Title: Geometry and topology for physicists 2021/2022 - Lecture 8

Speakers: Kevin Costello

Collection: Geometry and Topology for Physicists 2021/2022

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10:17

Lecture 4

Stokes' Theorem

M an oriented manifold with boundary.

$\omega \in \Omega^{n-1}(M)$. Then,

$$\int_M d\omega = \int_{\partial M} \omega$$

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<u>Completion of proof</u>

Divide M up into cubes C_i

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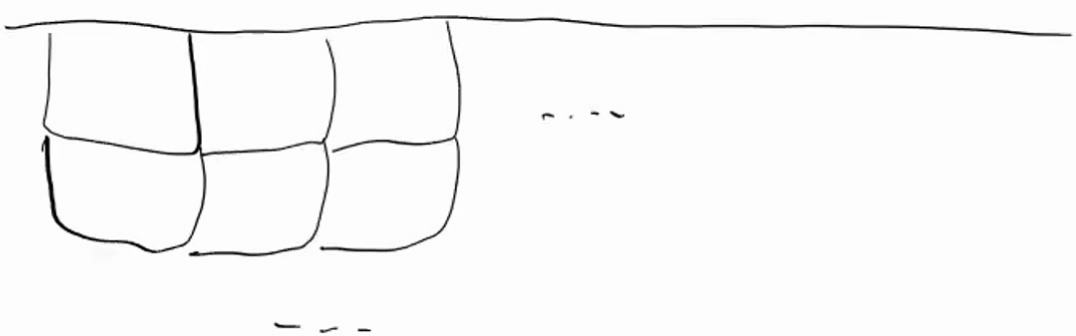
Lecture 4

$$\int_M d\omega = \int_{\partial M} \omega$$

Last time:
C cube
$$\int_C d\omega = \int_{\partial C} \omega$$

Completion of proof

Divide M up into cubes C_i



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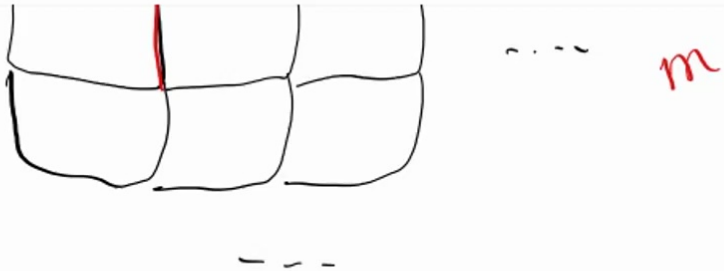
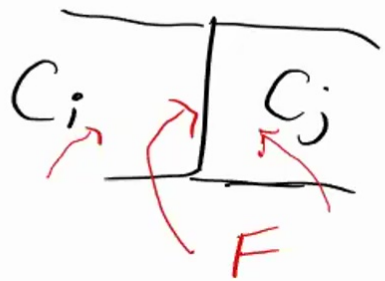


$$\int_{m} d\omega = \sum_i \int_{C_i} d\omega$$

We already know

$$\int_{C_i} d\omega = \int_{\partial C_i} \omega$$

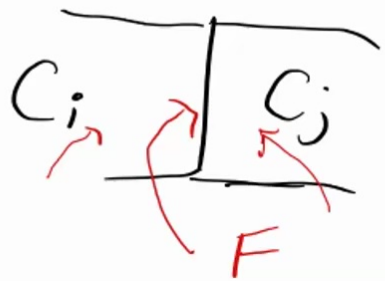
If two cubes share a face



then $\int_F \omega$ appears twice, with opposite signs.

The only terms that don't cancel are those faces [illegible]

If two cubes share a face



F is $x = 1$

C_i $x \leq 1$

C_j $x \geq 1$

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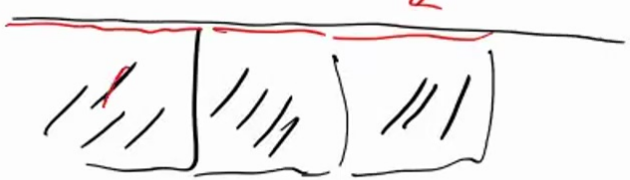
The only terms that don't cancel are those faces which live on the boundary of M:

(these faces don't cancel)

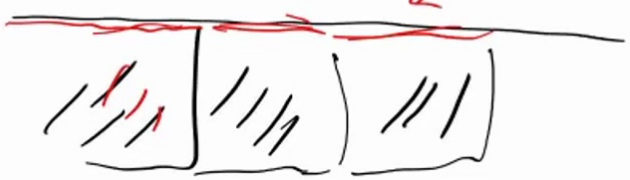


So,

$$\int_M d\omega = \sum_{\substack{\text{faces} \\ \text{in } \partial M}} \int_F \omega = \int_{\partial M} \omega$$

the boundary of M:

(these faces don't cancel)



So,
$$\int_M d\omega = \sum_{\substack{\text{faces} \\ \text{in } \partial M}} \int_F \omega = \int_{\partial M} \omega$$

as desired.

$\square$ Completes the proof.

Consequences of stokes theorem

If ω is an n-form on a closed manifold M (with no boundary) and $\int_M \omega \neq 0$, then ω is not exact.

For instance,

Consequences of stokes theorem

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If $\omega = d\eta$

$$\int_M d\eta = \int_{\partial M} \eta = 0$$ as ∂M is empty

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$d\omega = 0$

ω defines an element of $H^n(M)$ which is not

For instance,

$$\omega = x_1\, dx_2\, dx_3 - x_2\, dx_1\, dx_3 + x_3\, dx_1\, dx_2$$

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Lecture 4

$\underline{\underline{m}}$

exact.

$$\int_m d\eta = \int_{\partial m} \eta \quad \partial m \text{ is}$$

$d\omega = 0$

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For instance,

$$\omega = x_1\, dx_2\, dx_3 - x_2\, dx_1\, dx_3 + x_3\, dx_1\, dx_2$$

on S^2.

$$\int_{S^2} \omega = \int_{\|x\| \le 1} d\omega = 3 \int_{\|x\| \le 1} dx_1\, dx_2\, dx_3$$

$$= 3 \times (\text{Volume of a ball})$$

$M \subseteq \mathbb{R}^n$ closed and bounded

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$d\omega = 0$

ω defines an element of $H^n(M)$ which is not zero.

For instance,

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$M \subseteq \mathbb{R}^n$ closed and bounded also, $\partial M = \phi$ eg. S^n

Consequences of stokes theorem

If ω is an n-form on a closed manifold M (with no boundary) and $\int_M \omega \neq 0$, then ω is not exact.

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$$\omega = x_1\, dx_2\, dx_3 - x_2\, dx_1\, dx_3 + x_3\, dx_1\, dx_2$$

on S^2.

$d\omega = 3\, dx_1\, dx_2\, dx_3$ on $\mathbb{R}^3$

$$\int_{S^2} \omega = \int_{\|x\| \le 1} d\omega = 3 \int_{\|x\| \le 1} dx_1\, dx_2\, dx_3$$

$$= 3 \times (\text{Volume of a ball})$$

So, ω is not exact, and $H^2(S^2) \neq 0$.

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Suppose $\omega \in \Omega^k(M)$ is closed.
If $Y \subseteq M$ is a closed submanifold (oriented, no boundary) can consider

$$\int \omega$$

Suppose $\omega \in \Omega^k(M)$ is closed.

If $Y \subseteq M$ is a closed submanifold (oriented, no boundary) can consider

$$\int_Y \omega$$

Stokes theorem $\Rightarrow$ that this does not change if we move Y continuously.

$= k$ (oriented, no boundary) can consider

$$\int_{\underline{\underline{Y}}} \omega$$

$M = \mathbb{R}^2 \setminus 0$

$Y = S^1$

Stokes theorem $\Rightarrow$ that this does not change if we move Y continuously.

Suppose Y_t, $t \in [0,1]$ is a one-parameter family of such submanifolds connecting Y and Y

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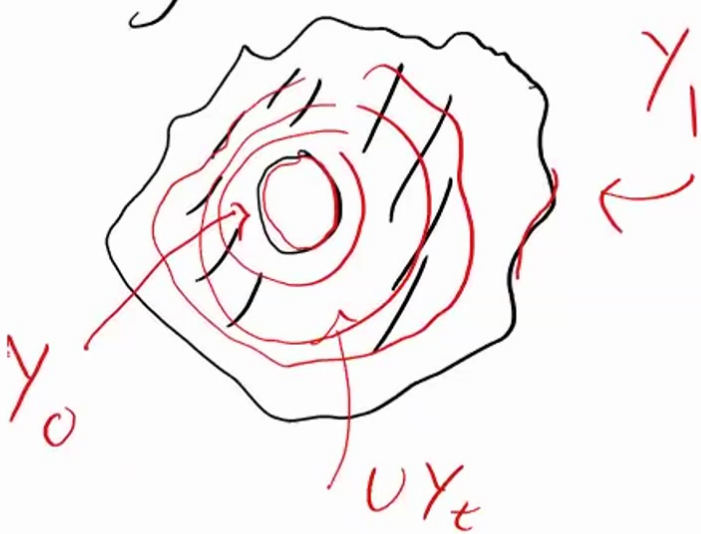
then $\bigcup_t Y_t$ is (usually) a $k+1$ dimensional submanifold of M with boundary $Y_0 \amalg Y_1$:

e.g.



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Then,

$d\omega \Rightarrow$

$$\int_{\cup Y_t} d\omega = 0 \quad \text{as } \omega \text{ is closed}$$

But by Stokes theorem it is

$$\int_{Y_0} \omega - \int_{Y_1} \omega = \int d\omega = 0$$

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$$\int_{Y_0} \qquad \int_{Y_1} \omega$$

(different signs are from the different orientations on the boundary)

So

$$\int_{Y_0} \omega = \int_{Y_1} \omega$$

Y_0, Y_1 can be continuously varied into each other then

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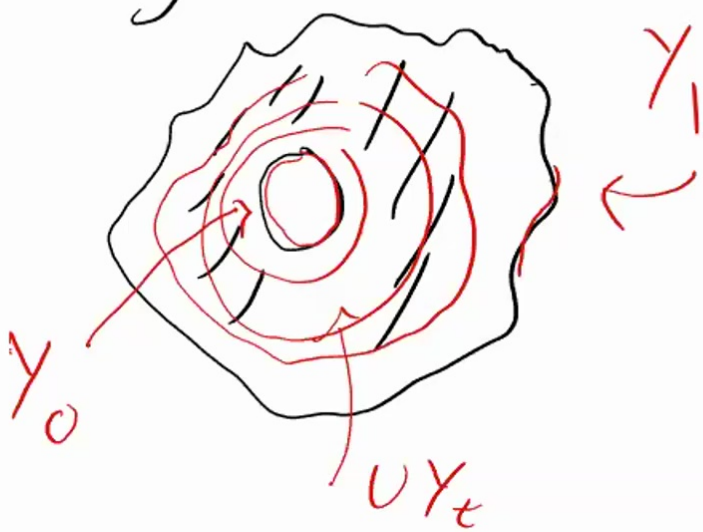
$k+1$ dimensional submanifold $\cup$ ...

with boundary $Y_0 \amalg Y_1$:

e.g.



extra ∂ component

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Lecture 4

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Cauchy's Theorem

$\mathbb{C} \setminus$ some points

$f(z)$ fn only of z, satisfies

CR equation $\frac{\partial f}{}$

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Lecture 4

U \ some points

$f(z)$ fn only of z, satisfies

CR equation $\frac{\partial f}{\partial \bar{z}} = 0$

$\int_{\text{Contour}} f \, dz$ doesn't change if we change the contour.

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Lecture 4

$f\,dz$ is a closed one-form

$\Longleftrightarrow$ f satisfies the Cauchy-Riemann equation. $z = x + iy$

$$f\,dz = f(dx + i\,dy)$$

$$d(f\,dz) = d(f\,dx + i\,dy)$$

$$=$$

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Lecture 4

$$f\,dz = f(dx + i\,dy)$$

$$d(f\,dz) = d(f\,dx + i\,dy)$$

$$= \frac{\partial f}{\partial y}\,dy \wedge dx + i\frac{\partial f}{\partial x}\,dx \wedge dy$$

$$= 0 \quad ;$$

equation. $z = x + iy$

$$f\,dz = f(dx + i\,dy)$$

$$d(f\,dz) = d(f\,dx + i\,dy)$$

$$= \frac{\partial f}{\partial y}\,dy \wedge dx + i\frac{\partial f}{\partial x}\,dx \wedge dy$$

$= 0$ if

$$-\frac{\partial f}{\partial y} + i\frac{\partial f}{\partial x} = 0$$ CR equation.

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Lecture 4

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$$\partial X = Y_0 - Y_1$$

(the $-$ sign means Y_1 with opposite orientation)

Then, if $\omega \in \Omega^4(M)$ is closed,

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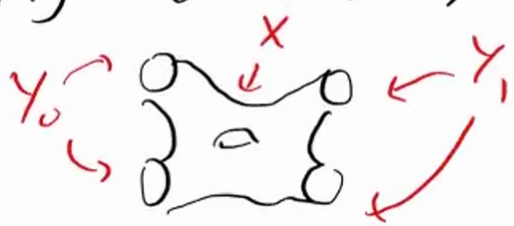
Lecture 4

$Y_0 \quad Y_1$

other then

More generally, if $Y_0, Y_1 \subseteq M$ are oriented submanifolds of dimension k, we say a cobordism from Y_0 to Y_1 is a submanifold $X \subseteq M$ of dimension $k+1$, oriented, with boundary



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Lecture 4

$\int_{\text{Contour}} f\, dz$ doesn't change if we change the contour.

$f\, dz$ is a closed one-form

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$$f\, dz = f(dx + i\, dy)$$

$$d(f\, dz) = d(f\, dx + i\, dy)$$

$$= \frac{\partial f}{\partial y}\, dy \wedge dx + i \frac{\partial f}{\partial x}\, dx \wedge dy$$

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Lecture 4

$$\partial X = Y_0 - Y_1$$

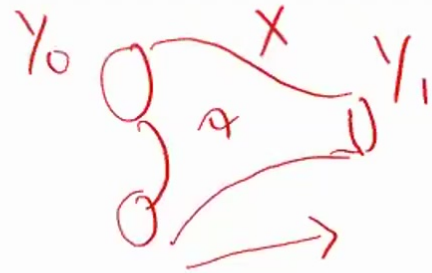
(the $-$ sign means Y_1 with opposite orientation)

Then, if $\omega \in \Omega^k(M)$ is closed,

$$\int_{Y_0} \omega = \int_{Y_1} \omega$$

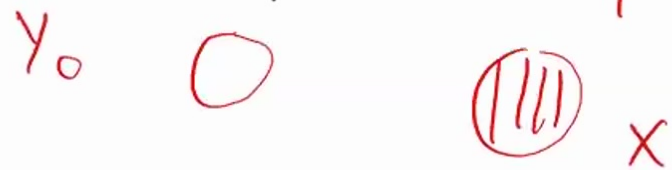
Similarly, since we assume Y_0, Y

$$\partial X = Y_0 - Y_1$$



(the $-$ sign means Y_1 with opposite orientation)

$Y_1 = \phi$



Then, if $\omega \in \Omega^k(M)$ is closed,

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Lecture 4

$$\partial X = Y_0 - Y_1$$

(the $-$ sign means Y_1 with opposite orientation)

$Y_1 = \phi$

Then, if $\omega \in \Omega^k(M)$ is closed,

$$\int_{Y_0} \omega = \int_{Y_1} \omega$$

$$\int_{Y_0} \omega - \int_{Y_1} \omega = \int_X d\omega = 0 \quad \text{as } d\omega = 0$$

Similarly, since we assume Y_0, Y_1 are closed (w boundary) if ω is

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Lecture 4

$\int_{Y_0} \quad \int_{Y_1} \qquad \int_{Y_0} \quad \int_{Y_1} \quad \int_X$ as $d\omega$

Similarly, since we assume Y_0, Y_1 are closed (no boundary) if ω is exact, $\omega = d\eta$, then $\int_{Y_0} \omega = 0$

$\int_{Y_1} \omega = 0$

Meaning of cohomology

If $\omega \in H^k(M)$ then we can

$Y,$

Meaning of cohomology

If $\omega \in H^k(M)$ then we can integrate ω over closed oriented k-dim^n submanifolds $Y \subseteq M$

$\int_Y \omega$ does not depend on the choice of $\tilde{\omega} \in \Omega^k(M)$ representing ω as $\int d\eta = 0$

Meaning of cohomology

If $\omega \in H^k(M)$ then we can integrate ω over closed oriented k-dim^n submanifolds $Y \subseteq M$

$\int_Y \tilde{\omega}$ does not depend on the choice of $\tilde{\omega} \in \Omega^k(M)$ representing ω

as $\int_Y d\eta = 0$ $\quad \int_Y \tilde{\omega} + d\eta = \int_Y \tilde{\omega}$

if we perturb Y, or replace Y by a manifold connected by a cobordism.

Roughly:

$$H^k(M) \longleftrightarrow \frac{\left\{ \begin{array}{c} k \text{ dim}^{\text{n}} \text{ oriented} \\ \text{submanifolds} \end{array} \right\}}{\text{Boundaries of}}$$

by a manifold connected by a cobordism.

Roughly:

$$H^k(M) \longleftrightarrow \frac{\left\{ \begin{array}{c} k \text{ dim}^n \text{ oriented} \\ \text{submanifolds} \end{array} \right\}}{\text{Boundaries of } k+1 \text{ dim}^n \text{ manifolds}}$$

cobordism.

More precise defn will lead to singular cohomology

Roughly:

$$H^k(M) \longleftrightarrow \left\{ \begin{array}{c} k \text{ dim}^n \text{ oriented} \\ \text{submanifolds} \end{array} \right\} \Big/ \begin{array}{c} \text{Boundaries of} \\ k+1 \text{ dim}^n \text{ manifolds} \end{array}$$

Homotopy Equivalence

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Lecture 4

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$$\begin{tikzcd} \{k+1 \text{ dim submanifolds}\} \arrow[r, "\partial"] & \{k \text{ dim submanifolds}\} \arrow[d, "\partial"'] \\ & \{k-1 \text{ dim submanifolds}\} \end{tikzcd}$$

Just like $d \circ d = 0$,

$\partial \circ \partial = 0$. Boundary of a bu

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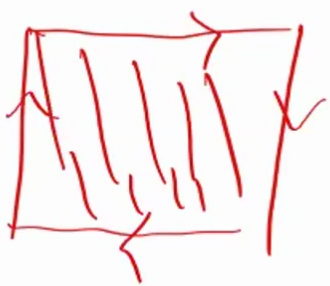
$\partial \downarrow$

$\{$ k-1 dim submanifolds $\}$

Just like $d \circ d = 0$,

$\partial \circ \partial = 0$. Boundary of a boundary

is zero.

∂ 

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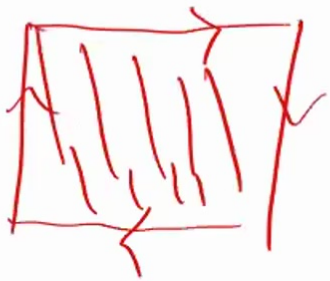
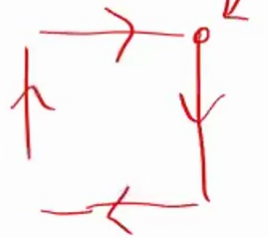
$\partial \circ \partial = 0$. Boundary of a [illegible]

is zero.

∂ 



appears twice w. opposite signs

$\partial \rightarrow$ is $-$ $+$

$\partial \downarrow$ is $-$ $+$

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$\partial \psi$ is

$\cdot t$

$$\Omega^{k-1}(M) \to \Omega^{k}(M) \to \Omega^{k+1}(M)$$

$$\left\{ \begin{matrix} k-1 \\ \text{dim} \end{matrix} \right\} \leftarrow \left\{ k \text{ dim} \right\} \xleftarrow{\partial} \left\{ k+1 \text{ dim} \right\}$$

Roughly $\Omega^{k-1}(M)$ is dual to a v. space

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Lecture 4

Roughly $\Omega^{k-1}(M)$ is dual to a v. space spanned by $k-1$ dim'n submanifolds

Stokes theorem says d and ∂ are adjoint:

$\{Y,$

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interpolating between f_0, f_1

(Strictly speaking, if we set

10:57

Lecture 4

$$\left\{ \begin{matrix} k-1 \\ \text{dim} \end{matrix} \right\} \xleftarrow{\partial} \left\{ k \text{ dim} \right\} \xleftarrow{\partial} \left\{ k+1 \text{ dim} \right\}$$

Roughly $\Omega^{k-1}(M)$ is dual to a v. space spanned by $k-1$ dim^n submanifolds

Stokes theorem says d and ∂ are adjoint:

$$\langle \partial Y, \omega \rangle = \langle Y, d\omega \rangle$$

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interpolating between f_0, f_1

$$\{k+1 \text{ dim submanifolds}\} \xrightarrow{\partial} \{k \text{ dim submanifolds}\}$$

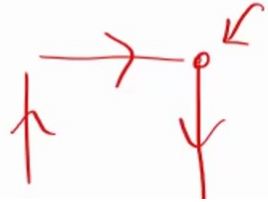
$$\partial \downarrow$$

$$\{k-1 \text{ dim submanifolds}\}$$

Just like $d \circ d = 0$,

$\partial \circ \partial = 0$. Boundary of a boundary is zero.

∂



appears twice w. opposite signs

Homotopy Equivalence

Let M, N be manifolds and $f_0, f_1 : M \to N$ be a pair of smooth maps.

A homotopy $f_0 \simeq f_1$ is a one parameter family of maps $f_t : M \to N$

Homotopy Equivalence

Let M, N be manifolds and $f_0, f_1 : M \to N$ be a pair of smooth maps.

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interpolating between f_0, f_1

(Strictly speaking, if we set

$F(x,t) = f_t(x)$, $x \in M$

we need $F(x,t)$ to be smooth in both x, t)

Example:

The two maps $\mathbb{R}^n \to \mathbb{R}^n$ given

Example:

The two maps $\mathbb{R}^n \to \mathbb{R}^n$ given by

$$f_0(x) = 0$$

$$f_1(x) = x$$

are connected by the homotopy

$$f_t(x) = tx$$

Def^n Two manifolds M, N are homotopy equivalent if there are

$$H(x) = tx$$

Def^n Two manifolds M, N are homotopy equivalent if there are maps

$$f : M \to N$$
$$g : N \to M$$

so that $f \circ g$ is homotopic to the identity on N,

11:02

Lecture 4

maps $f: M \to N$

$g: N \to M$

so that $f \circ g$ is homotopic to the identity on N, $g \circ f \simeq \mathrm{Id}_M$

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$g \circ f$ to the identity on M.

In this case we write $M \simeq N$

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In this case we write $M \simeq N$

Topology does not distinguish between homotopy equivalent manifolds.

E.g.

$S^{n-1} \simeq \mathbb{R}^n \setminus 0$

11:03

Lecture 4

$g \circ f$ to the identity on M.

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Topology does not distinguish between homotopy equivalent manifolds.

E.g.

$S^{n-1} \simeq \mathbb{R}^n \setminus 0$

E.g.

$$S^{n-1} \simeq \mathbb{R}^n \setminus 0$$

$$f: S^{n-1} \to \mathbb{R}^n \setminus 0$$

$$x \to x$$

$$g: \mathbb{R}^n \setminus 0 \to S^{n-1}$$

$$y \to \frac{y}{\|y\|} \in S^{n-1}$$

$$g \circ f(x) = x$$

11:05

Lecture 4

$$y \to \frac{y}{\|y\|} \in S^{n-1}$$

$$g \circ f(x) = x \quad \checkmark$$

$$f \circ g(y) = \frac{y}{\|y\|}$$

Homotopy $f \circ g \simeq Id_{\mathbb{R}^n \setminus 0}$

is $F_t(y) = \frac{y}{\|y\|}\left(t\|y\| + 1 - t\right)$

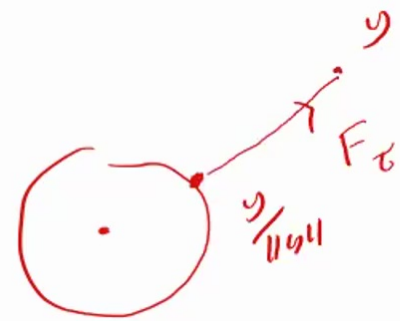
11:06

Lecture 4

$$y \to \frac{y}{\|y\|} \in S^{n-1}$$

$$g \circ f(x) = x \quad \checkmark$$

$$f \circ g(y) = \frac{y}{\|y\|}$$



Homotopy $f \circ g \simeq Id_{\mathbb{R}^n \setminus 0}$

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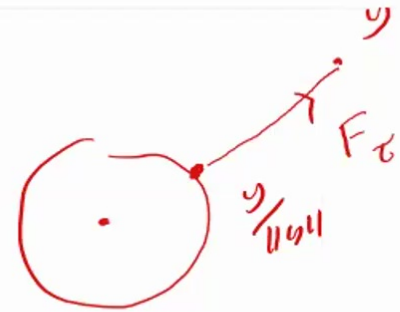
Lecture 4

$\|y\|$

$$g \circ f(x) = x$$



$$f \circ g(y) = \frac{y}{\|y\|}$$

Homotopy $f \circ g \simeq Id_{\mathbb{R}^n \setminus 0}$

is $F_t(y) = \frac{y}{\|y\|}\left(t\|y\| + 1 - t\right)$

S^1, $\mathbb{R}^2 \setminus 0$ are "the same" for topology.

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Lecture 4

Theorem If M, N are homotopy equivalent, then

$$H^k(M) = H^k(N) \text{ for all } k.$$

This will help us compute cohomology.

11:09

Lecture 4

Next time: we will develop a techniqe for computing $H^{\cdot}(\text{manifold})$

we will find that

$$H^k(S^2) = \begin{cases} \mathbb{R} & k=0 \\ \mathbb{R} & k=2 \\ 0 & \text{otherwise} \end{cases}$$

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