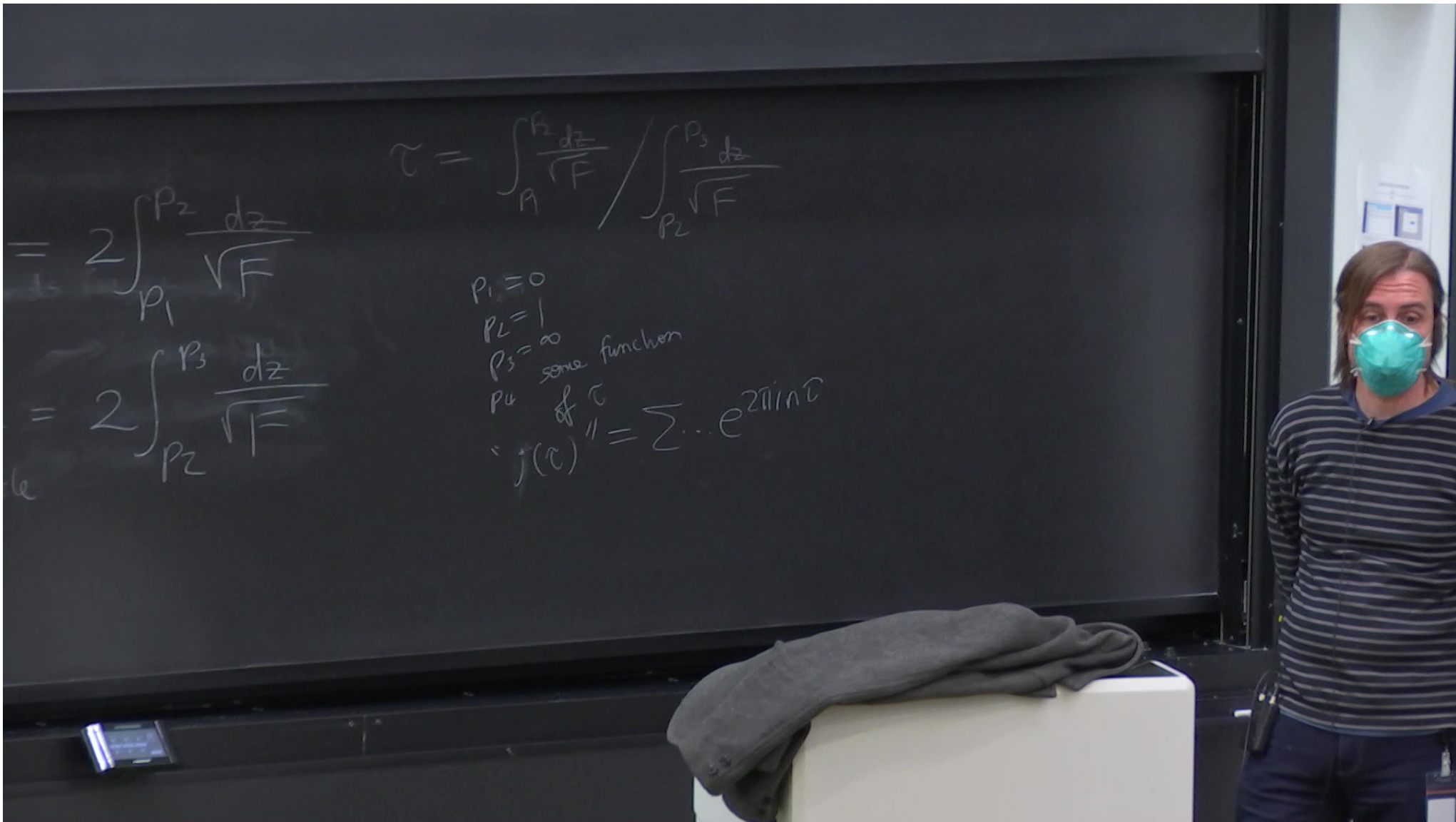


$$p_1, p_2 \in \mathbb{R}$$

$$\begin{array}{ccc} p_1 & -\sqrt{F} & p_2 \\ x & \xrightarrow{\quad} & x \\ & \sqrt{F} & \end{array}$$

A basis
for cycles:

$$A, B \text{ or } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$



What are A, B cycles?

