Title: Geometry and topology for physicists 2021/2022 - Lecture 9

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Collection: Geometry and Topology for Physicists 2021/2022

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Today

Simple techniques for computing cohomology

## Today

- Simple techniques for computing cohomology
- Proof of hairy ball theorem

Theorem (Poincaré duality)

If $M$ is an oriented $n$-manifold, without boundary, closed (compact, or closed + bounded in $\mathbb{R}^k$)

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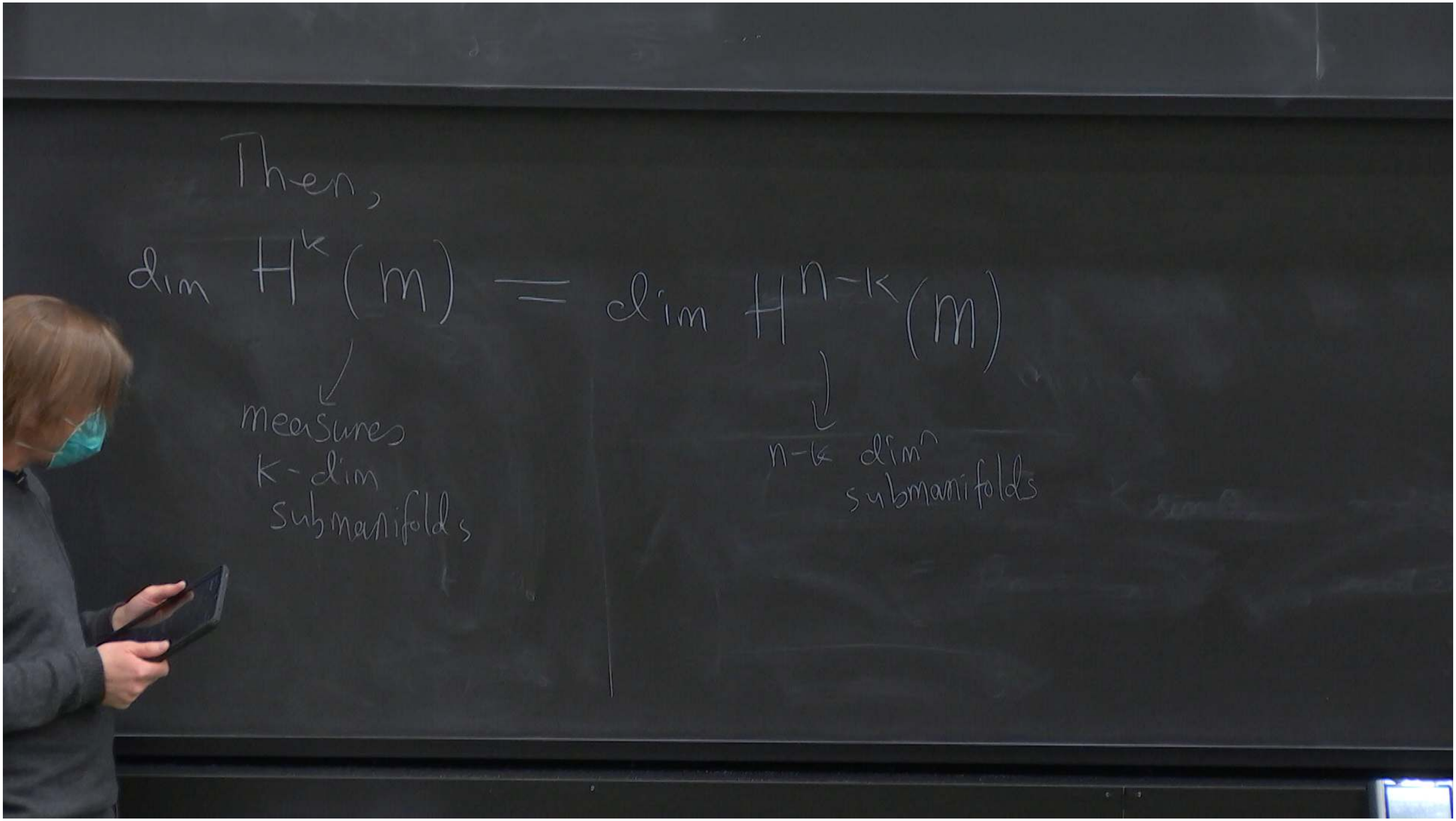
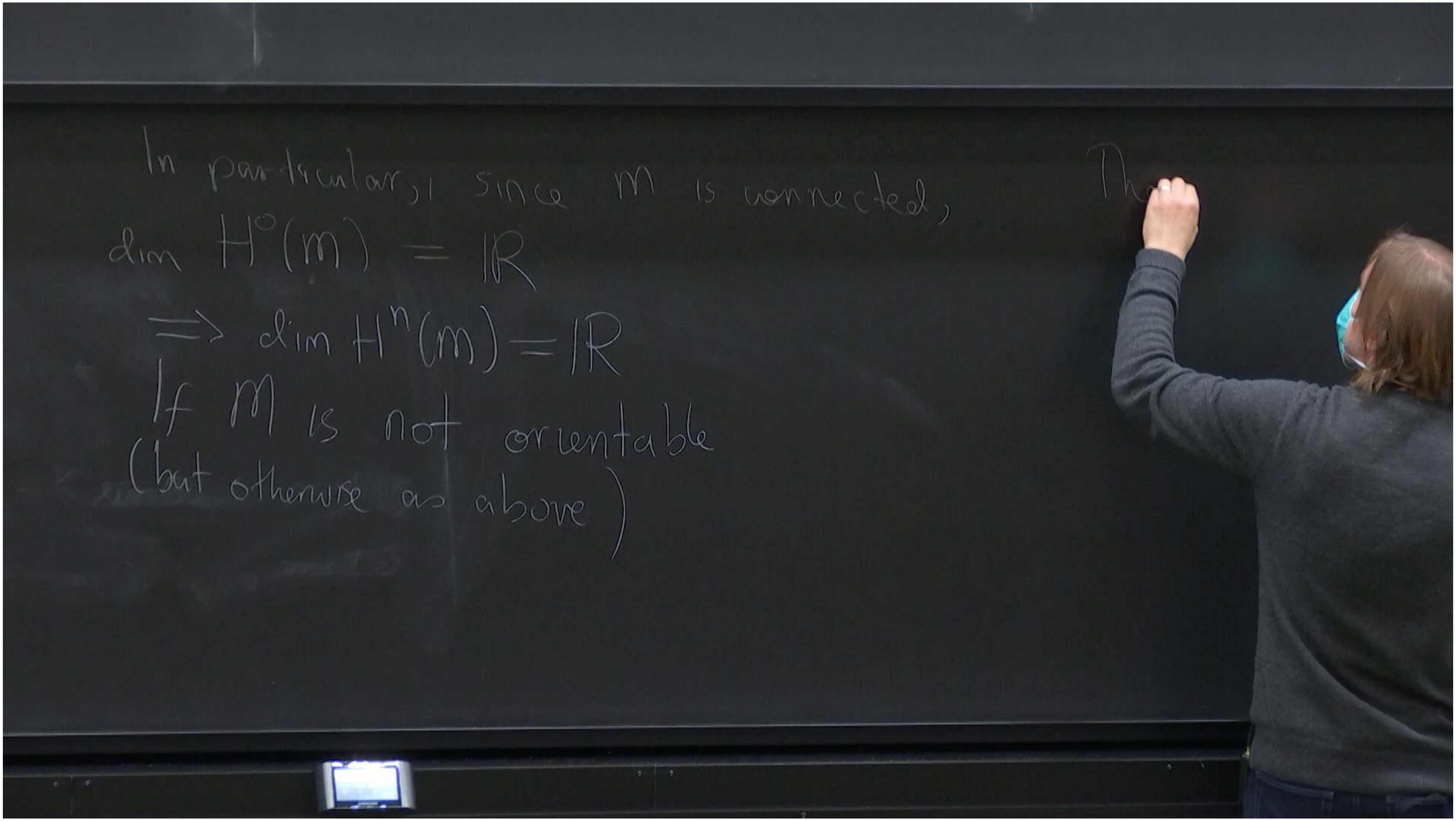
**Theorem** (Poincaré duality)

If $M$ is an oriented $n$-manifold, without boundary, closed (compact, or closed + bounded in $\mathbb{R}^k$)

+ connected


Then,
$\dim H^k(M) = \dim H^{n-k}(M)$
measures
k-dim
submanifolds
n-k dim^n
submanifolds


In particular, since M is connected,
dim H°(M) = IR
⟹ dim Hⁿ(M) = IR
If M is not orientable
(but otherwise as above)

is connected, Then, $\dim H^n M = 0$

uble

**Def^n** If $M$ is a manifold,

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim H^k(M)$$

this is an integer.

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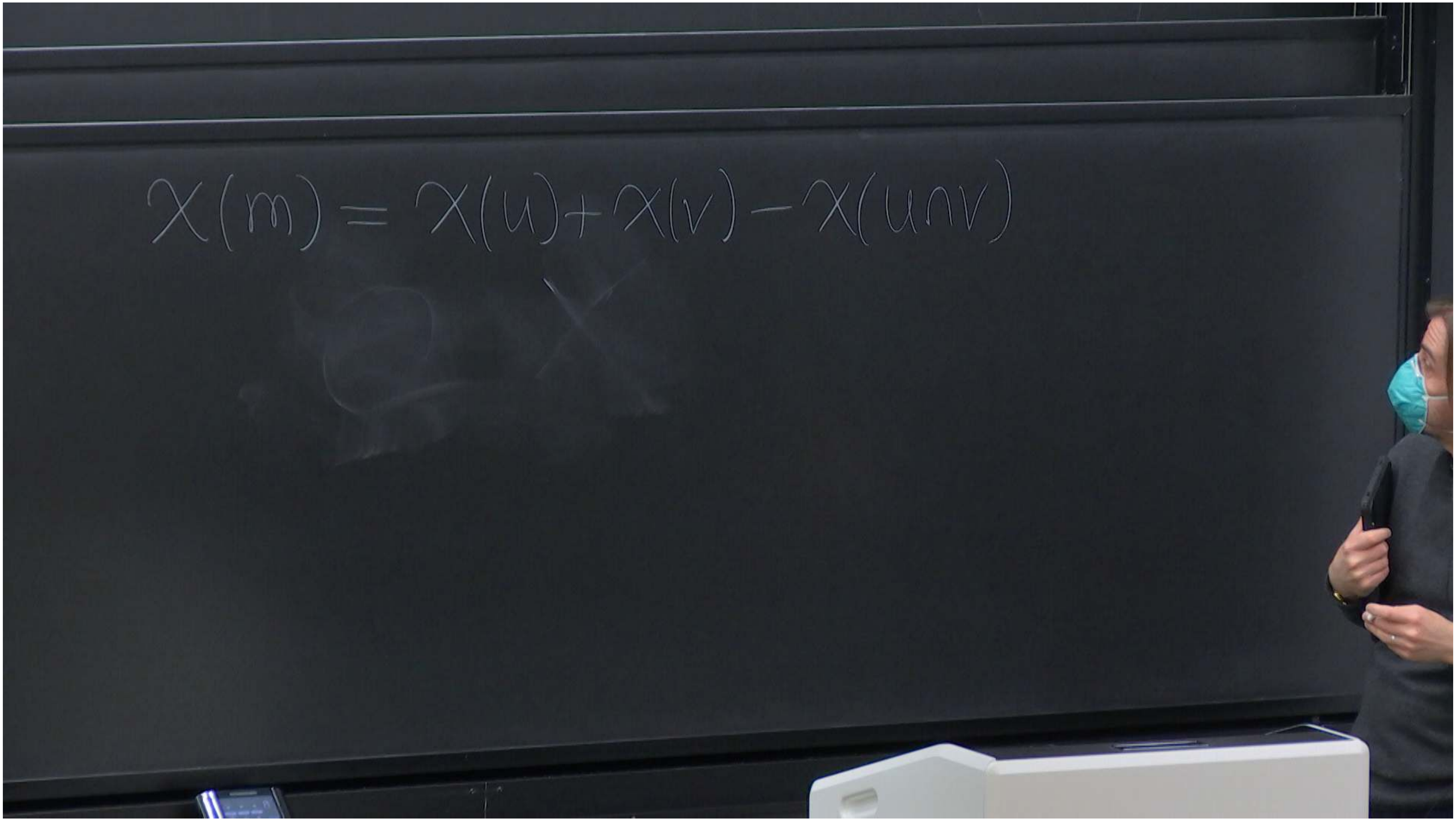
**Theorem** If $M = U \cup V$, where $U, V$ are open in $M$, then

$\chi($


$\chi(M) = \chi(U) + \chi(V) - \chi(U \cap V)$

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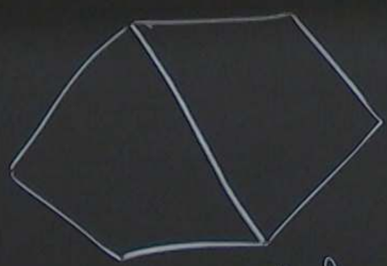
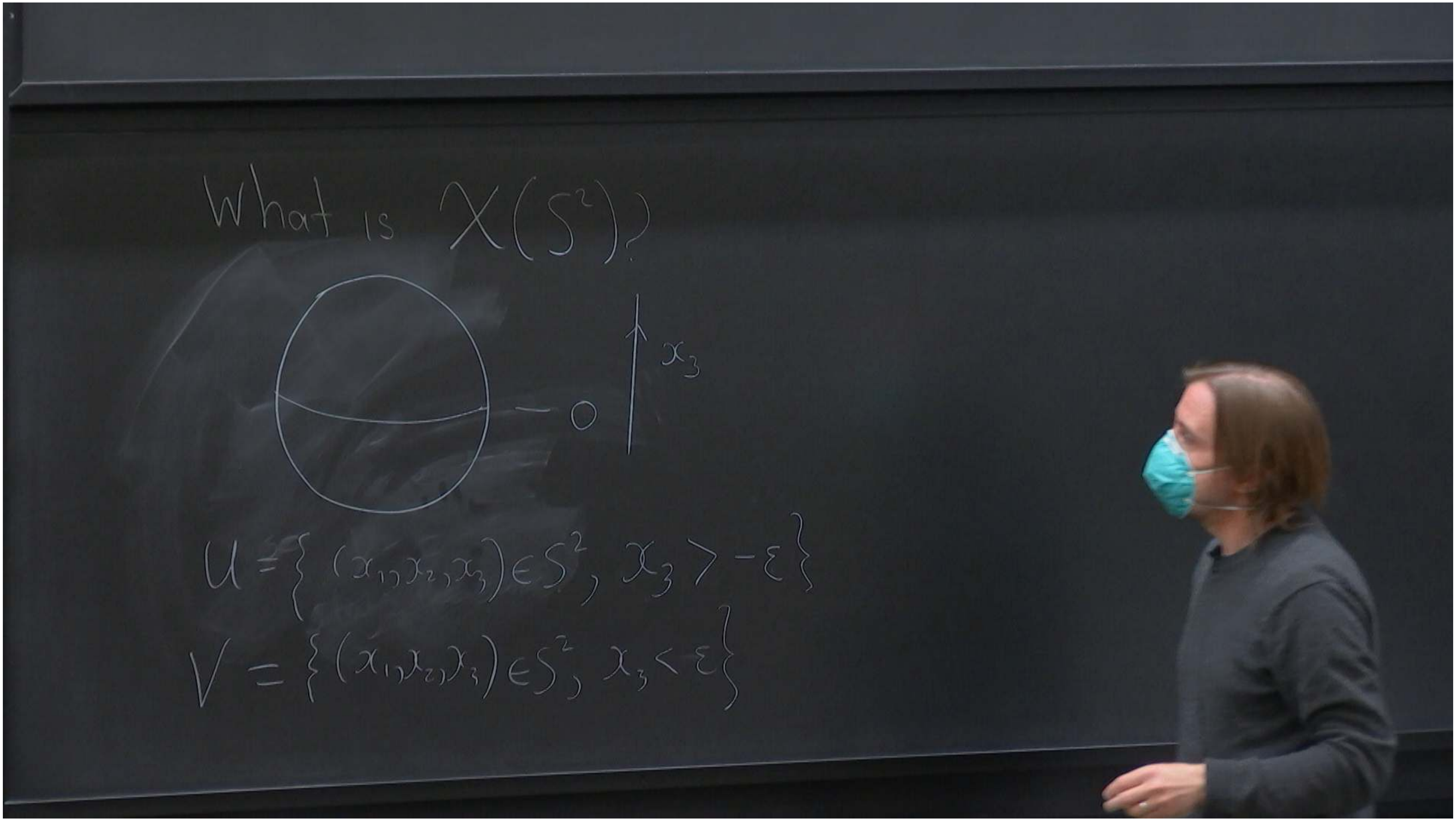
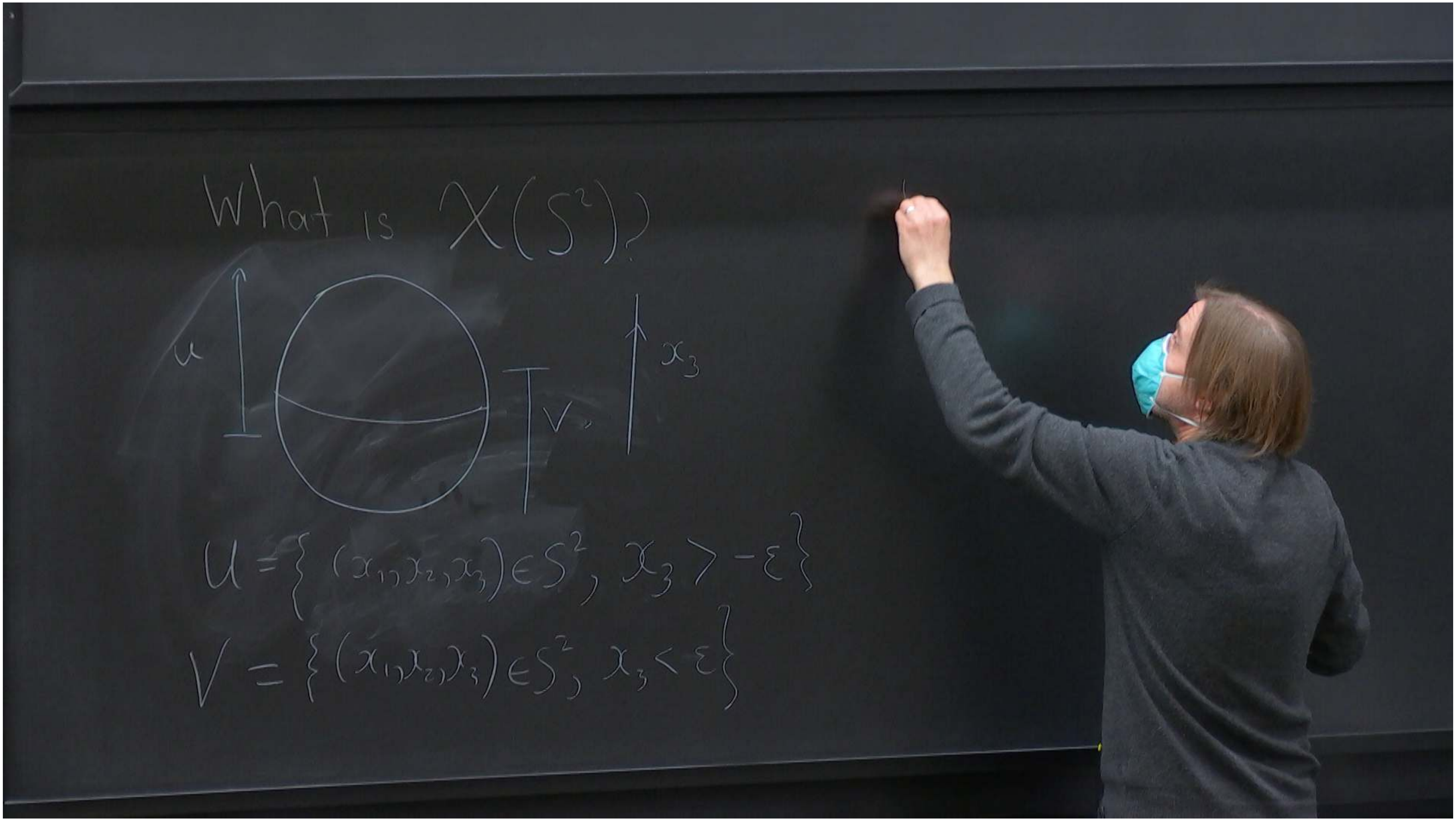
(is a consequence of a more complicated result showing how $H^{\bullet}M$ is built from $H^{\bullet}U$, $H^{\bullet}V$, $H^{\bullet}U \cap V$



added one edge
one face

so $V - E + F$
stays the same.


What is $\chi(S^2)$?
$x_3$
$0$
$U = \{(x_1, x_2, x_3) \in S^2, x_3 > -\varepsilon\}$
$V = \{(x_1, x_2, x_3) \in S^2, x_3 < \varepsilon\}$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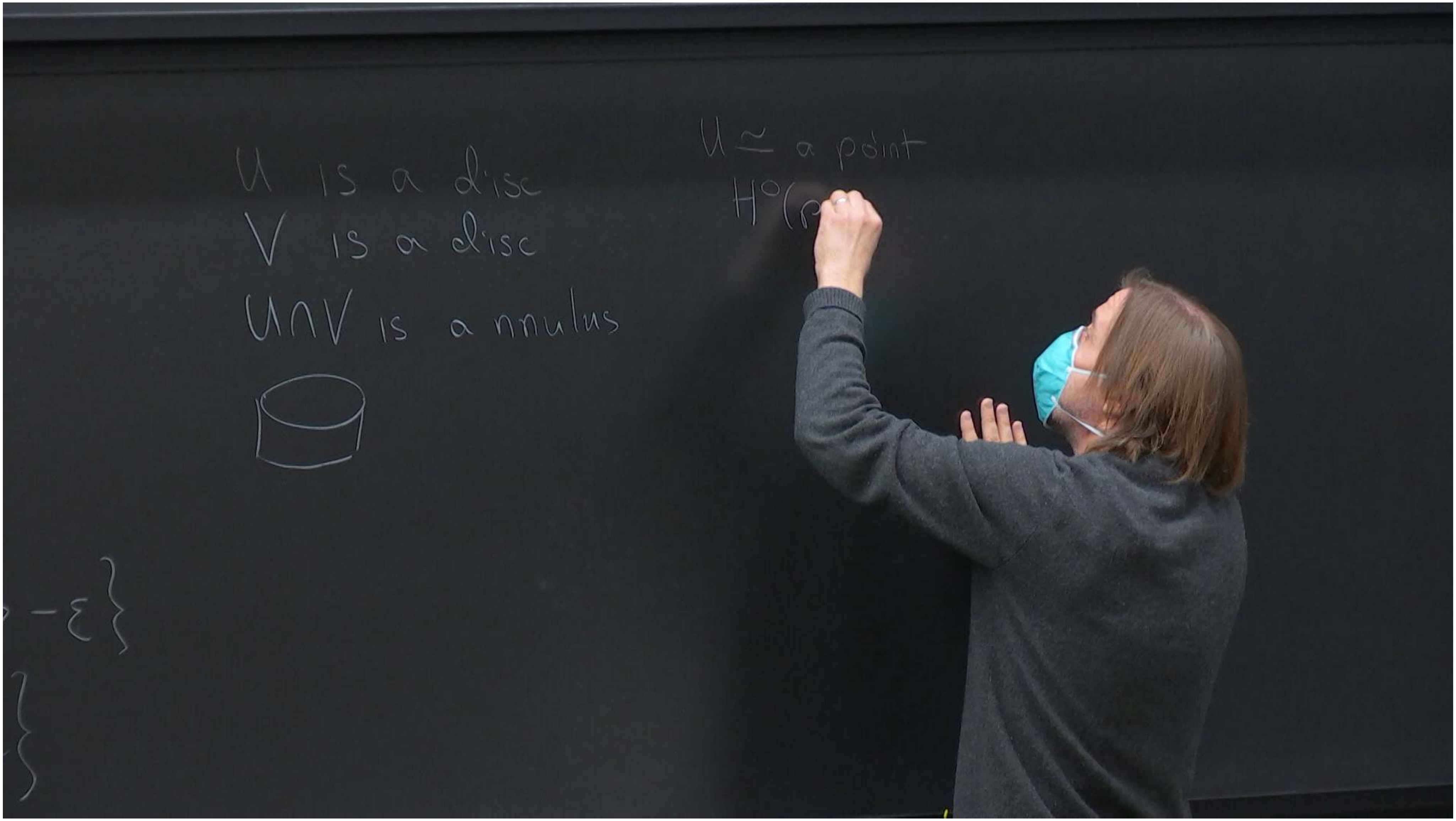
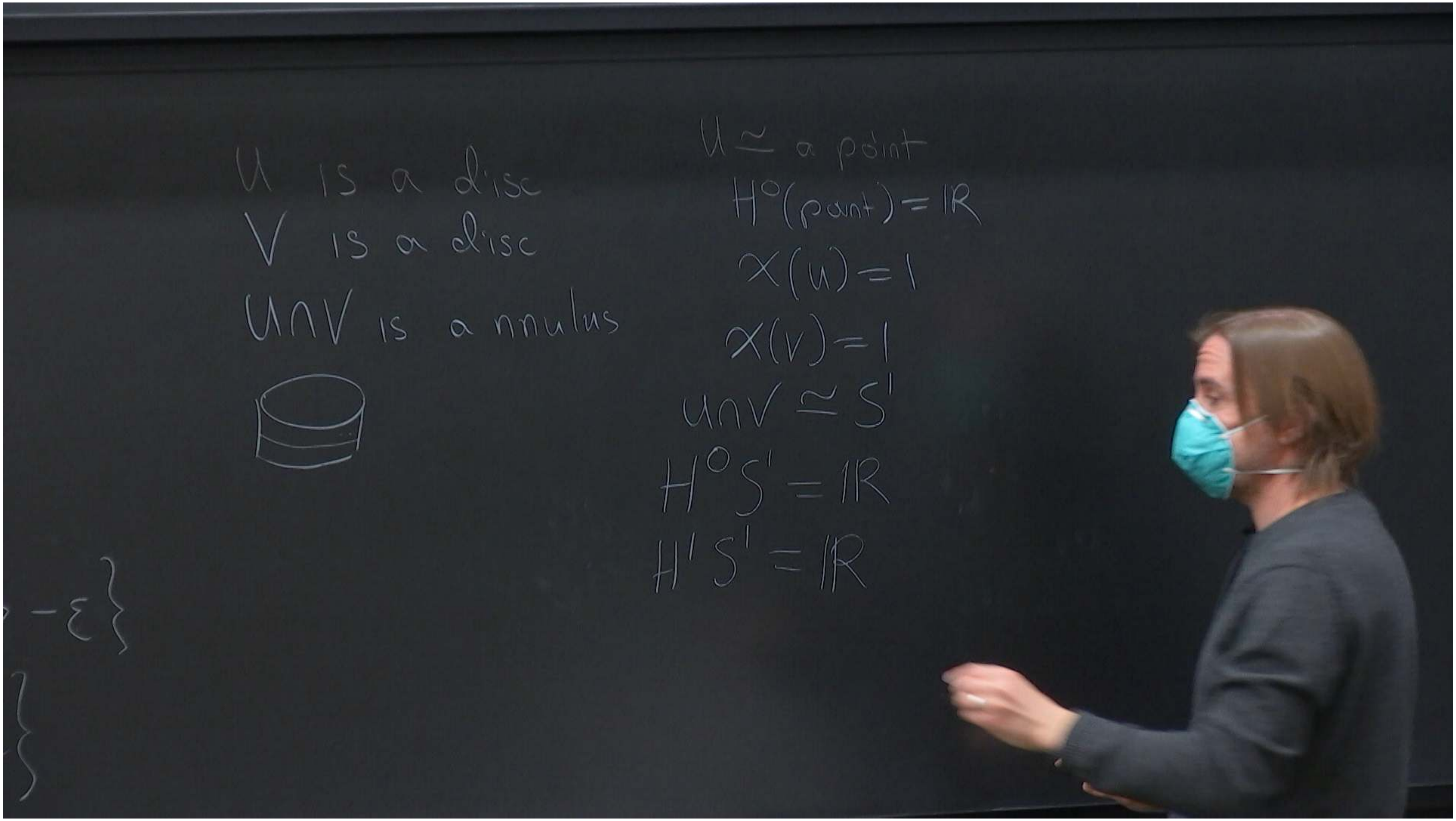
$U$ is a disc

$V$ is a disc

$U \cap V$ is annulus

$-\varepsilon\}$


U is a disc
V is a disc
U∩V is annulus
U ≃ a point


$U$ is a disc
$V$ is a disc
$U \cap V$ is a nnulus
$U \simeq$ a point
$H^0(\text{point}) = \mathbb{R}$
$\chi(U) = 1$
$\chi(V) = 1$
$U \cap V \simeq S^1$
$H^0 S^1 = \mathbb{R}$
$H^1 S^1 = \mathbb{R}$

$-\varepsilon\}$

$U$ is a disc

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$U \simeq$ a point

$H^0(\text{point}) = \mathbb{R}$

$\chi(U) = 1$

$\chi(V) = 1$

$U \cap V \simeq S^1$

$H^0 S^1 = \mathbb{R}$

$H^1 S^1 = \mathbb{R}$

$\chi(S^1) = 0$

$\chi(S^2) =$

$$\chi(S^2) = \chi(u) + \chi(v) - \chi(u \cap v)$$

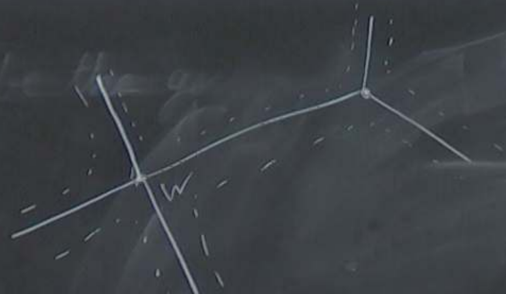
$$= 1 + 1 - 0 = 2.$$

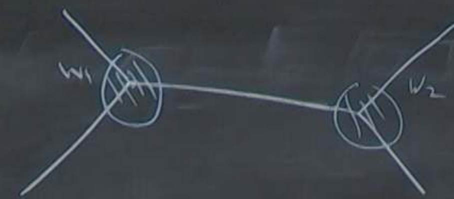
Suppose $\Sigma$ is a 2d manifold build by gluing polygons
$V, E, F$ be #'s of vertices, edges, faces

Lemma

$$\chi(\Sigma) = V - E + F$$

Proof Step I. Let $W \subseteq \Sigma$
be an open which is a small neighbourhood
of edges + vertices

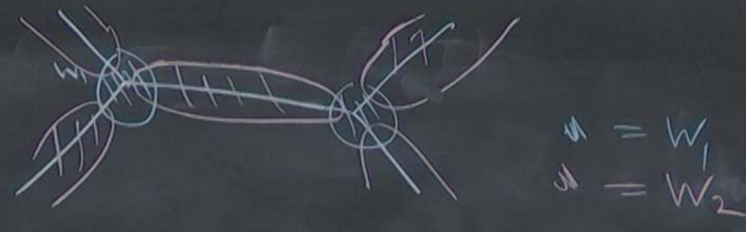



First, we show $\chi(W) = V - E$

$W_1$ = small nbd of the vertices

$W_2$ = small nbd of interior of edges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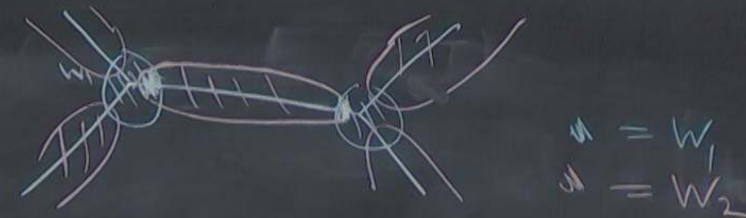
$\chi(W_1) = V \qquad \chi(W_2) = E \qquad \chi(W_1 \cap W_2)$


First, we show $\chi(W) = V - E$

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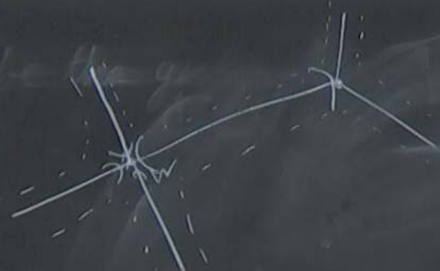
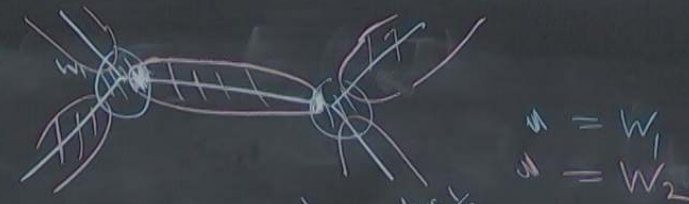
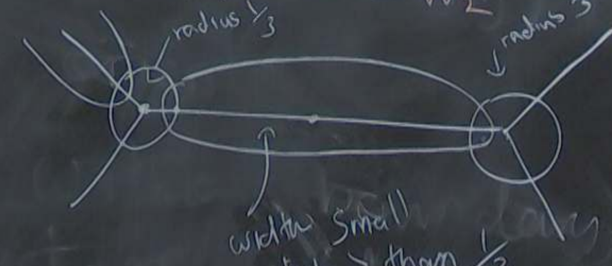
$\chi(W_1) = V \qquad \chi(W_2) = E \qquad \chi(W_1 \cap W_2) = 2E$

$\chi(W) = \chi(W_1) + \chi(W_2) - \chi(W_1 \cap W_2) = V - E$











First, we show $\chi(W) = V - E$

$W_1$ = small nbd of the vertices

$W_2$ = small nbd of mid point of edges

$\chi(W_1) = V \qquad \chi(W_2) = E \qquad \chi(W_1 \cap W_2) = 2E$

$\chi(W) = \chi(W_1) + \chi(W_2) - \chi(W_1 \cap W_2) = V - E$

**Def$^n$** If $M$ is a manifold,

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim H^k(M)$$

this is an integer.

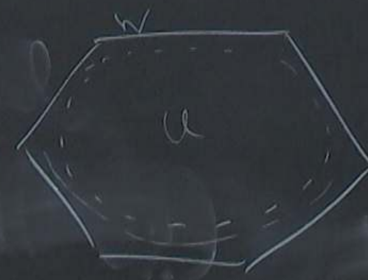
$$\sum_{n=0}^{\infty} 1 - \sum_{0}^{\infty} 1$$

$\Sigma$

$\cup|$

$W = \{\text{nbd of edges+vertices}\}$

$U = \{\text{Interior of faces}\}$

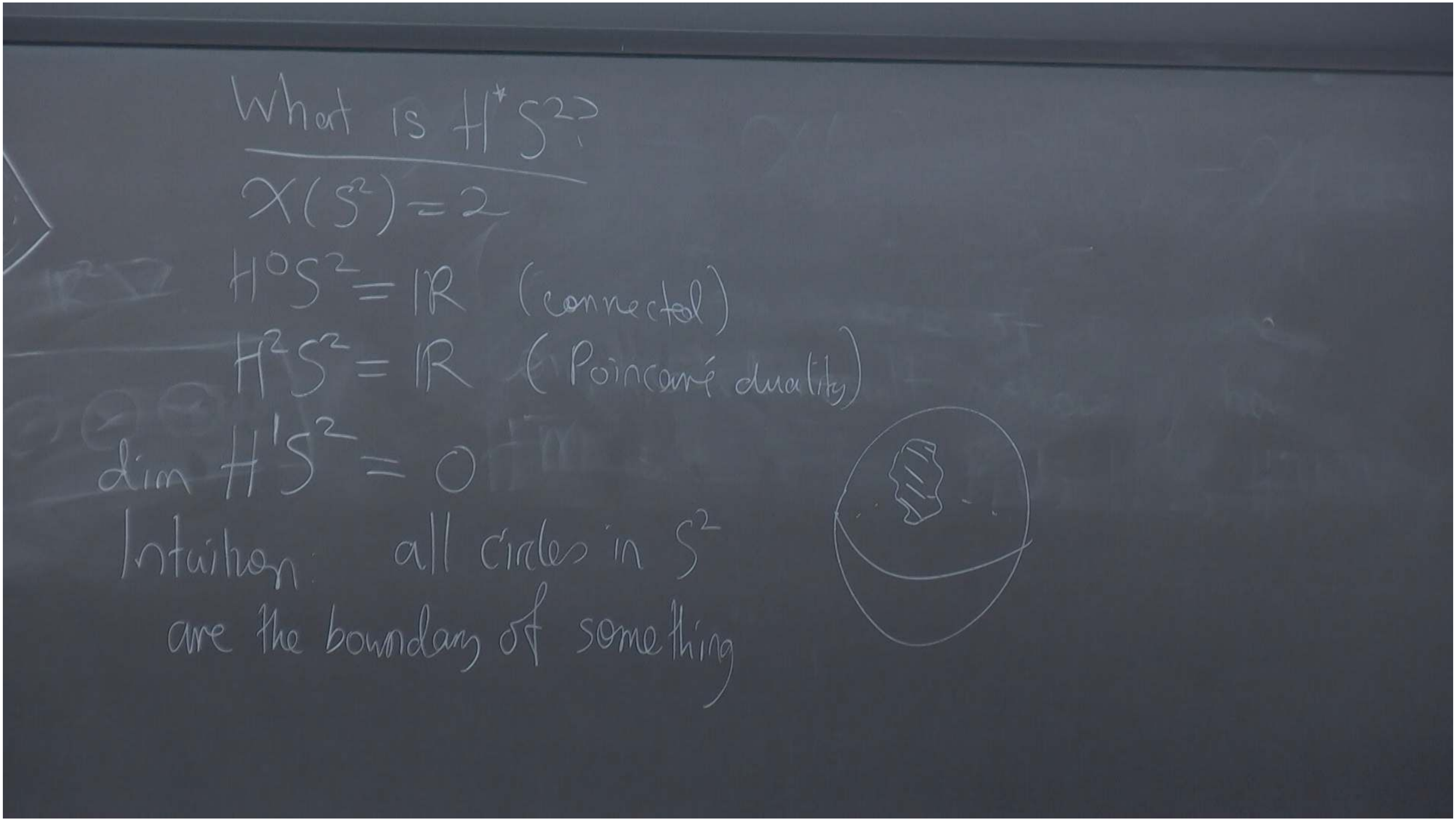



$\chi(W) = V - E$

$\chi(U) = F$ $\quad U \cap W \simeq$ an $S^1$ for each face

$\chi(U \cap W) = 0$

$\chi(\Sigma) = V + F - E$


What is $H^* S^2$?
$\chi(S^2) = 2$
$H^0 S^2 = \mathbb{R}$ (connected)
$H^2 S^2 = \mathbb{R}$ (Poincaré duality)
$\dim H^1 S^2 = 0$
Intuition: all circles in $S^2$ are the boundary of something

Let $\Sigma$ be a surface

$f : \Sigma \longrightarrow \mathbb{R}$

$\Sigma$ has Riemannian metric

Set $V = \text{Grad}\, f = g_{ij} \dfrac{\partial f}{\partial x_i} \dfrac{\partial}{\partial x_j}$

this is a vector field

Assume $f$ is "generic"

$V = 0$ at points $p$ where

$$\frac{\partial f}{\partial x_i}(p) = 0$$

(critical points of $f$)

Assume

$$\det \frac{\partial^2 f}{\partial x_i \partial x_j}(p) \neq 0$$

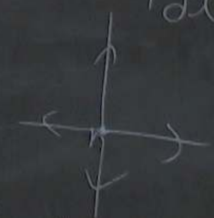
at each such $p$

Possibilities

1) p is a local maximum

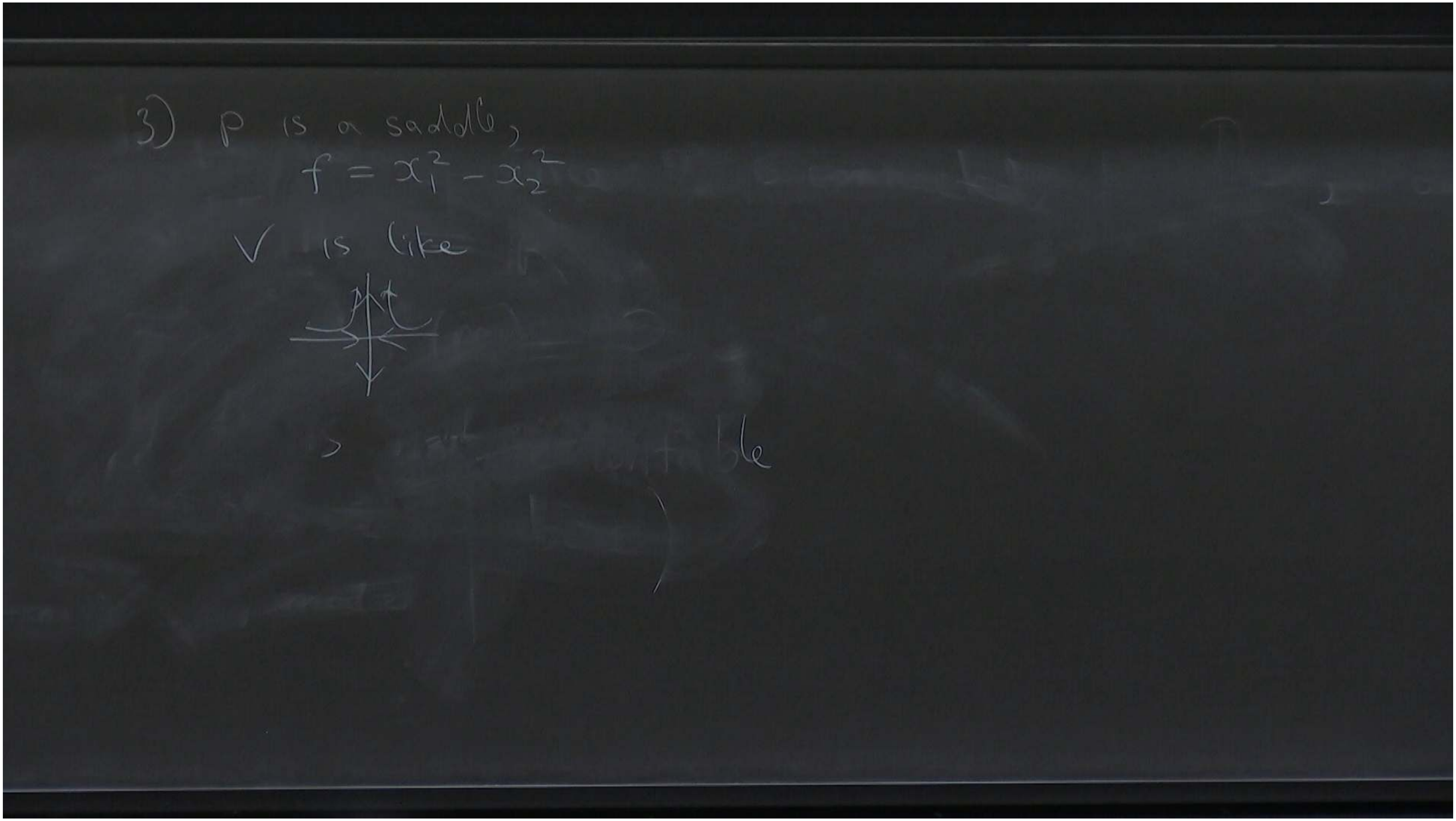
f looks like $x_1^2 + x_2^2$ near p

$V = 2x_1 \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_2}$ is a source



2) p local minimum, p is a sink for V

3) p


3) p is a saddle,
$f = x_1^2 - x_2^2$
V is like

Theorem

$$\chi(\Sigma) = \#\text{Sinks} + \#\text{Sources} - \#\text{Saddles}.$$

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Proof Note $f$ decreases along flows of $V$
(no periodic orbits)

Cut $\Sigma$ up into faces, edges, vertices by:

Faces $= \{ x \in \Sigma$ that flow to a sink $\}$

Theorem

$\chi(\Sigma) = \#\text{Sinks} + \#\text{Sources} - \#\text{Saddles}$

Proof Note $f$ decreases along flows of $V$
(no periodic orbits)

Cut $\Sigma$ up into faces, edges, vertices by:

$\text{Faces} = \{ x \in \Sigma \text{ that flow to a sink} \}$

For each sink, there is one face.

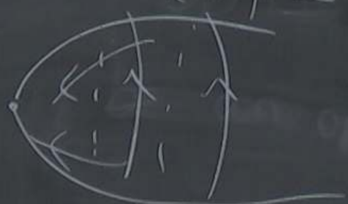
$$\nabla \cdot f = g_{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} > 0$$

For e

Why is a face a disc?

Face has a coord. given by f

Locus where $f =$ const is a circle



$r^2 = f$

For each saddle, build an edge, $\{x \in \Sigma, x \text{ flows to the saddle}\}$

For each source, have a vertex - the source

For each saddle, build an edge, $\{ x \in \Sigma, x \text{ flows to the saddle} \}$

For each source, have a vertex - the source

So,

$$\chi(\Sigma) = V - E + F$$

$$= \# \text{Sources} + \# \text{Sinks} - \# \text{Saddles}.$$

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Set $V = \mathrm{Grad}\, f = g_{ij} \dfrac{\partial f}{\partial x_i} \dfrac{\partial}{\partial x_j}$

this is a vector field

$\dfrac{x\,dy + y\,dx}{x^2+y^2}$

$d\omega = 0$

$d{*}\omega = 0$

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