Title: Geometry and topology for physicists 2021/2022 - Lecture 10

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Collection: Geometry and Topology for Physicists 2021/2022

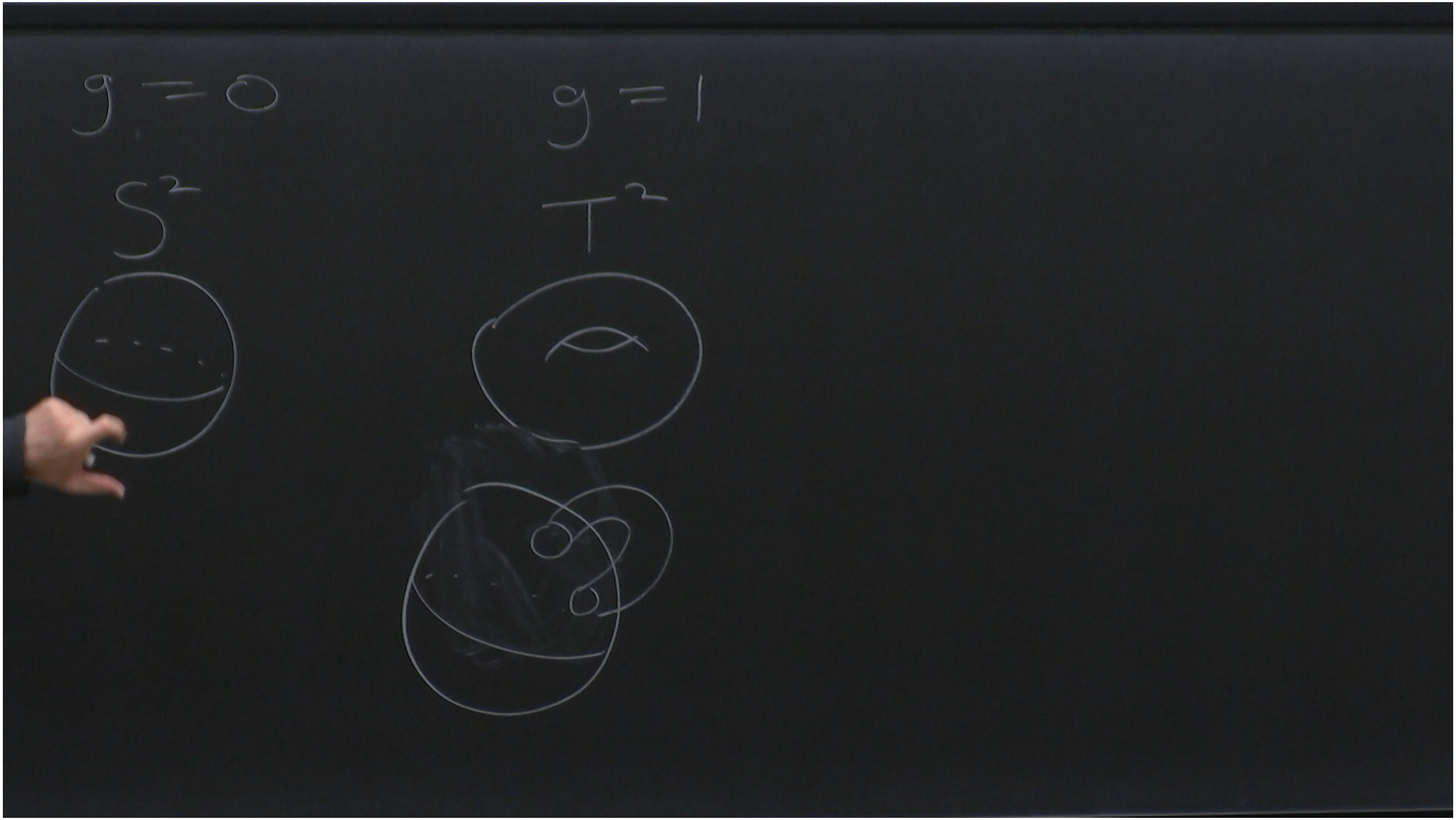
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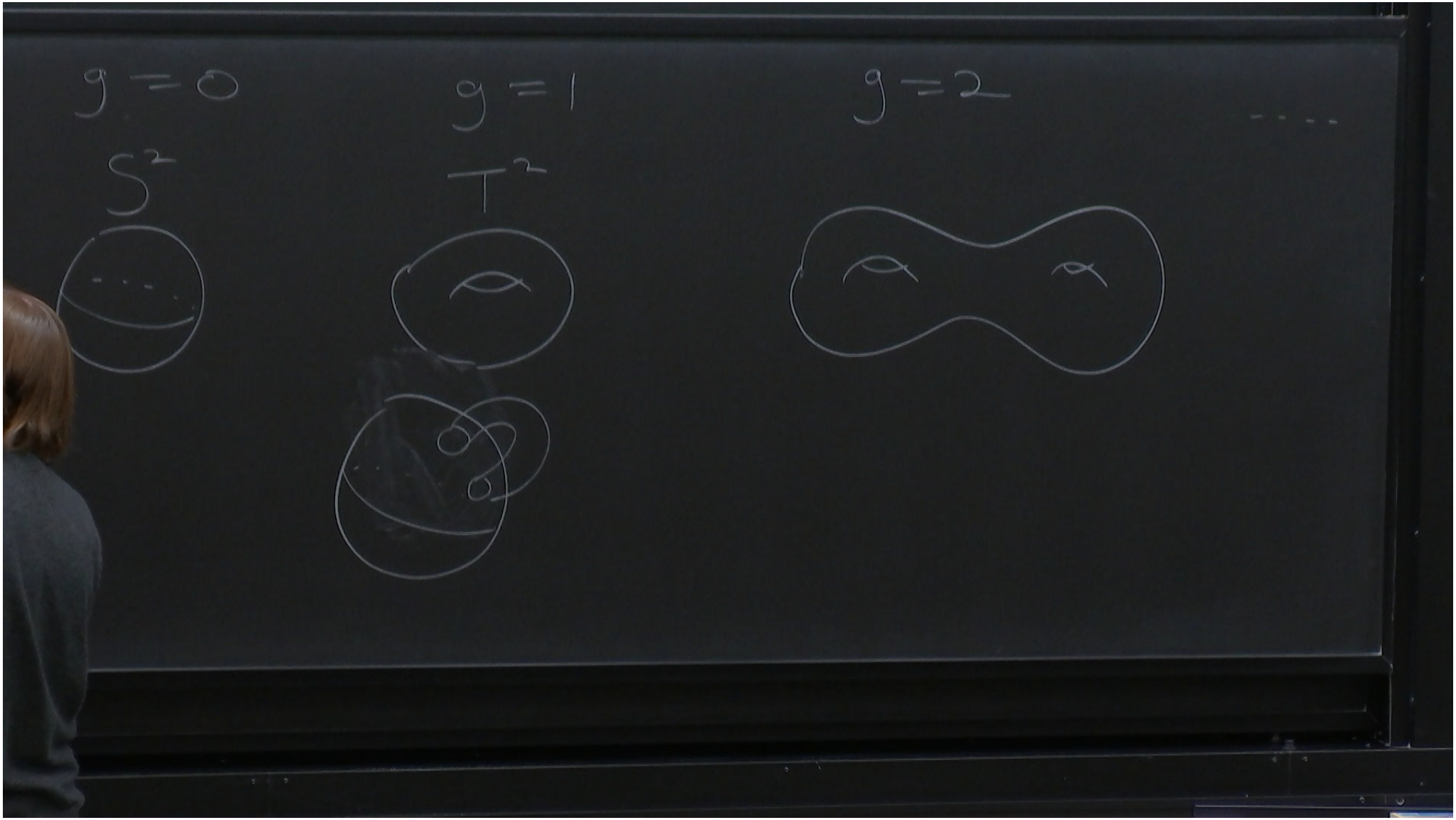
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Classification of Surfaces

Consider compact surfaces with no boundary. These are classified as follows:

1) Orientable surfaces are all obtained by adding g "handles" to S^2.


g = 0
S^2
g = 1
T^2


g = 0
S^2
g = 1
T^2
g = 2
....

Further, $\chi(\Sigma_g) = 2 - 2g$.

2) In the non-orientable case, every surface is obtained from S^2 by removing k discs, and sewing in k Möbius bands.

$\chi(S_k) = 2 - k$

$k=1$

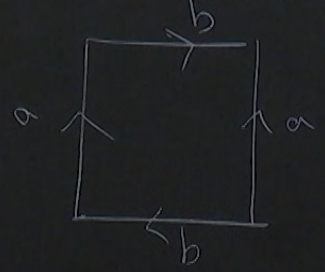
$\mathbb{RP}^2$

$= S^2$ / antipodal points identified

$= (S^2 \setminus \text{a disc}) \cup (\text{Möbius band})$

$k=2$:

Klein bottle



$k=1$

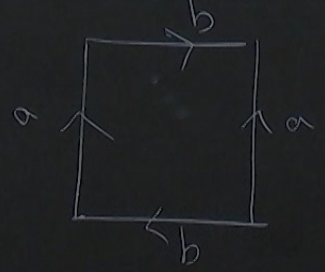
$\mathbb{RP}^2$

$= S^2$ / antipodal points identified

$= (S^2 \setminus$ a disc$) \cup$ (Möbius band)

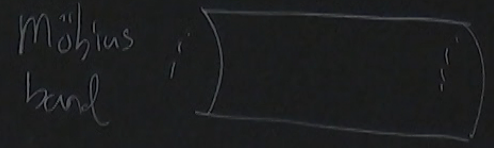
$k=2$:

Klein bottle



$S^2 \setminus 2$ discs

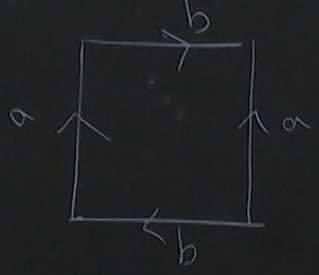
= a cylinder



$=$ 2 Möbius bands se

K=2:

Klein bottle

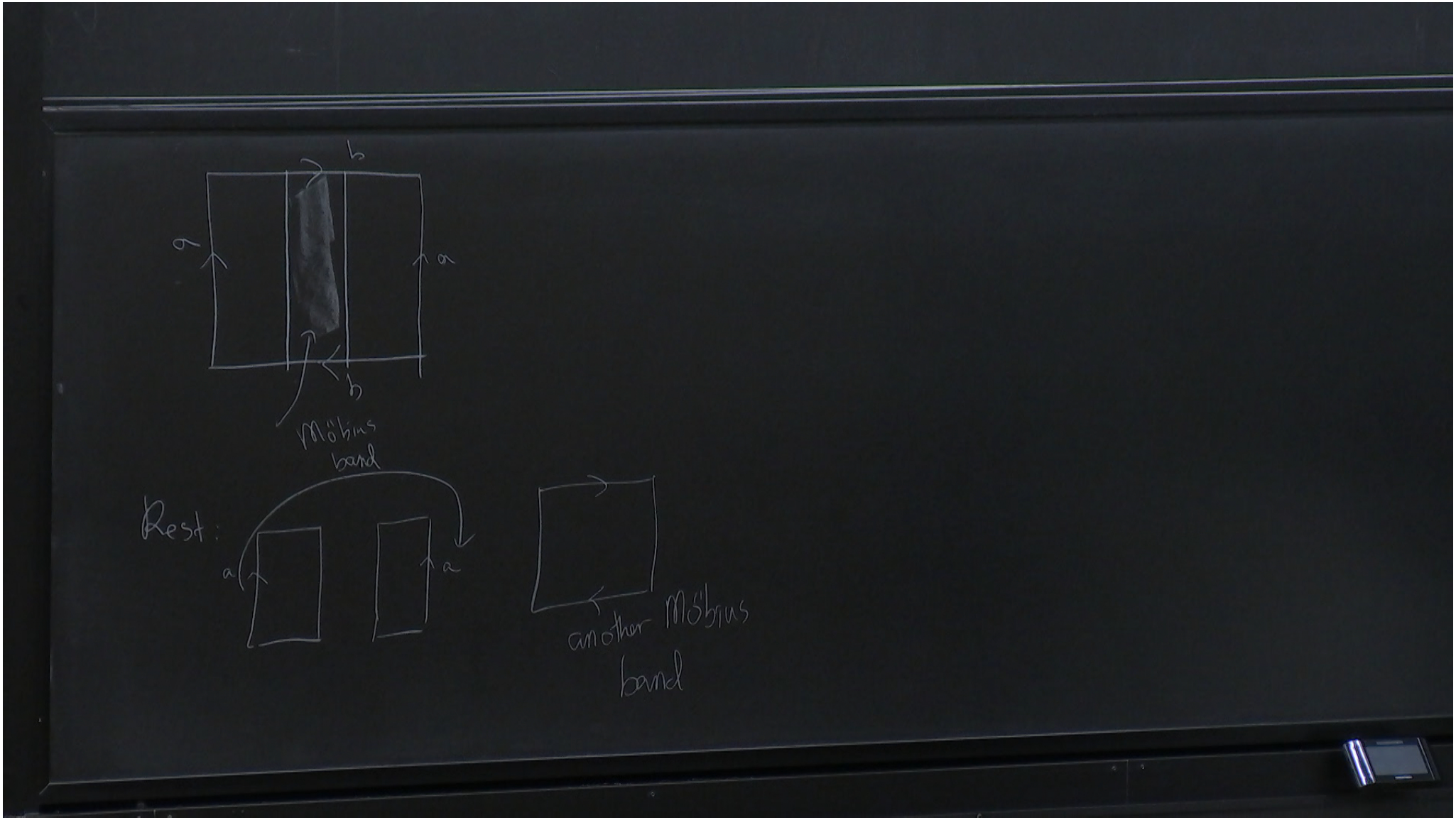


tified

(Möbius band)

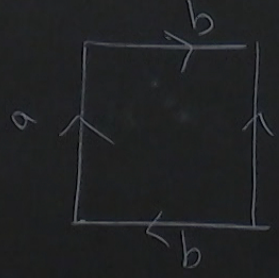
$S^2/2$ discs = a cylinder

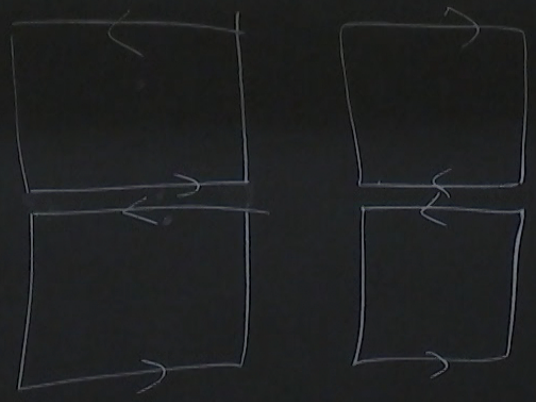
Möbius band ; Möbius band = 2 Möbius bands sewn together along boundary


b
a
a
b
Möbius band
Rest:
a
a
another Möbius band

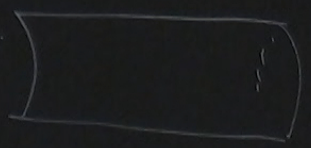
K=2:

Klein bottle



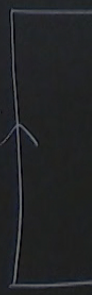


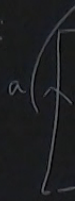
$S^2 \setminus 2$ discs
= a cylinder



Möbius band = 2 Möbius bands sewn together along boundary

Rest:

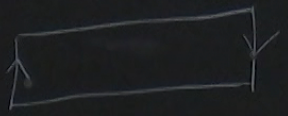




Oriented double cover of Σ is

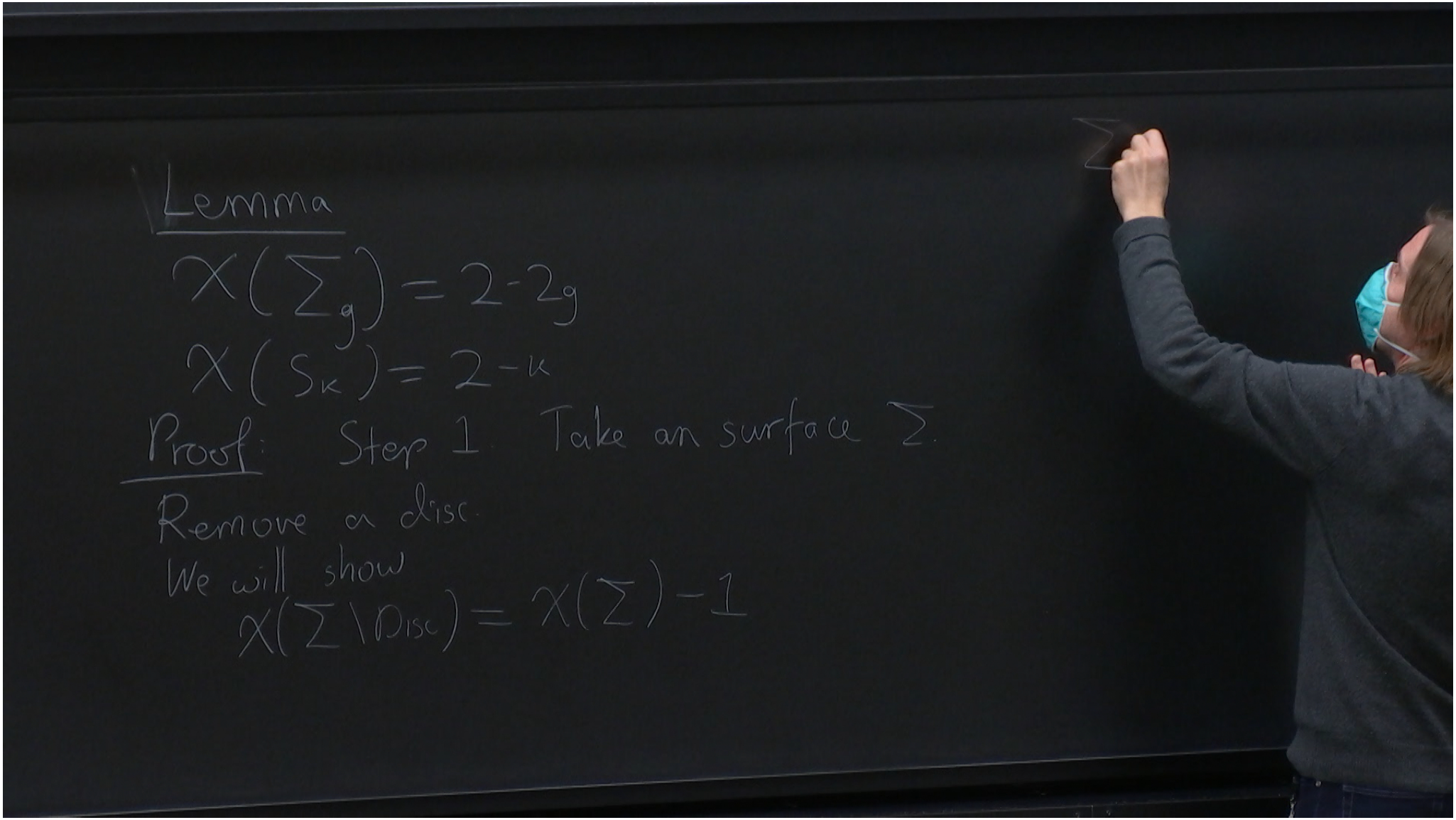
$$\widetilde{\Sigma} = \{ p \in \Sigma, \text{ and an orientation at } p \}$$

Oriented double cover of Möbius band is an annulus



$\theta \sim \theta + 2\pi$
$t \sim -t$

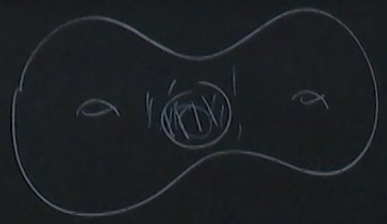
$\theta \sim \theta + 4\pi$
$t \sim t$


Lemma
$\chi(\Sigma_g) = 2-2g$
$\chi(S_k) = 2-k$
Proof: Step 1. Take an surface Σ.
Remove a disc.
We will show
$\chi(\Sigma \setminus Disc) = \chi(\Sigma) - 1$

$\Sigma = U \cup V$

$U = \Sigma \setminus$ a disc

$V =$ a bigger disc



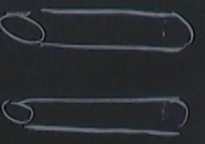
$U \cap V =$ an annulus $\simeq S^1$

$\chi(U \cap V) = 0$

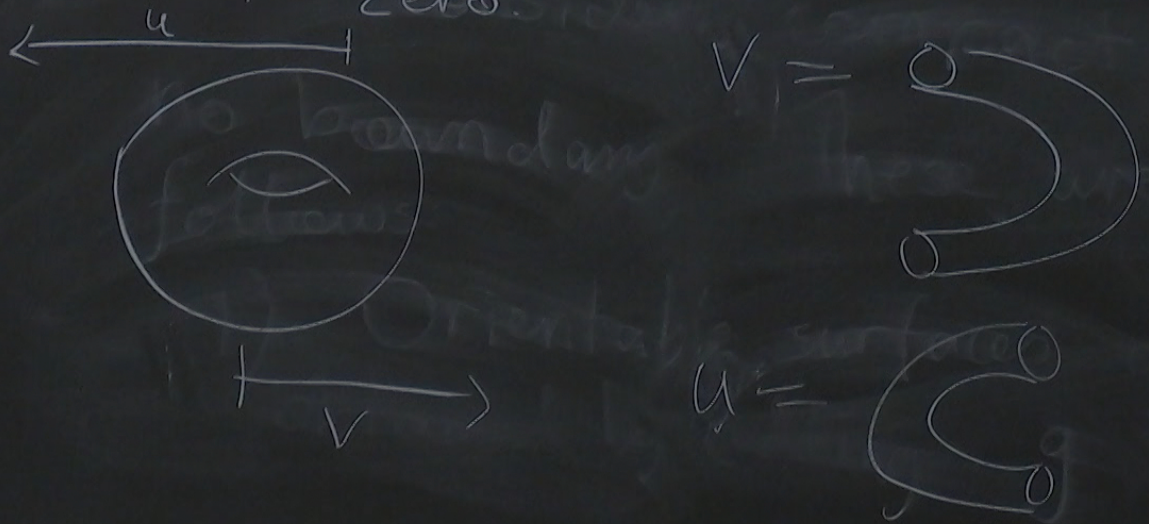
$\chi(\Sigma) = \chi(\Sigma \setminus \text{a disc}) + \chi(\text{disc})$

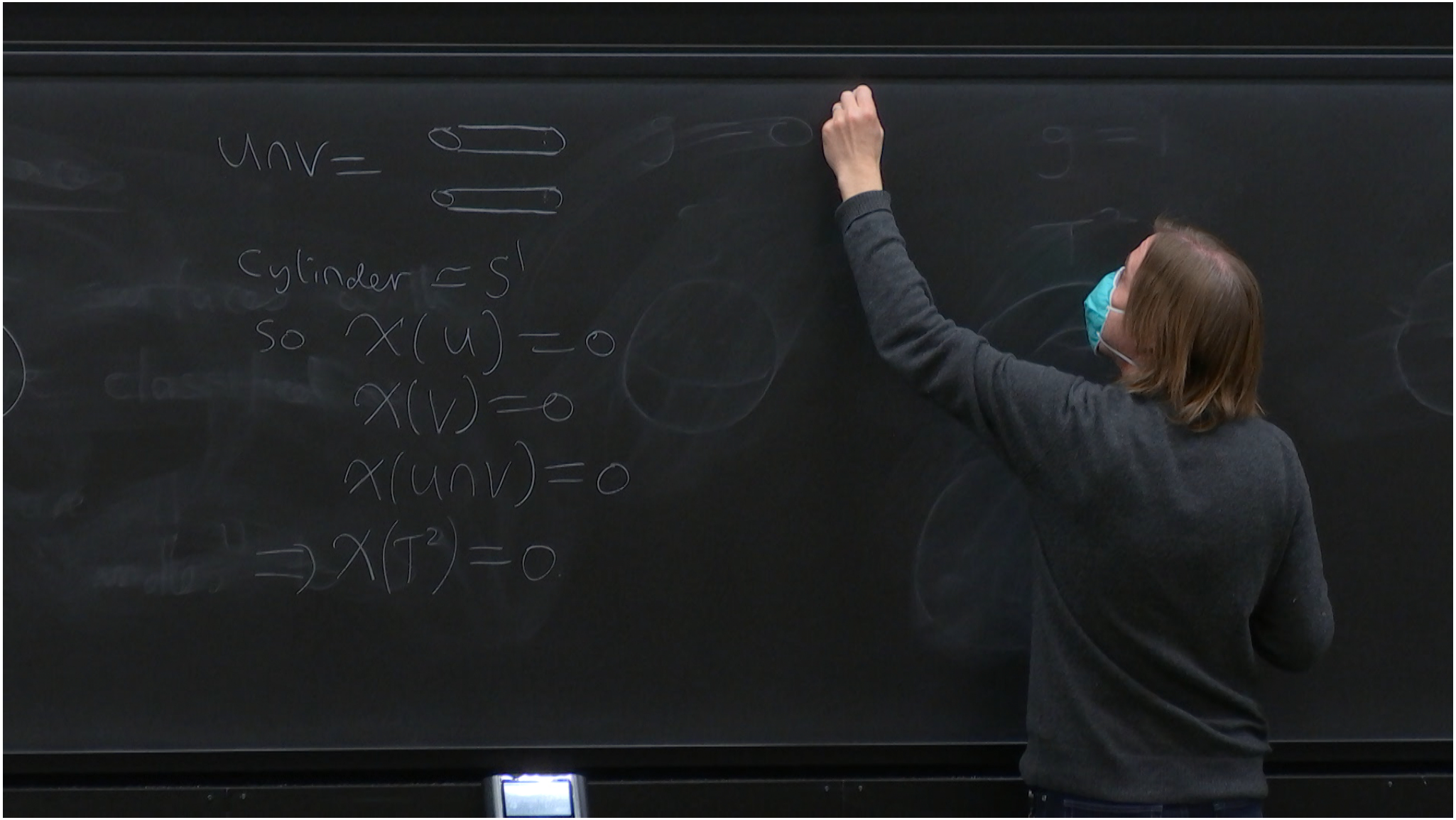
$= \chi(\Sigma \setminus \text{a disc}) + 1$

Next, compute $\chi(T^2)$

$U \cap V =$ 

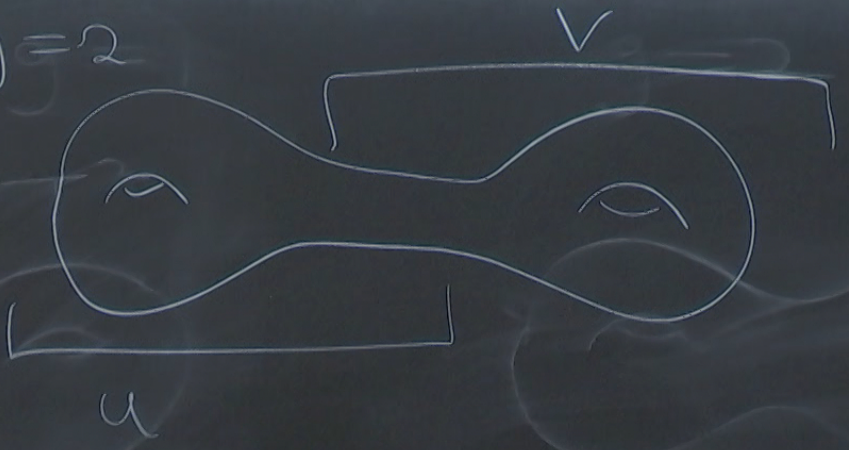
This is zero:




UnV=
Cylinder ≃ S¹
so χ(U)=0
χ(V)=0
χ(UnV)=0
⇒ χ(T²)=0

Next, $g=2$

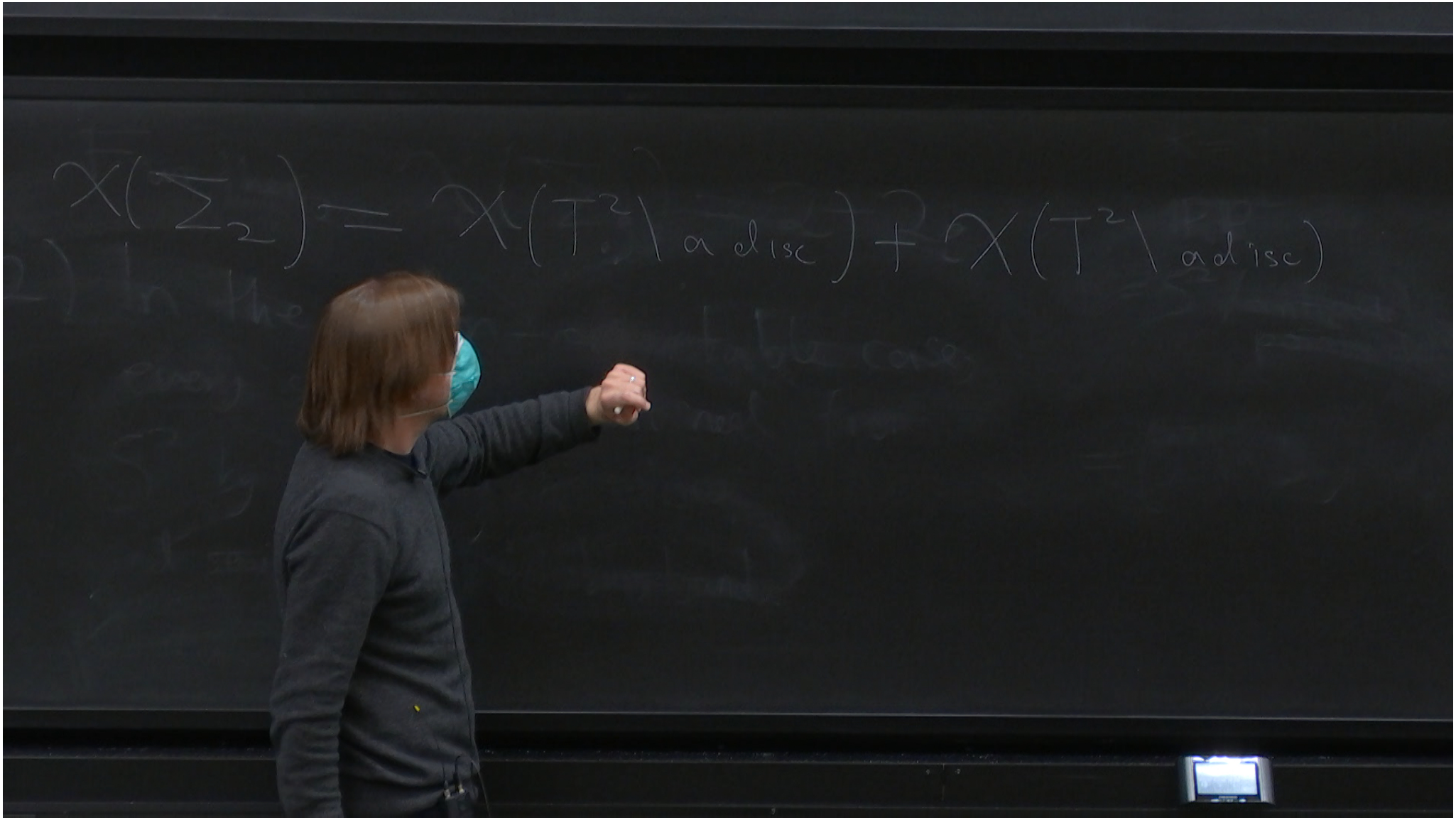
Σ_2

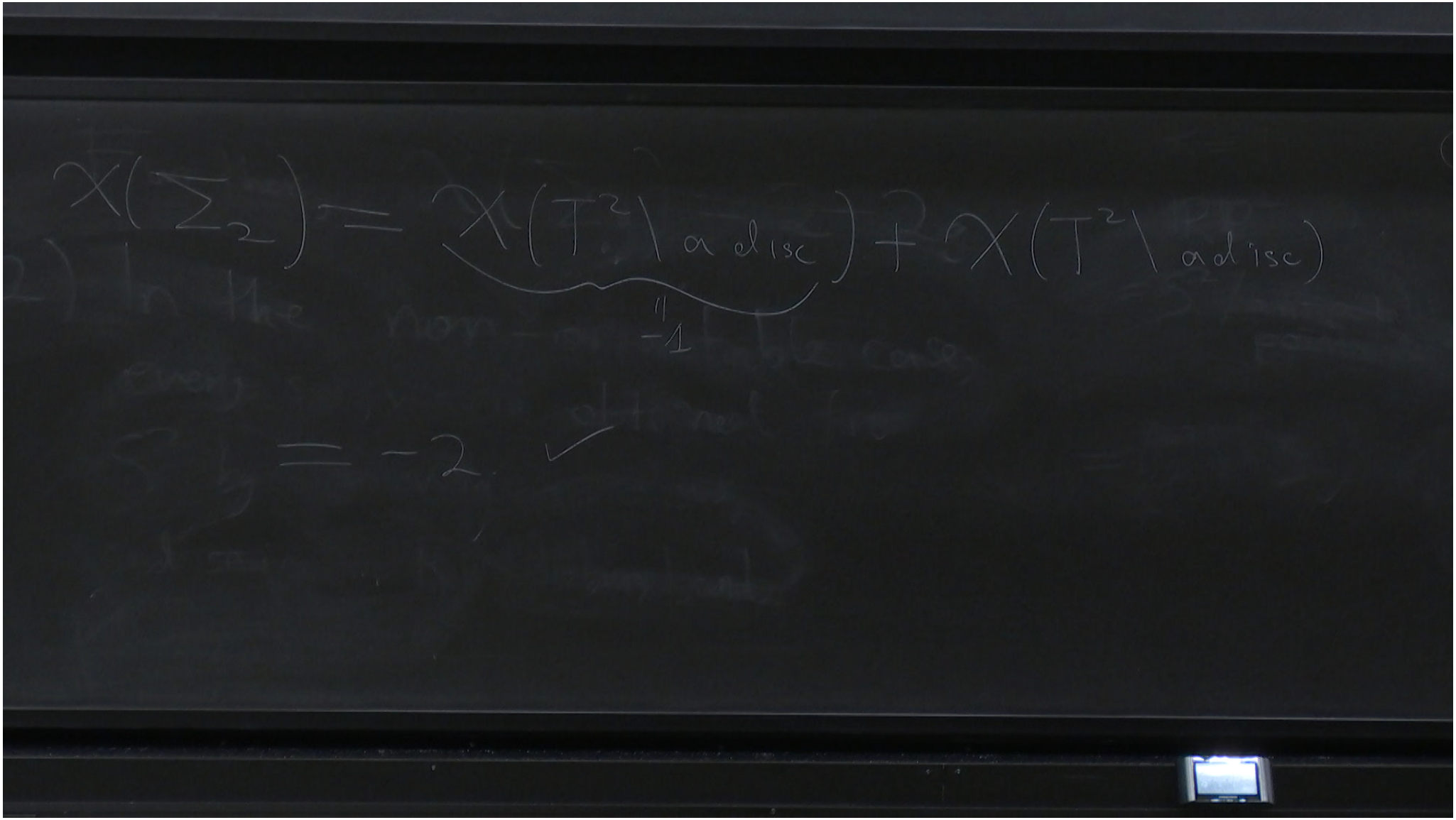


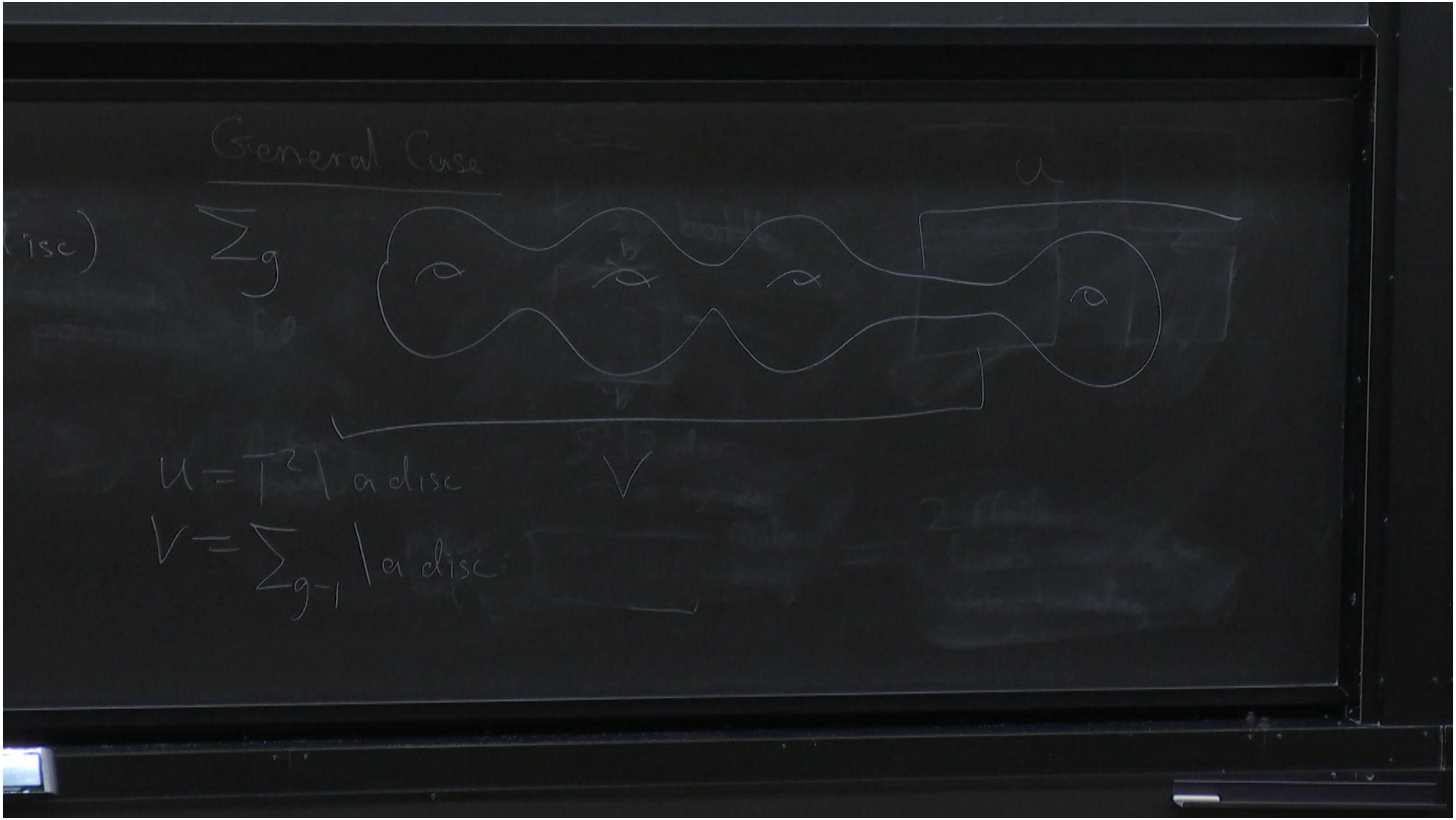
$U = T^2 \setminus$ a disc

$V = T^2 \setminus$ a disc

$U \cap V =$ cylinder


$\chi(\Sigma_2) = \chi(T^2 \setminus a\ disc) + \chi(T^2 \setminus a\ disc)$


$\chi(\Sigma_2) = \chi(T^2 \setminus \text{a disc}) + \chi(T^2 \setminus \text{a disc})$
-1
$= -2$


General Case
isc)
Σ_g
a
b
a
a
U
V
$U = T^2 \setminus$ a disc
$V = \Sigma_{g-1} \setminus$ a disc

$U \cap V$ = Cylinder (doesn't contribute)

$\chi(U) = -1$

$\chi(V) = \chi(\Sigma_{g-1}) - 1$

= (by induction)

$2 - 2(g-1) - 1$

$\chi(\Sigma_g) = 2 - 2(g-1) - 2$

$= 2 - 2g.$

S_k (unoriented surface)

$$\chi(S^2 \setminus k \text{ discs}) = \chi(S^2) - k$$
$$= 2 - k$$

Gluing in a Möbius band doesn't change χ, as $\chi(\text{Möbius band}) = 0$
and we glue along a cylinder with $\chi = 0$.

$\chi(S_k) = 2 - k$.

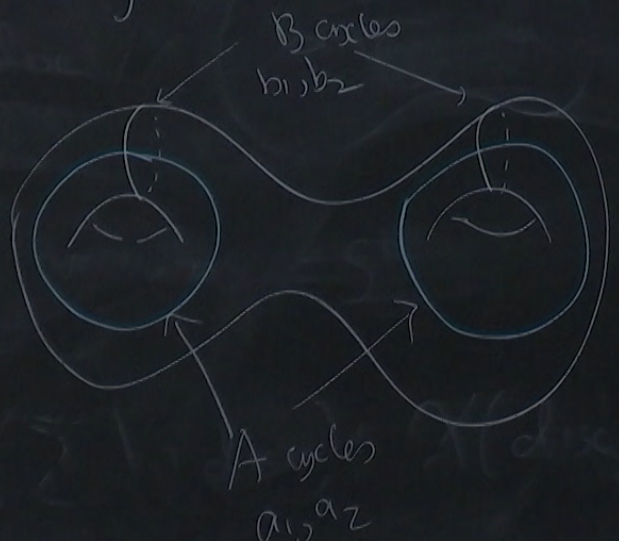
What is the cohomology of Σ_g?

$H^0(\Sigma_g) = \mathbb{R}$

$H^2(\Sigma_g) = \mathbb{R}$ by Poincaré duality

$H^1(\Sigma_g) = \mathbb{R}^{2g}$ from the Euler characteristic.

$H^1(\Sigma_g)$ is related to 1d submanifolds:



On a surface of genus g, are
g A cycles
g B cycles

$$\int_{\text{Path}} \omega$$

$\omega \mapsto \omega + d\eta$

changes to

$$\int_{\text{Path}} d\eta = \eta(\text{final}) - \eta(\text{initial})$$

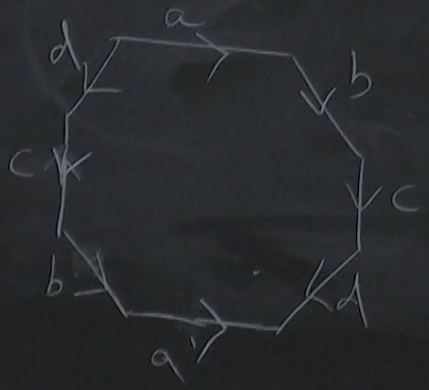
Can find a basis

$\omega_A^i \quad , \quad \omega_B^j$

where $\int_{a_k} \omega_A^i = \delta_k^i \qquad \int_{b_k} \omega_B^i = \delta_k^i$

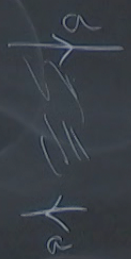
$\int_{b_k} \omega_A^i = 0 \qquad \int_{a_k} \omega_B^i = 0$

Every surface can be obtained by taking a polygon and identifying pairs of sides.



1) When is this orientable?

Answer: if it doesn't contain a Möbius band



2) What is χ?

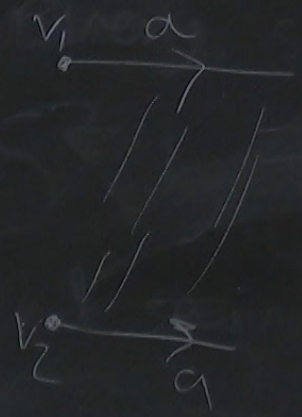
$\chi = V - E + F$

$F = 1$

$E = (\text{edges of polygon})/2$

V = # vertices that remain distinct on surface.

E.g. if we have

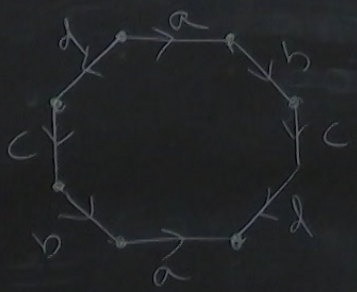


$V_1 = V_2$ on the surface.

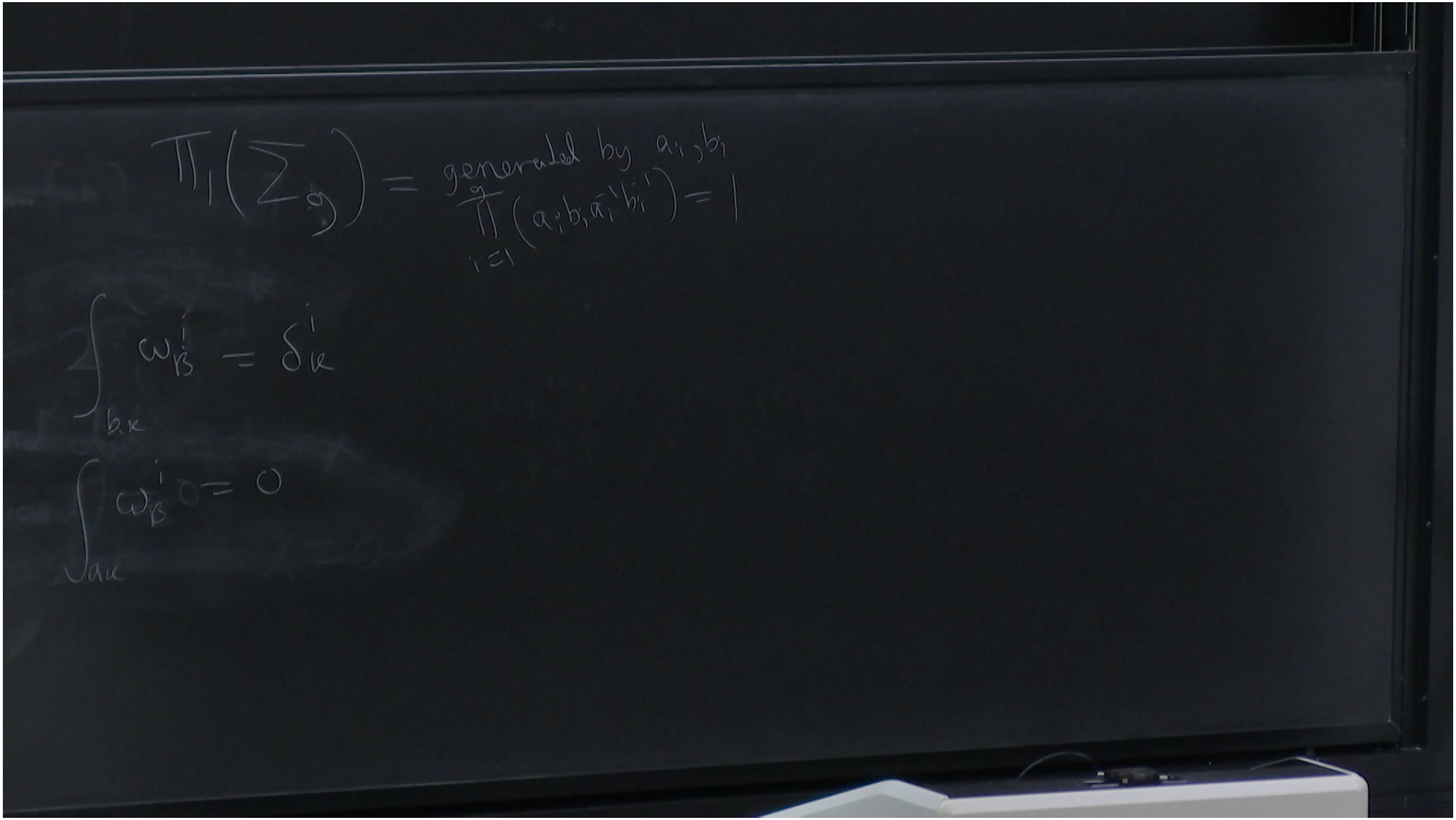
Example of octagon:

$F = 1, E = 4, V = 1$

$\Rightarrow \chi = -2$



and we find $\mathbb{Z}_2$.


$\pi_1(\Sigma_g)$ = generated by a_i, b_i
$\prod_{i=1}^{g} (a_i b_i a_i^{-1} b_i^{-1}) = 1$
$\int_{b_k} \omega_\beta^i = \delta_k^i$
$\int_{a_k} \omega_\beta^i = 0$