Title: Geometry and topology for physicists 2021/2022 - Lecture 12

Speakers: Kevin Costello, Giuseppe Sellaroli

Collection: Geometry and Topology for Physicists 2021/2022

Date: March 25, 2022 - 10:15 AM

URL: https://pirsa.org/22030070

1. "Remarks on the notion of quantum integrability" arXiv: 1012.3587 (2010)

2. "A bound on chaos", arXiv: 1503.01409 (2015)

$g \longmapsto e^{-2\varphi} g$ is called a Weyl transformation.

A conf. transformation is a coord. change where g transforms to $e^{-2\varphi} g$ for some φ

Flat metric

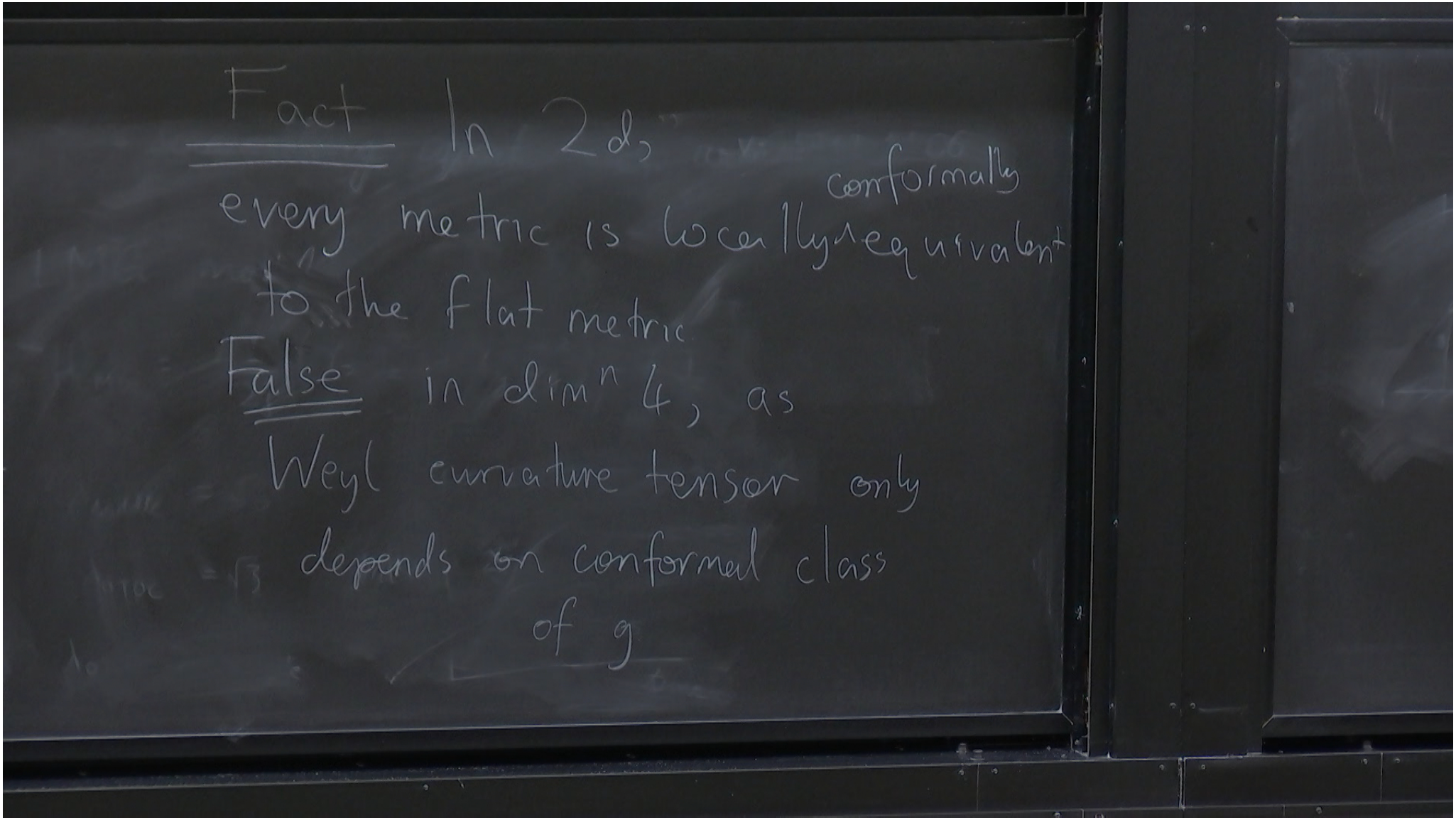
$$dx^2 + dy^2$$

$x \to 10x$

$y \to 10y$

is a conf. transformation

(but not an isometry)


Fact In 2d, every metric is locally conformally equivalent to the flat metric
False in dim^n 4, as Weyl curvature tensor only depends on conformal class of g

In $\dim^n 2$, the curvature is just Scalar curvature S

$$S(e^{-2\varphi} g) = e^{-2\varphi}(S - \Box\varphi)$$

If we can find φ so $\Box\varphi = S$

then we can set S to 0.

Locally we can always do this.

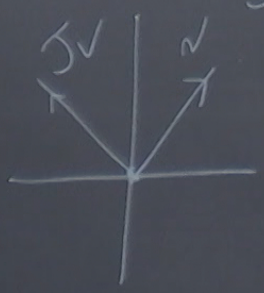
Alternative formulation of conformal geometry

If Σ has a conformal class of metric, + oriented,

for each $p \in \Sigma$ define

$J : T_p\Sigma \to T_p\Sigma$

$J(v) =$ "rotate v counter-clockwise by $90°$"



I.e. Jv is unique vector so

$$g(v, Jv) = 0$$

$$g(v, v) = g(Jv, Jv)$$

and v, Jv form an orientated basis.

If $g \to cg$, equations don't change.

J only only depends on conformal class.

i.e. Jv is unique vector so

$$g(v, Jv) = 0$$

$$g(v, v) = g(Jv, Jv)$$

and v, Jv form an orientated basis.

If $g \to e^{-2\alpha} g$, equations don't change.

J only only depends on conformal class.

Converse is true:

If $\forall p \in \Sigma$, we have

$J : T_p\Sigma \to T_p\Sigma$

$J^2 = -Id$

varies smoothly with p in surface

$\Longleftrightarrow$ as a conformal class of metric.

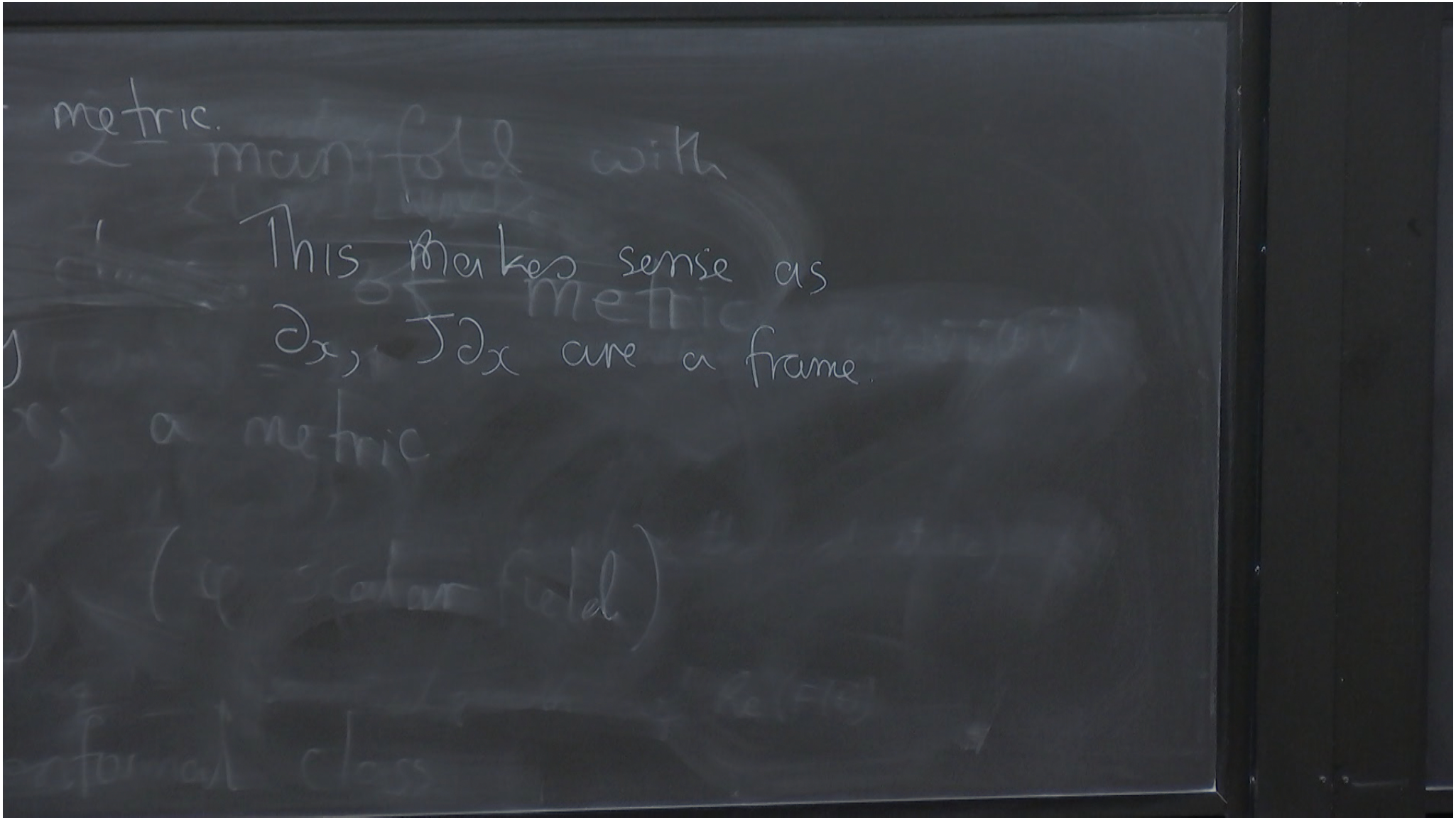
Proof:

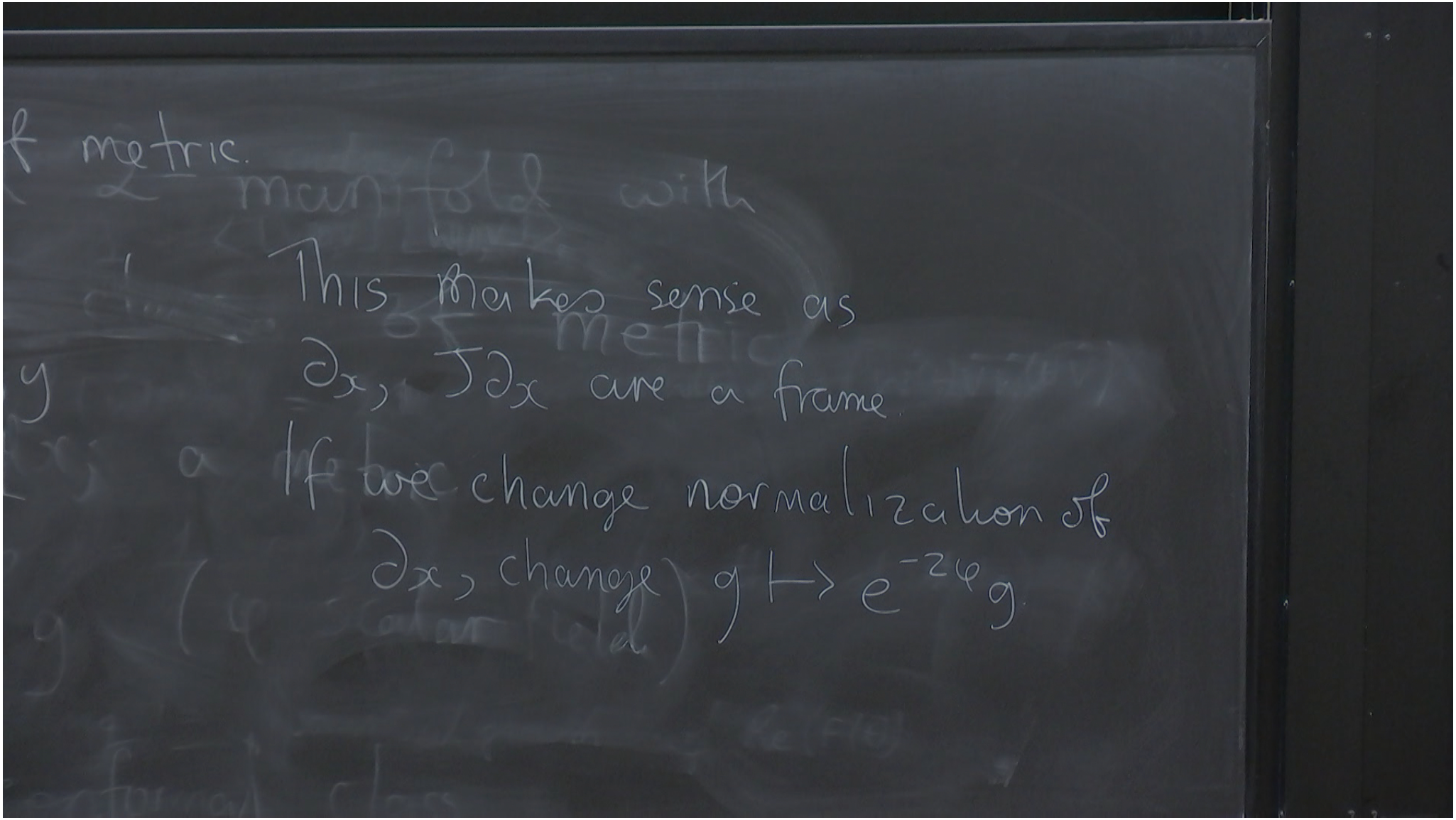
Choose coordinates x, y

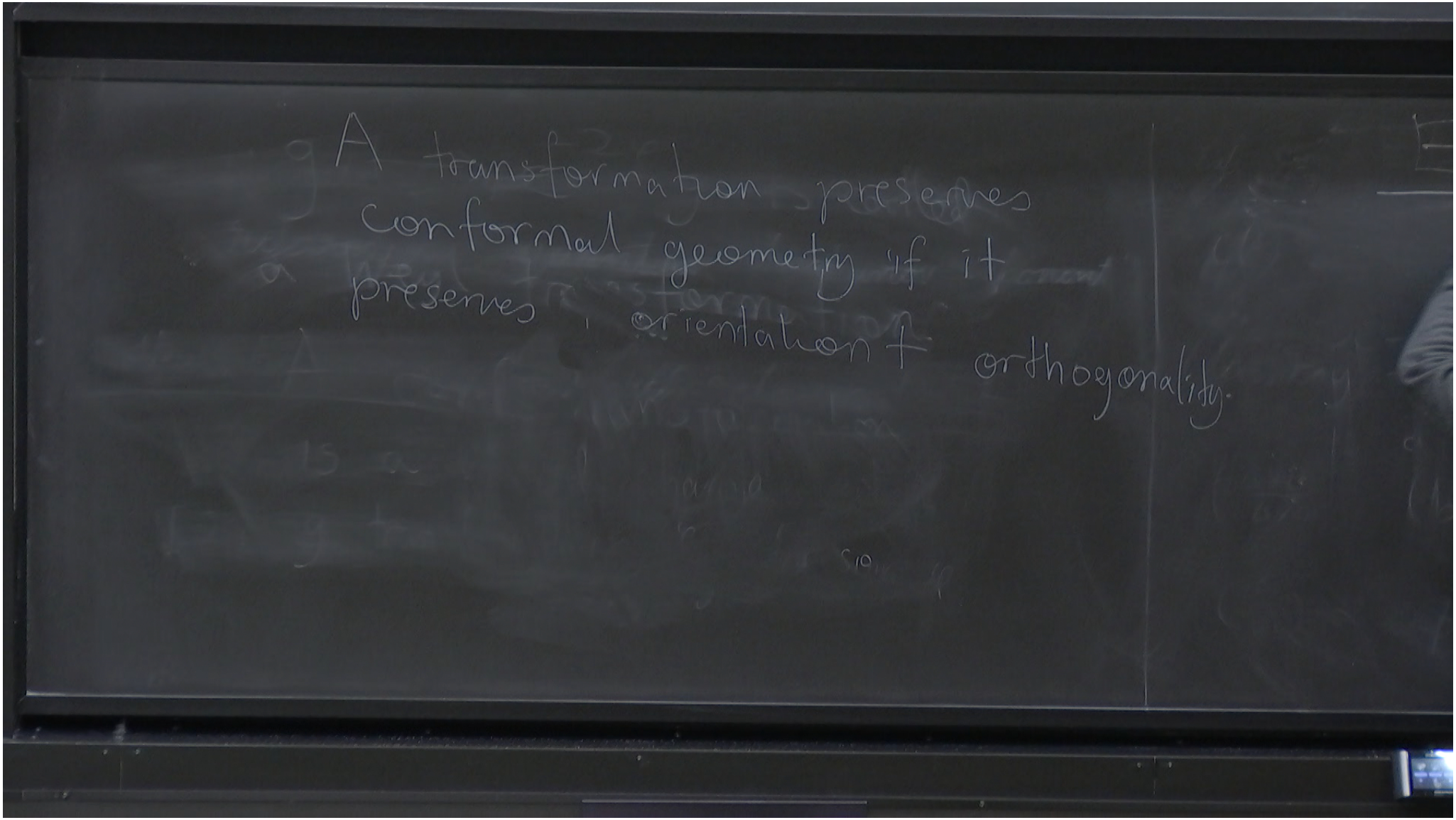
Define $g(\partial_x, \partial_x) = 1$

$g(\partial_x, J\partial_x) = 0$

$g(J\partial_x, J\partial_x) = 1$

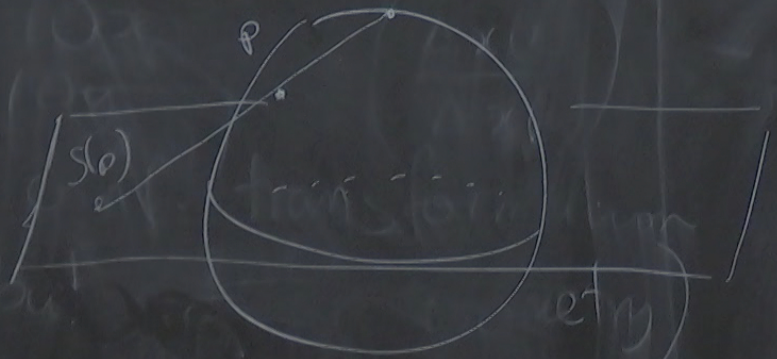

metric.
manifold with
This makes sense as
∂_x, $J\partial_x$ are a frame.
a metric


metric.
manifold with
This makes sense as
∂_x, $J\partial_x$ are a frame.
If we change normalization of
∂_x, change $g \mapsto e^{-2\varphi} g$.


A transformation preserves
conformal geometry if it
preserves orientation & orthogonality.

Examples

Stereographic Projection:



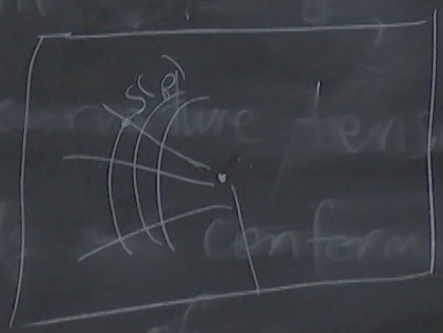
every metric is ... to the flat metric

False in ... Weyl curvature tens... depends ... Conform...

Projection:

This preserves orthogonality

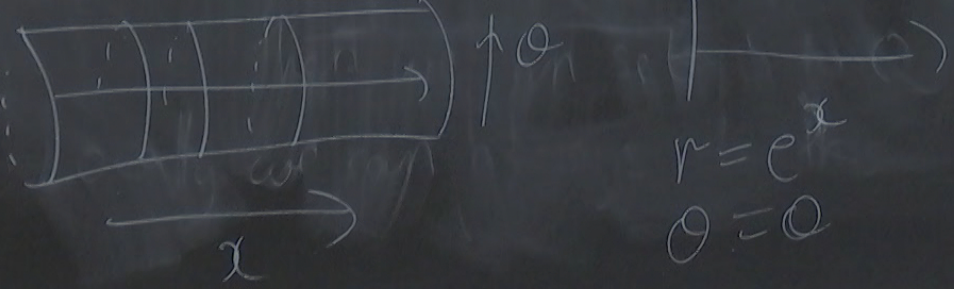
Orthogonal lines on S^2 are sent to

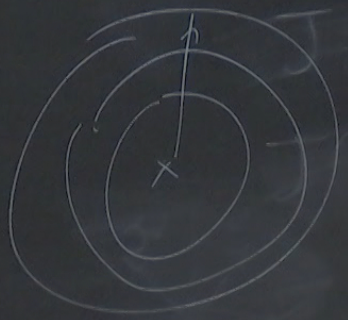


Other examples

$\mathbb{R}^2 \setminus 0$

is conformally equivalent to the cylinder.



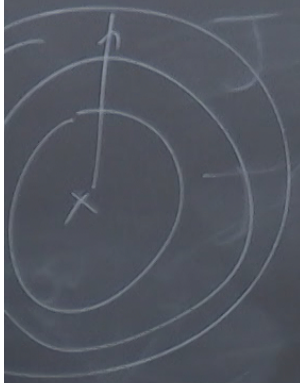


One more example:

Transformation $\mathbb{R}^2 \setminus \text{Origin}$ ↻

$(r, \theta) \longrightarrow \left(\frac{1}{r}, -\theta\right)$

preserves orientation and orthogonality



$\langle x, y \rangle$ a number

eqr

$\langle x, y \rangle = 0$

doesn't change if we scale $\langle\ \rangle$

Complex Geometry Interpretation

J acts on $T_p\Sigma$

and so on the dual, $T_p^*\Sigma$ by transpose

If in coordinates $J\partial_x = \partial_y$

$J\partial_y = -\partial_x$

then,

$J\,dy = dx$

$J\,dx = -dy$

Then, Also $J^2 = -1$

$J\,dy = dx$

$J\,dx = -dy$

J acts on $\Omega^1(\Sigma)$

Eigenvalues of J are $\pm i$

eigenvectors must have complex coefficients.

$\Omega^1(\Sigma, \mathbb{C})$ = 1-forms with complex coefficients

$\Omega^{1,0}(\Sigma, \mathbb{C}) = +i$ eigenspace of J

$\Omega^{0,1}(\Sigma, \mathbb{C}) = -i$ eigenspace of J.

In coords Since g is locally conformally flat, always find coords so

$$J\partial_x = \partial_y$$

$$J\partial_y = -\partial_x$$

Then, on a 1-form

$$J(f\,dx + g\,dy) = -f\,dy + g\,dx$$

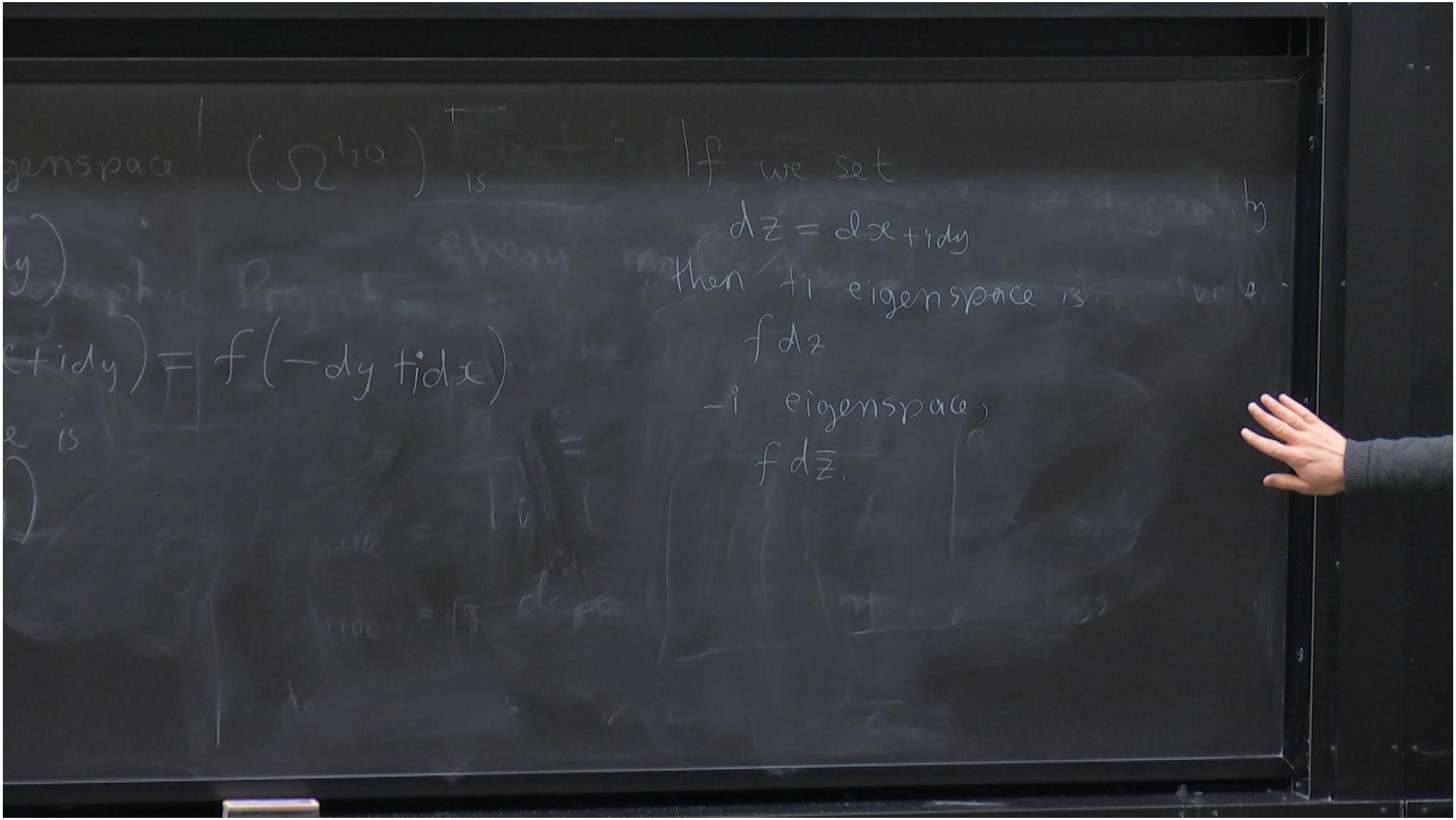
The $+i$ eigenspace of $(\Omega^{1,0})^*$ is

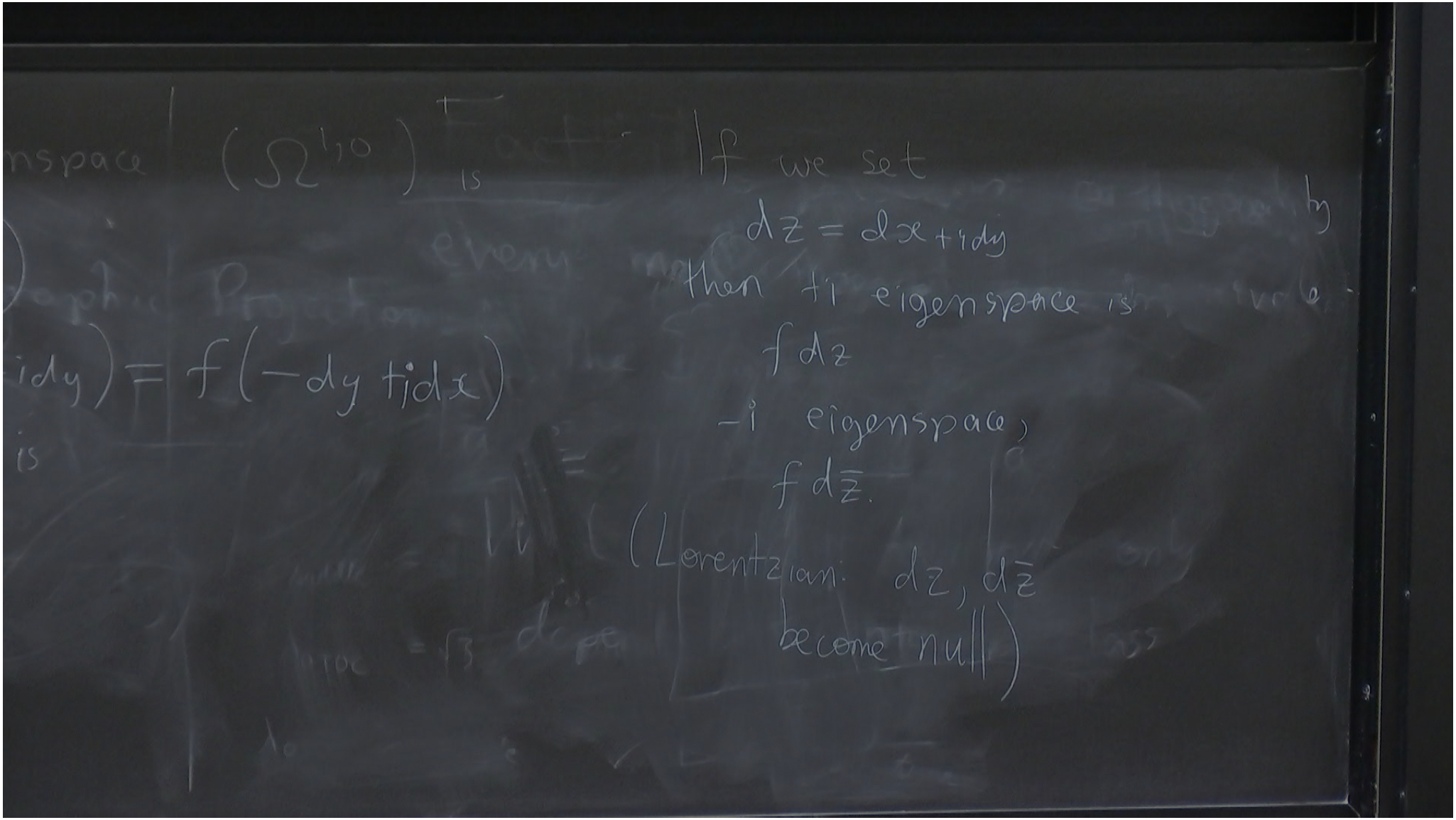
$f(dx + i\,dy)$

As, $J\,f(dx + i\,dy) = f(-dy + i\,dx)$

$-i$ eigenspace is

$f(dx - i\,dy)$


genspace
$(\Omega^{1,0})^*$ is
If we set
$dz = dx + i dy$
then $+i$ eigenspace is
$f\,dz$
$-i$ eigenspace
$f\,d\bar{z}$.


nspace
$(\Omega^{1,0})^{\top}$ is
If we set
$dz = dx + i\,dy$
then $+i$ eigenspace is
$f\,dz$
$-i$ eigenspace,
$f\,d\bar{z}$.
(Lorentzian: dz, $d\bar{z}$
become null)
$idy) = f(-dy + i\,dx)$
is

Cauchy Riemann equations

On a surface Σ, if

$f \in \Omega^0(\Sigma, \mathbb{C})$ $\quad \in \Omega^{0,1}(\Sigma)$

define $\bar{\partial} f = \frac{1}{2i}(J - i)\, df$ = projection to $-i$ eigenspace

$\partial f = \frac{1}{2i}(J + i)\, df \in \Omega^{1,0}(\Sigma)$ " $+i$ "

$$df = \partial f + \bar{\partial} f$$

We say f is holomorphic if

$$\bar{\partial} f = 0$$

In coords:

$$df = dx\,\partial_x f + dy\,\partial_y f$$

$$\bar{\partial} f = \frac{1}{2i}\left(-dy\,\partial_x f + dx\,\partial_y f - i\,dx\,\partial_x f - i\,dy\,\partial_y f\right)$$

$$= \frac{1}{2i}\left(\partial_y f - i\partial_x f\right)(dx - i\,dy)$$

In coordinates, $\bar{\partial} f = 0$ is the

are usual CR equation.

$$= \frac{1}{2i}\left(\partial_y f - i\partial_x f\right)\left(dx - i dy\right)$$

In coordinates, $\bar{\partial} f = 0$ is the

are usual CR equation.

Proof:

g is locally conformally flat.

□

Corollary The transition functions between patches are also holomorphic functions.

Riemann Surface:

1) g_{ij} up to Weyl transform.

2) J

3) Locally modelled on $\mathbb{C}$, with holomorphic transition functions