Title: Geometry and topology for physicists 2021/2022 - Lecture 13

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Collection: Geometry and Topology for Physicists 2021/2022

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# More examples of Riemann surfaces

$S^2$ with the standard metric

$= \mathbb{C} \cup \{\infty\}$

Called Riemann sphere.

$S^2 \setminus N \cong \mathbb{C}$, coord $z$

Stereographic Projection

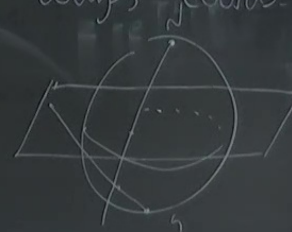
$S^2 \setminus S \cong \mathbb{C}$

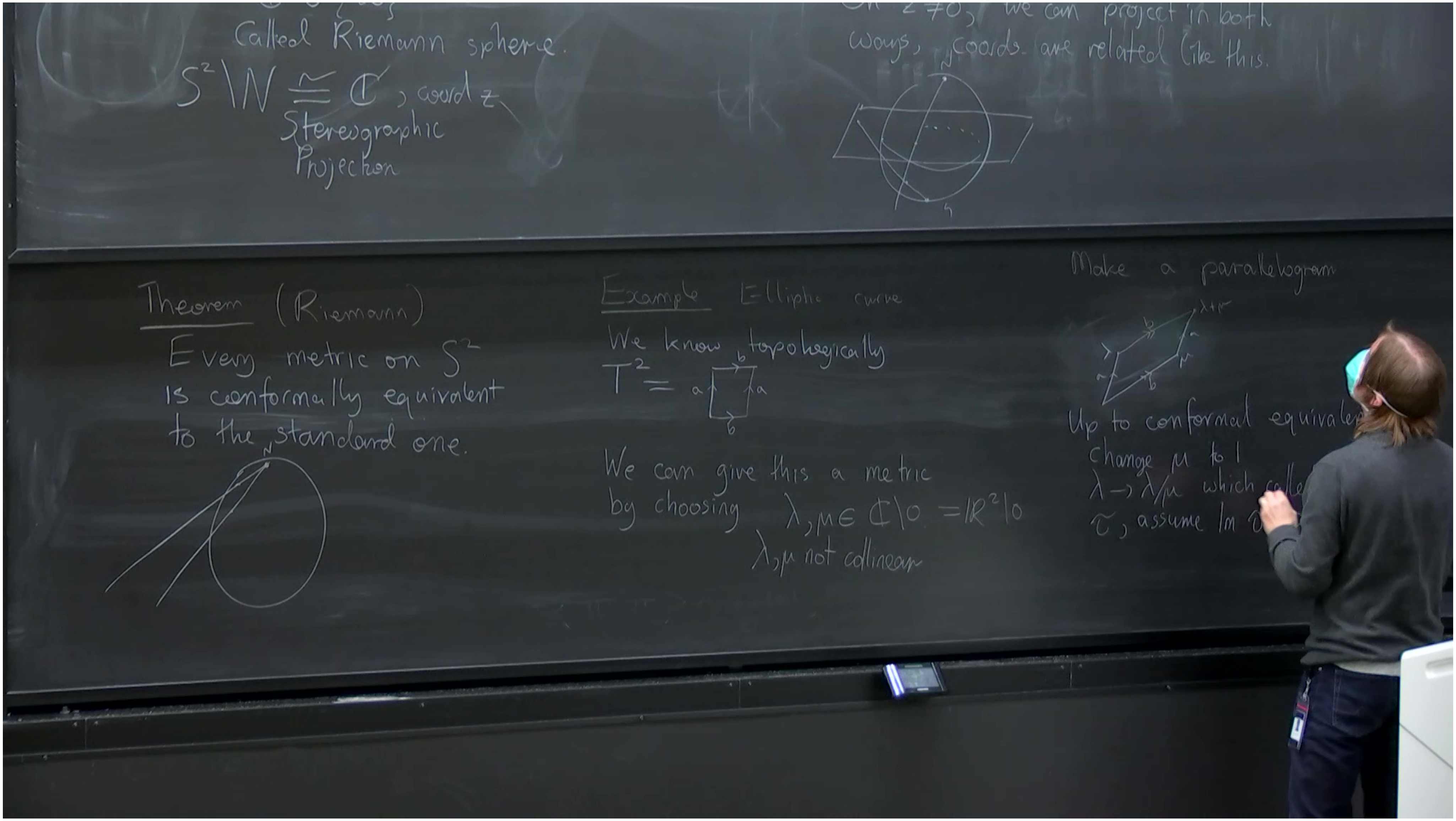
by stereographic proj.

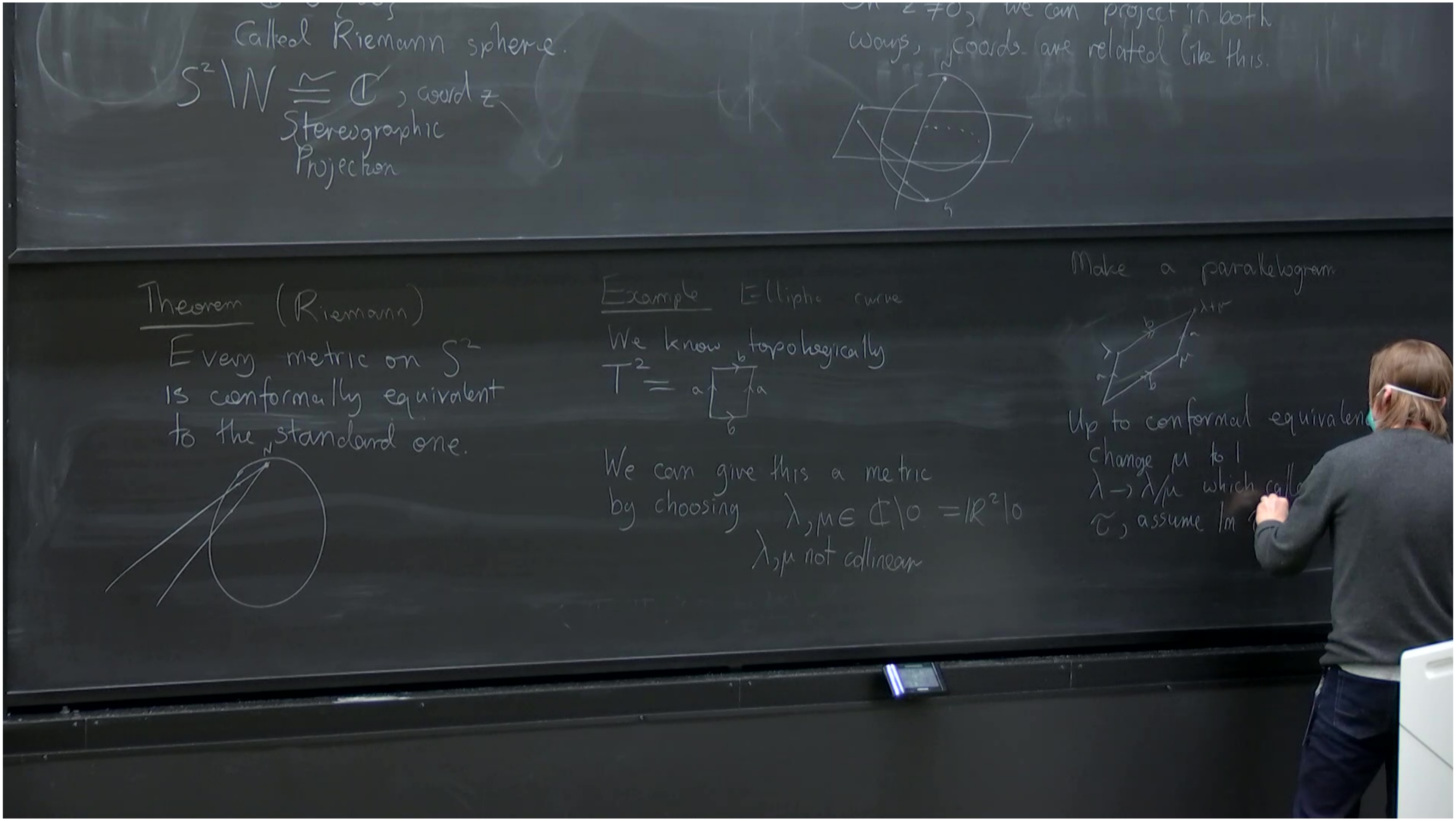
coord $w$

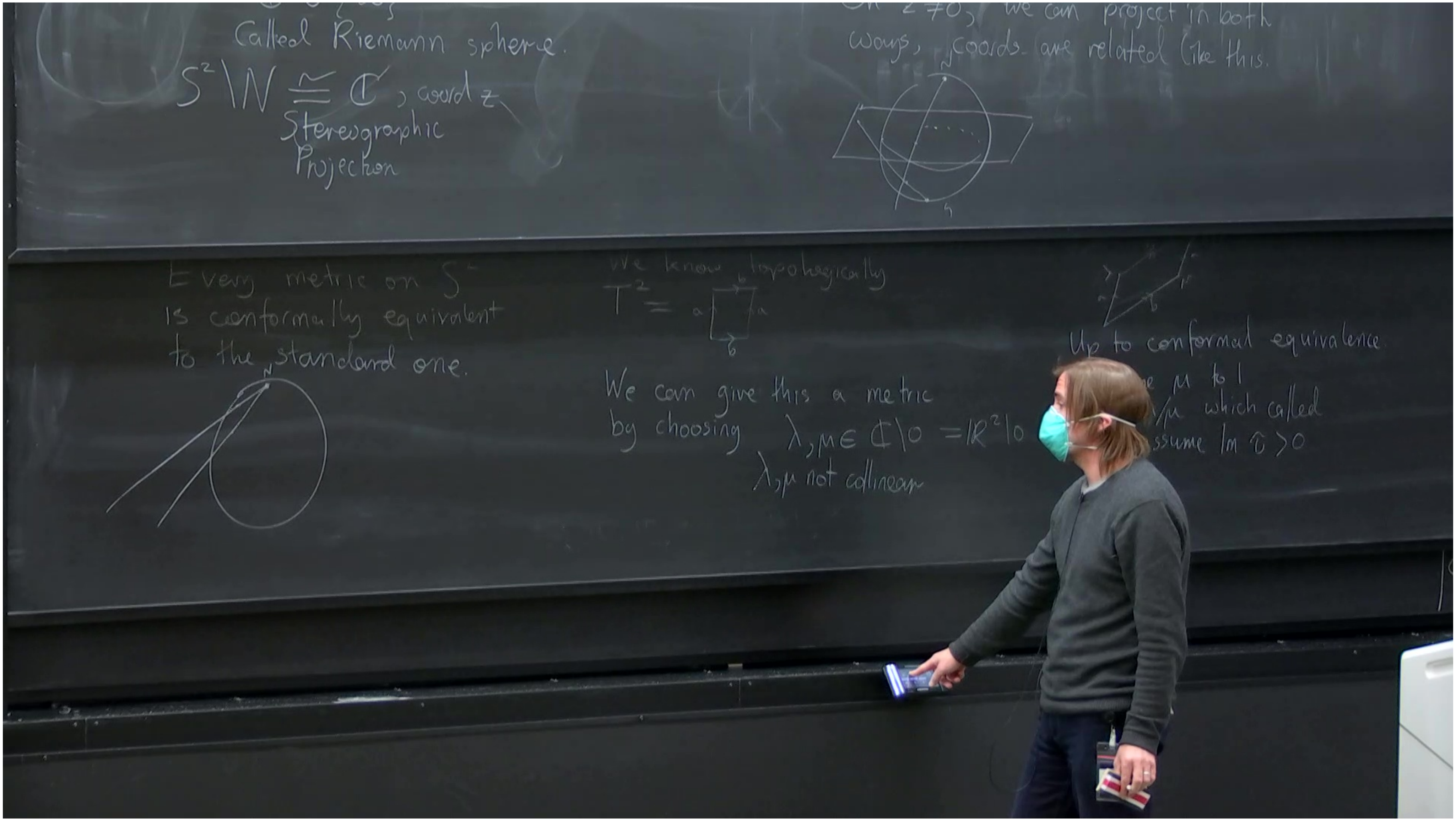
$w = 1/z$

On $z \neq 0$, we can project in both ways, coords are related like this.




Called Riemann sphere.
S²\N ≅ ℂ, coord z
Stereographic Projection
ways, coords are related like this.
Theorem (Riemann)
Every metric on S² is conformally equivalent to the standard one.
Example Elliptic curve
We know topologically
T² =
We can give this a metric
by choosing λ,μ ∈ ℂ\0 = ℝ²\0
λ,μ not collinear
Make a parallelogram
λ+μ
Up to conformal equivale
Change μ to 1
λ → λ/μ which calle
τ, assume Im


Called Riemann sphere.
$S^2 \setminus N \cong \mathbb{C}$, coord z
Stereographic Projection
we can project in both ways, coords are related like this.
Theorem (Riemann)
Every metric on $S^2$ is conformally equivalent to the standard one.
Example Elliptic curve
We know topologically
$T^2 =$
We can give this a metric by choosing $\lambda, \mu \in \mathbb{C} \setminus 0 = \mathbb{R}^2 \setminus 0$
$\lambda, \mu$ not collinear
Make a parallelogram
$\lambda + \mu$
Up to conformal equivalen
change $\mu$ to 1
$\lambda \to \lambda/\mu$ which call
$\tau$, assume Im



Called Riemann sphere.
S²\N ≅ ℂ, coord z
Stereographic Projection
On z ≠ 0, we can project in both ways, coords are related like this.
Every metric on S² is conformally equivalent to the standard one.
We know topologically
T² =
We can give this a metric
by choosing λ, μ ∈ ℂ\0 = ℝ²\0
λ, μ not collinear
Up to conformal equivalence
which called
ssume Im τ > 0

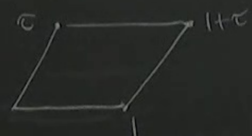
Called Riemann sphere.

$S^2 \setminus N \cong \mathbb{C}$, coord $z$

Stereographic Projection

$z \neq 0$, we can project in both ways, coords are related like this.


So metric is




We can think of this as

$\mathbb{C}$ where $z \sim z+1$

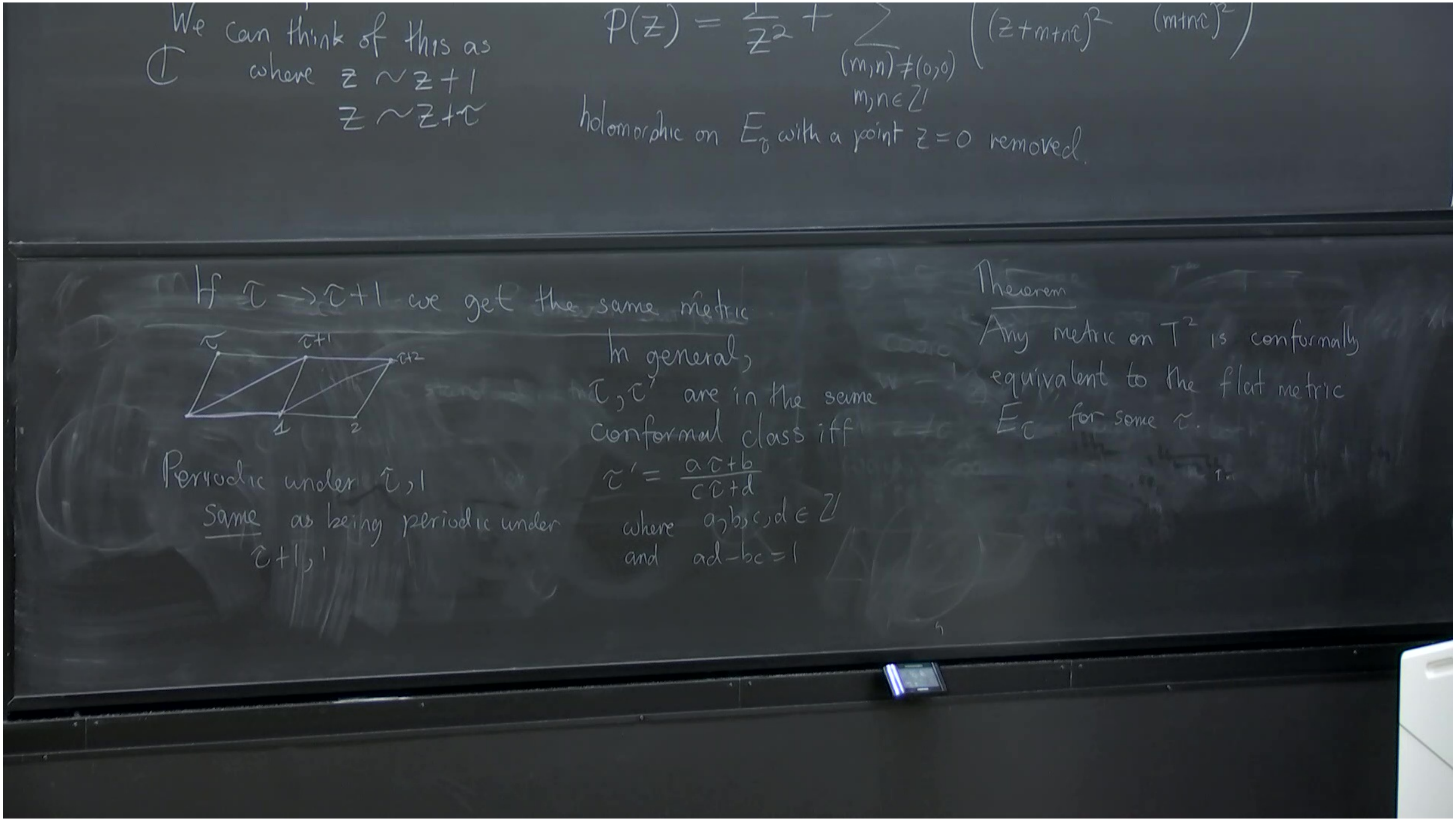
$z \sim z+\tau$

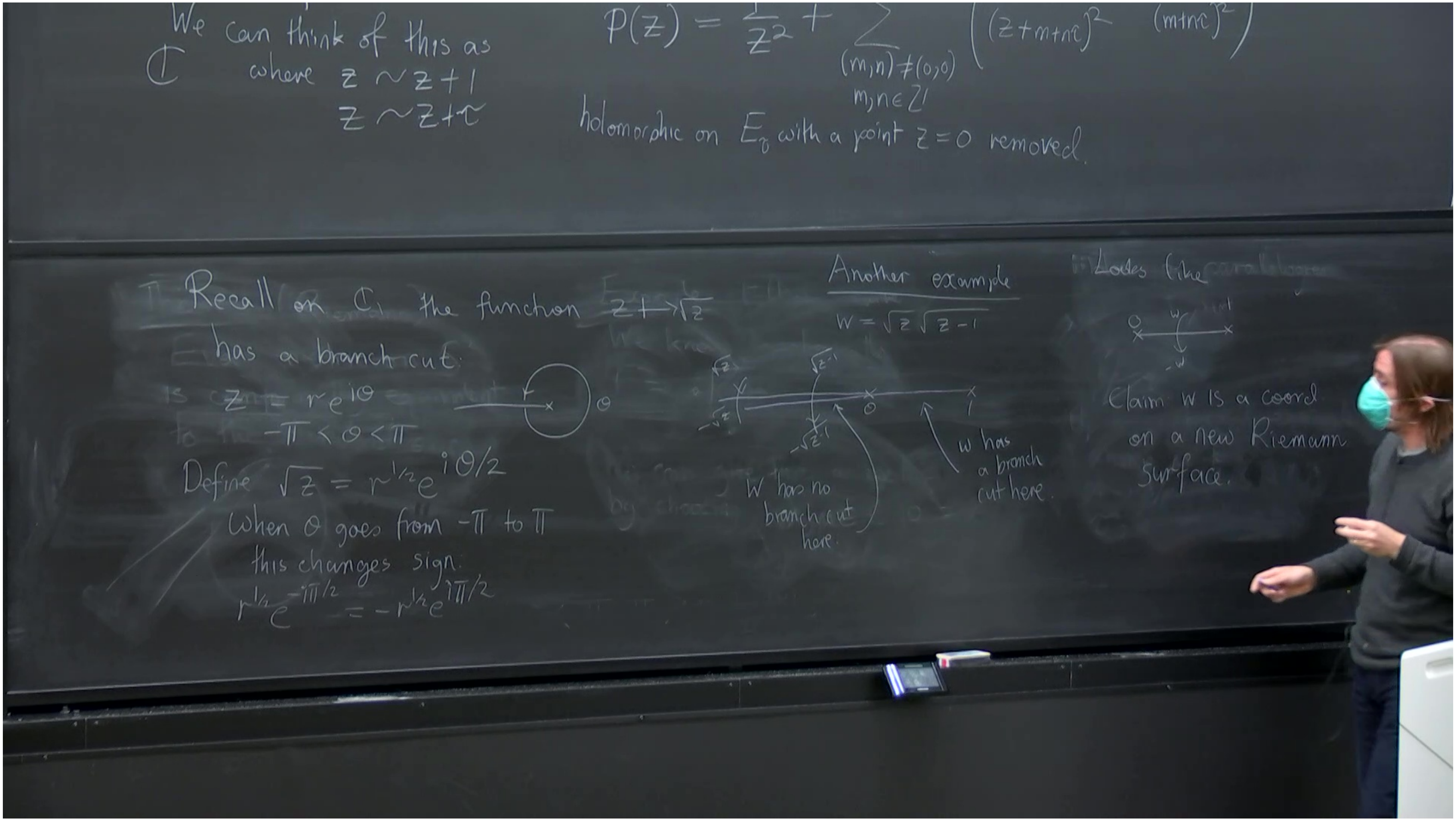
A holomorphic function is a doubly periodic function (elliptic functions)

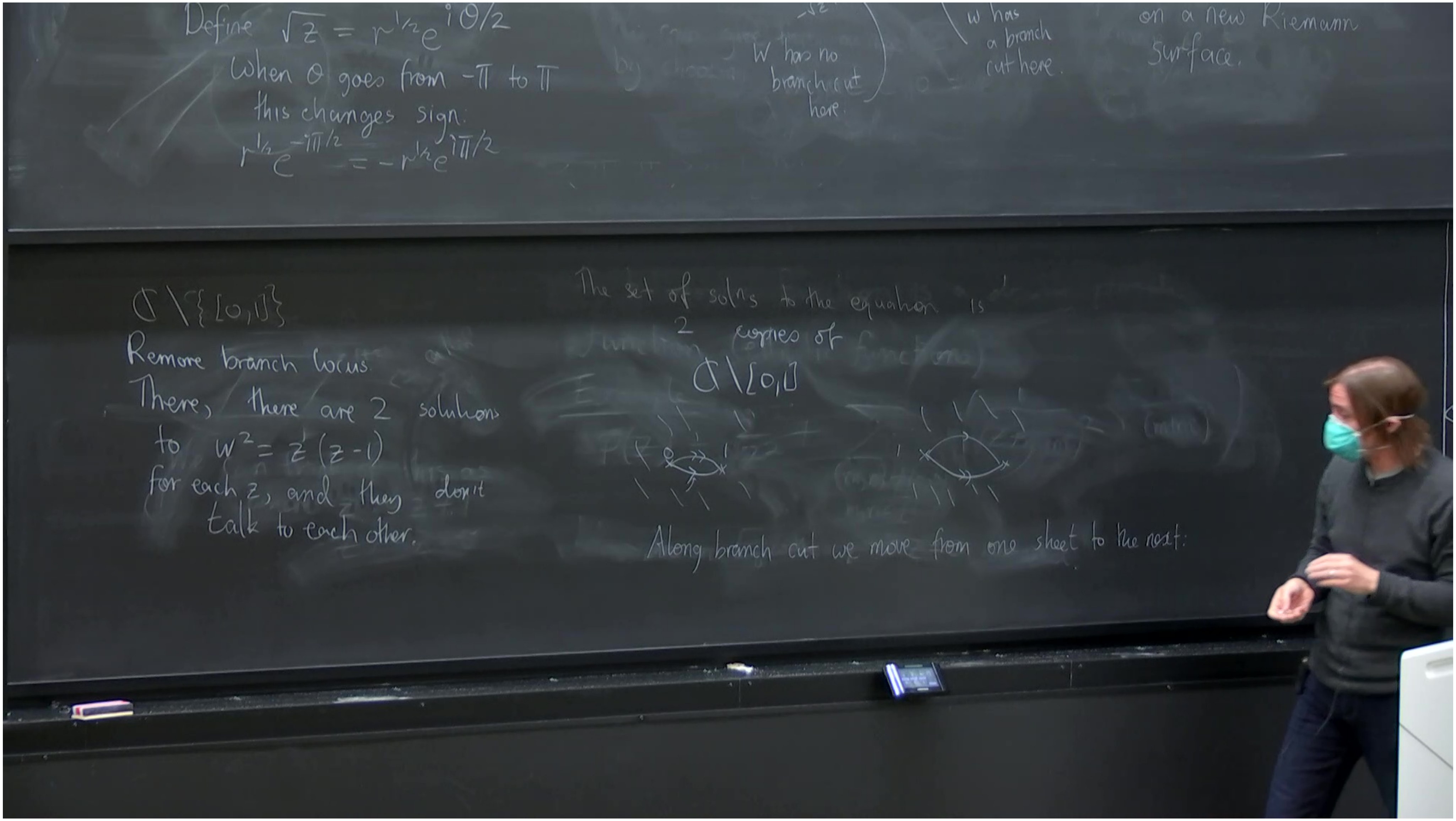
Example

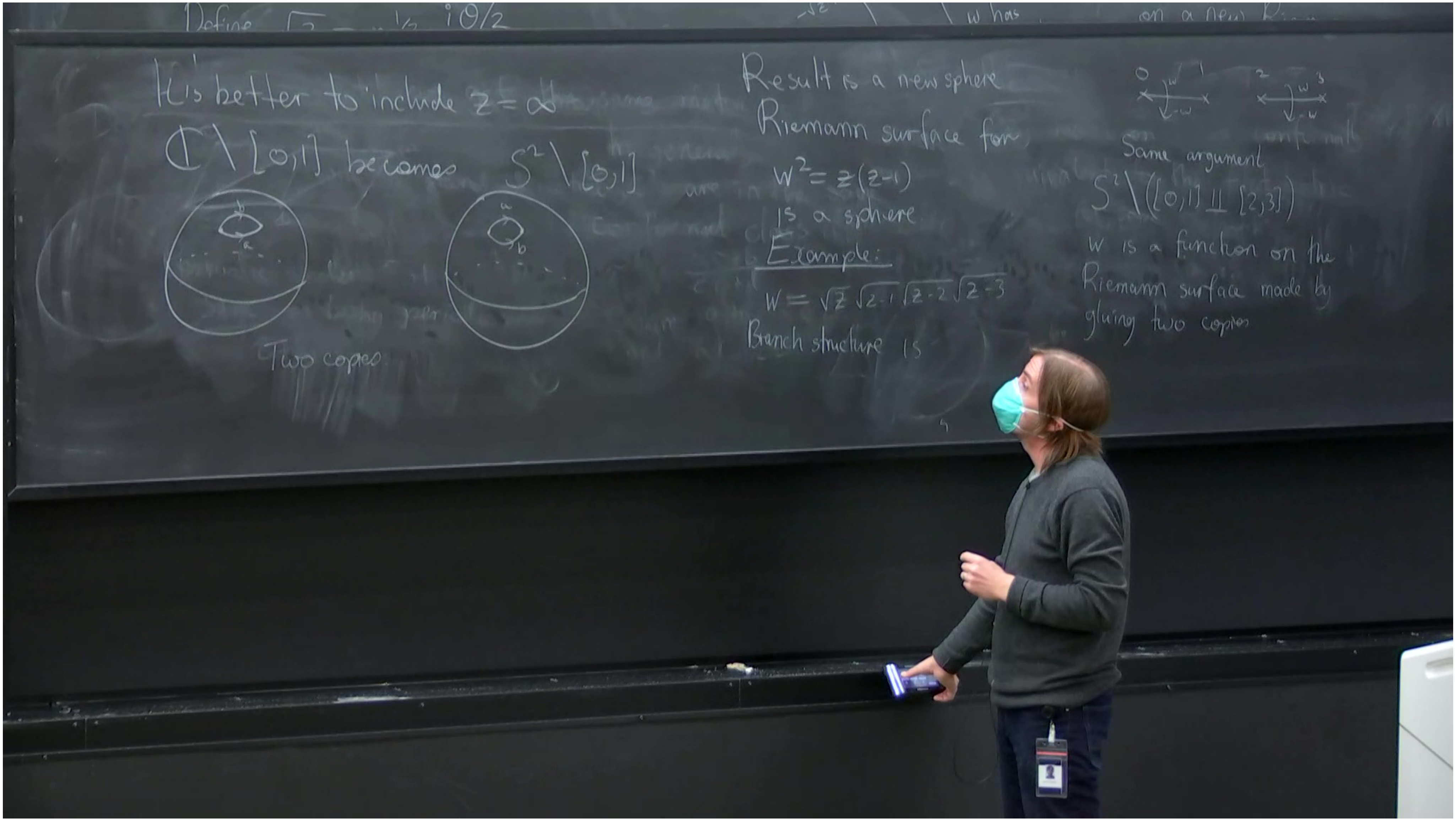
$$P(z) = \frac{1}{z^2} + \sum_{\substack{(m,n)\neq(0,0)\\ m,n\in\mathbb{Z}}} \left( \frac{1}{(z+m+n\tau)^2} - \frac{1}{(m+n\tau)^2} \right)$$

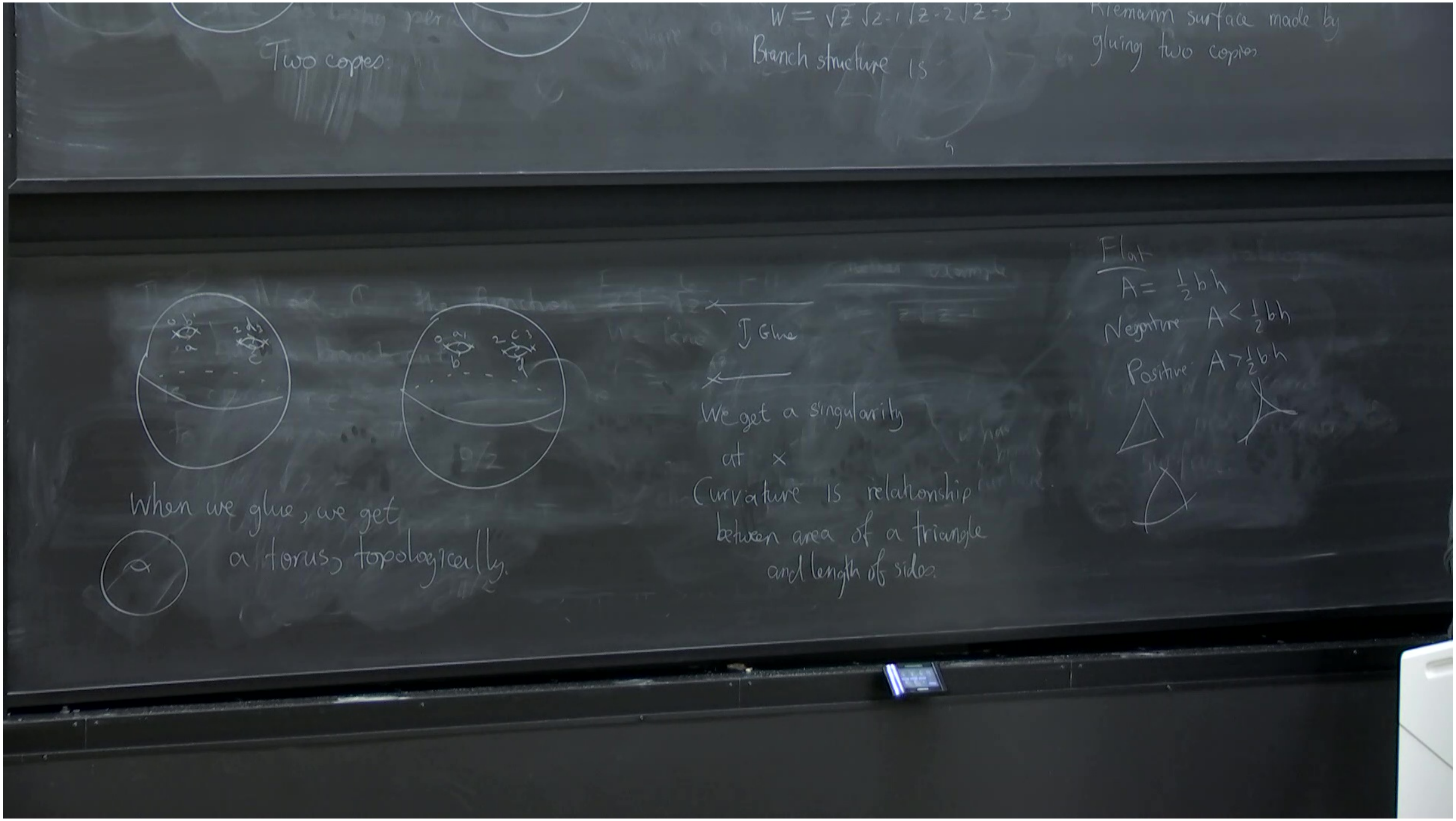
holomorphic on $E$ with a point $z=0$ removed.

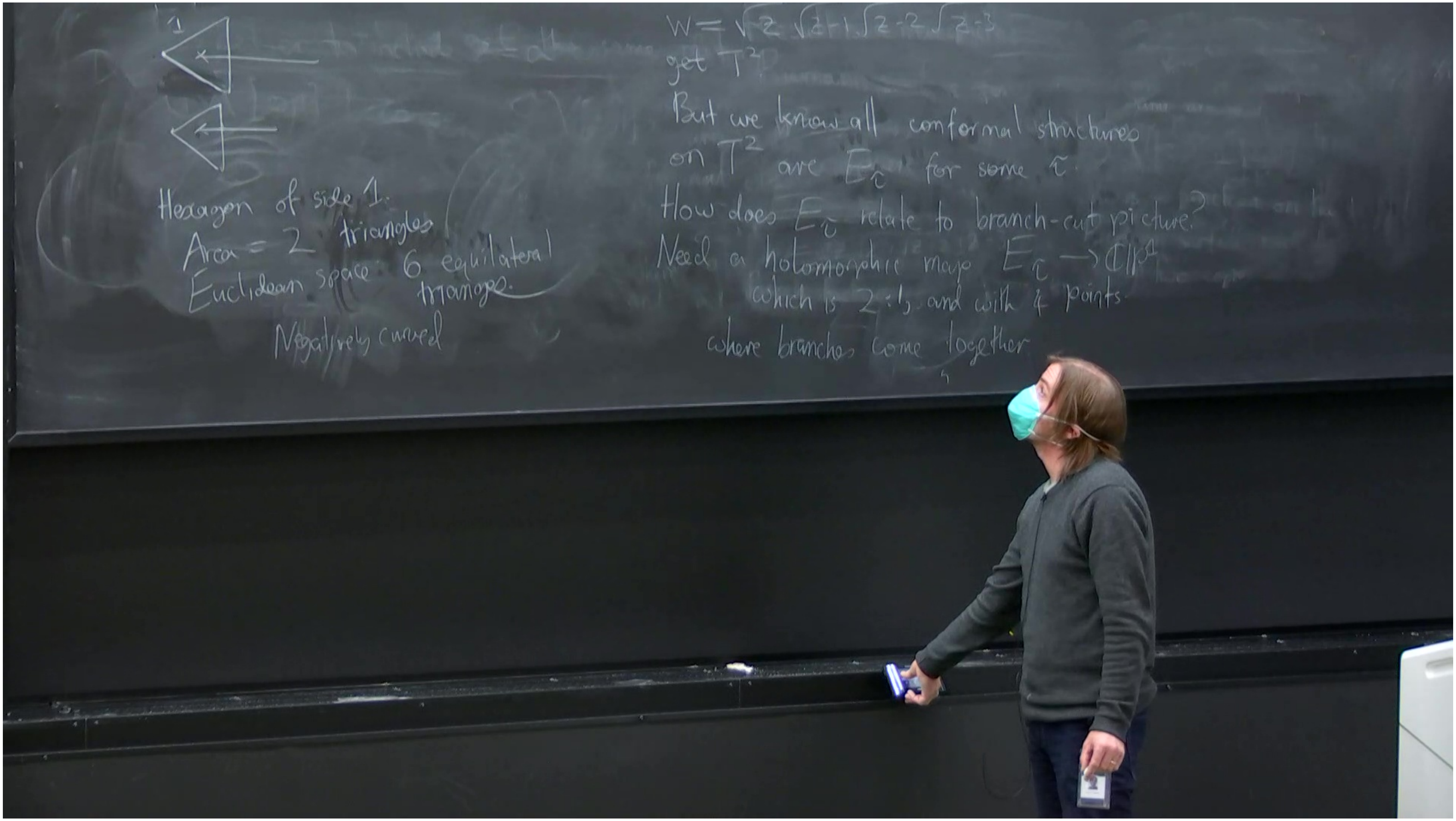

We can think of this as
ℂ where z ~ z+1
z ~ z+τ
P(z) = 1/z² + Σ_{(m,n)≠(0,0), m,n∈ℤ} ( 1/(z+m+nτ)² − 1/(m+nτ)² )
holomorphic on E_τ with a point z=0 removed.
If τ → τ+1 we get the same metric
τ τ+1 τ+2
1 2
Periodic under τ, 1
Same as being periodic under τ+1, 1
In general,
τ, τ' are in the same conformal class iff
τ' = (aτ+b)/(cτ+d)
where a,b,c,d ∈ ℤ
and ad−bc=1
Theorem
Any metric on T² is conformally equivalent to the flat metric E_τ for some τ.


We can think of this as
ℂ where z ~ z + 1
z ~ z + τ
P(z) = 1/z² + Σ_{(m,n)≠(0,0), m,n∈ℤ} ( 1/(z+m+nτ)² − 1/(m+nτ)² )
holomorphic on E_τ with a point z = 0 removed.
Recall on ℂ, the function z ↦ √z
has a branch cut.
z = re^{iθ}
−π < θ < π
Define √z = r^{1/2} e^{iθ/2}
When θ goes from −π to π
this changes sign.
r^{1/2} e^{−iπ/2} = −r^{1/2} e^{iπ/2}
Another example
w = √z √(z−1)
w has no branch cut here.
w has a branch cut here.
Claim w is a coord on a new Riemann surface.


Define $\sqrt{z} = r^{1/2} e^{i\theta/2}$
When $\theta$ goes from $-\pi$ to $\pi$
this changes sign:
$r^{1/2} e^{-i\pi/2} = -r^{1/2} e^{i\pi/2}$
W has no branch cut here.
w has a branch cut here.
on a new Riemann surface.
$\mathbb{C} \setminus \{[0,1]\}$
Remove branch locus.
There, there are 2 solutions
to $w^2 = z(z-1)$
for each z, and they don't
talk to each other.
The set of sol'ns to the equation is
2 copies of
$\mathbb{C} \setminus [0,1]$
Along branch cut we move from one sheet to the next:


It's better to include z = ∞
ℂ \ [0,1] becomes S² \ [0,1]
Two copies
Result is a new sphere
Riemann surface for
w² = z(z-1)
is a sphere
Example:
w = √z √(z-1) √(z-2) √(z-3)
Branch structure is
Same argument
S² \ ([0,1] ⊔ [2,3])
w is a function on the
Riemann surface made by
gluing two copies


Two copies
W = √z √z-1 √z-2 √z-3
Branch structure is
Riemann surface made by gluing two copies
Glue
We get a singularity at x
When we glue, we get a torus, topologically.
Curvature is relationship between area of a triangle and length of sides.
Flat
A = 1/2 bh
Negative A < 1/2 bh
Positive A > 1/2 bh

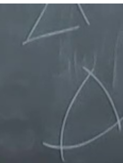


W = √z √z-1 √z-2 √z-3
get T²?
But we know all conformal structures
on T² are E_τ for some τ.
How does E_τ relate to branch-cut picture?
Need a holomorphic map E_τ → CP¹
which is 2:1, and with 4 points
where branches come together
Hexagon of side 1.
Area = 2 triangles
Euclidean space: 6 equilateral triangles.
Negatively curved

at $\times$

When we glue, we get a torus, topologically.

Curvature is relationship between area of a triangle and length of sides.



$\wp(z)$ is doubly periodic, double pole at 0

Define a map

$$E_\tau \longrightarrow \mathbb{C}\mathbb{P}^1$$

$$z \longmapsto \frac{1}{\wp(z)}$$

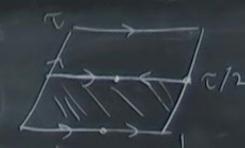
This will do what we want.

$$\wp(z) = \wp(-z)$$

The two branches are related by $z \longrightarrow -z$

This suggests $\mathbb{C}\mathbb{P}^1$ is $E/z\sim -z$.

To see this




$z \sim z+1$

$z \sim z+\tau$

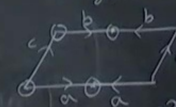
$z \sim -z$

Always move $z$ to the shaded region

$\frac{\tau}{2}+\frac{1}{2}$ is fixed under $z \to -z$

$\frac{\tau}{2}+c \to \frac{\tau}{2}+(1-c)$

We get




This is $S^2$

4 corners are special points