Title: Geometry and topology for physicists 2021/2022 - Lecture 7

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Collection: Geometry and Topology for Physicists 2021/2022

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Orientations and Integrations

On a manifold M

orientation: charts so that in patches y_i, x_i we have

$$\det\left(\frac{\partial y_i}{\partial x_i}\right)$$

$$\det\left(\frac{\partial y_i}{\partial x_i}\right) > 0$$

Or If $\omega \in \Omega^n(M)$,

ω is never zero (i.e. locally $\omega = f(x)\, dx_1 \wedge \dots \wedge dx_n$, $f(x) \neq 0$)

then it gives an orientation:

x_i are oriented if

$$\omega = f(x)\, dx_1 \wedge \dots \wedge dx_n, \quad f > 0$$

If replace ω by $\omega \cdot g$

$g > 0$ is a positive function, get same orientation.

If $n=1$:

An orientation is a direction

If $n=2$: If we have an orientation, in coords. $dx_1 \wedge dx_2$ is compatible with orientation $dx_2 \wedge dx_1$ is not

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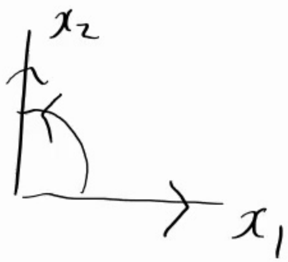
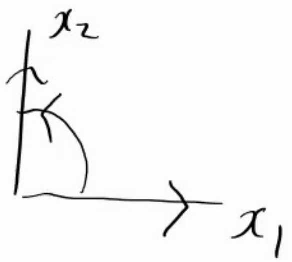


Move counter-clockwise from x_1 to x_2

If $n=2$: If we have an orientation, in coords. $dx_1 \wedge dx_2$ is compatible with orientation $dx_2 \wedge dx_1$ is not

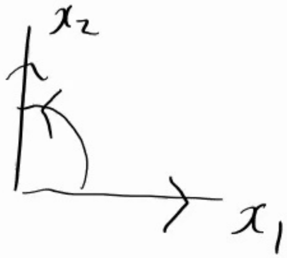


Move counter-clockwise from x_1 to x_2

$$d(-x_2) \wedge dx_1 = dx_1 \wedge dx_2$$

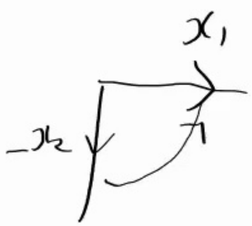
is compatible with orientation

$dx_2 \wedge dx_1$ is not



Move counter-clockwise from x_1 to x_2

$d(-x_2) \wedge dx_1 = dx_1 \wedge dx_2$



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An orientation in 2d, is a choice of clockwise vs. counter-clockwise rotation at each $p \in M$, varying continuously

<u>Or</u> At each $p \in M$, draw a small circle around p

Orientation at p gives an orientation on this circle

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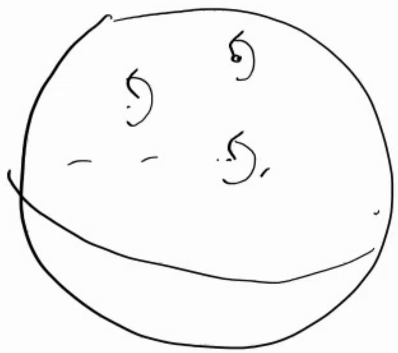

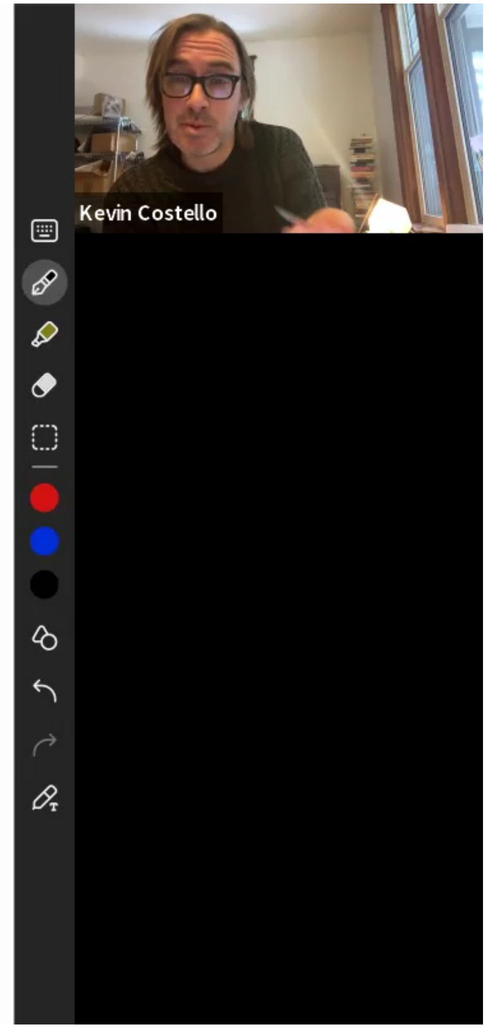

In 3d: $\forall\, p \in M$

S^2_p surrounding p

Want at each $q \in S^2_p$ a choice of clockwise vs. counter-clockwise (varying continuously)

Orientations and Integrations

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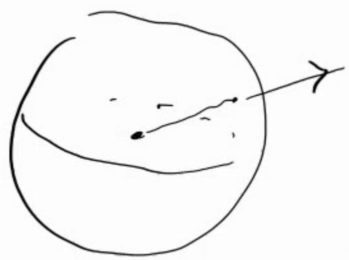
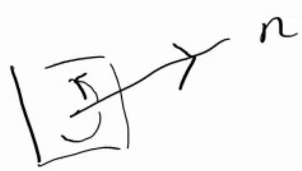
Is S^2 oriented?

Yes! Right hand rule:

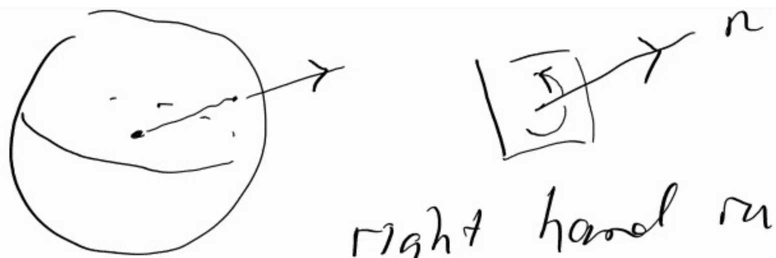
normal vector n to the 2-sphere





right hand rule tells us

how to rotate around n



right hand rule tells us how to rotate around n

More generally, if $\Sigma \subseteq \mathbb{R}^3$

an orientation $\Longleftrightarrow$ is a consistent choice of normal vector (up to scale)

More generally, if $\subset \mathbb{R}$

an orientation $\iff$ is a consistent choice of normal vector (up to scale)

Möbius band is not oriented.

1) There is no consistent normal vector (it has one side)

2)

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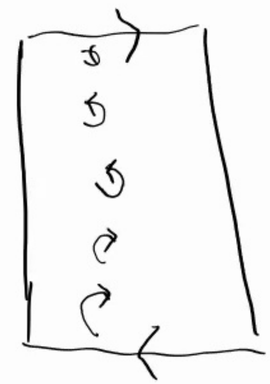

no consistent choice of clockwise vs. counter clockwise.

Another example:

$$\mathbb{RP}^2 = \{ \text{lines through } 0 \in \mathbb{R}^3 \}$$

$$= \{ x \in S^2 \} / x \sim -x$$

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$$\mathbb{R}\mathbb{P}^2 = \{ \text{lines through } 0 \in \mathbb{R}^3 \}$$

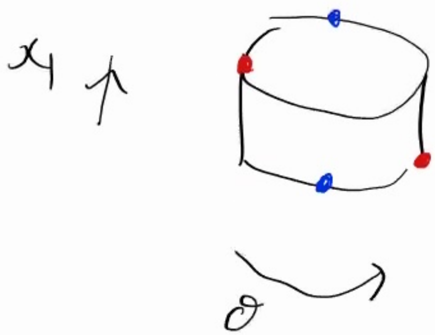
$$= \{ x \in S^2 \} / x \sim -x$$

This contains a Möbius band and so cannot have an orientation.

$x_1 = \varepsilon$

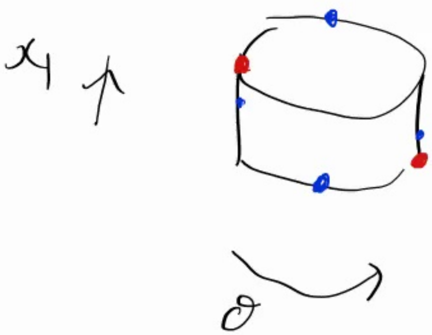
$x_1 = -\varepsilon$

Redundancy: antipodal points in



are identified

Redundancy: antipodal points in

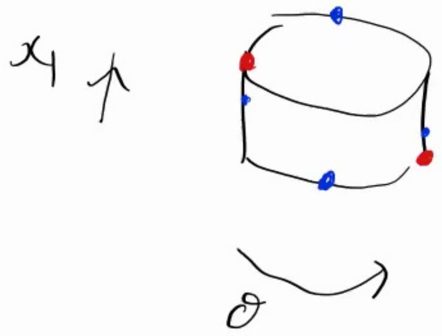


are identified

$$(x_1, \theta) \sim (-x_1, \theta + \pi)$$

Exactly what we saw for Möbius band!

Redundancy: antipodal points in



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$\mathbb{RP}^2$ = Disc sewed to a Möbius band; boundary of Möbius band is a cir

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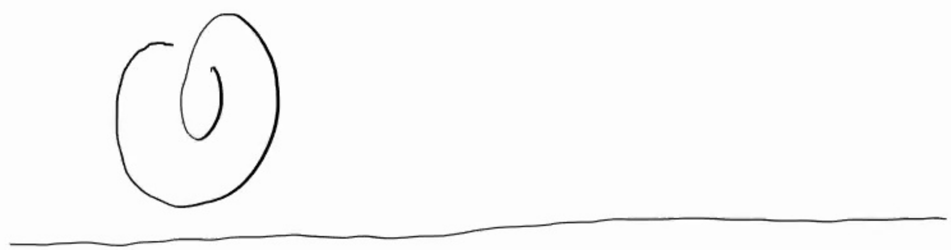

Integration

Locally, if $\omega \in \Omega^n(M)$

$R \subseteq U$ (coord. patch)

R some bounded regio

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R some bounded region

If x_i, y_i are oriented coords on U, then

$\omega = f(x)\, dx_1 \wedge \dots \wedge dx_n$

$= c$

If x_i, y_i are oriented coords on U, then

$$\omega = f(x)\, dx_1 \wedge \dots \wedge dx_n$$
$$= g(y)\, dy_1 \wedge \dots \wedge dy_n$$

$$\int_R f\, dx_1 \wedge \dots \wedge dx_n = \int_R g$$

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As, $dy_1 \wedge \cdots \wedge dy_n = \det\left(\frac{\partial y_i}{\partial x_j}\right) dx_1 \cdots dx_n$

Since this is positive, we have

$dy_1 \wedge \cdots \wedge dy_n = |\det(\partial y_i$

6/6

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$$dy_1 \wedge \dots \wedge dy_n = \left|\det\left(\frac{\partial y_i}{\partial x_j}\right)\right| dx_1 \wedge \dots \wedge dx_n$$

which appears in change of vars

6/6

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formula for integra

6/7

formula for integration.

E.g. x_i, y_i where $x_i = h_i(y)$

$$\int_{x_1=0}^{1} \cdots \int_{x_n=0}^{1} \omega$$

get same answer in x, y coords:

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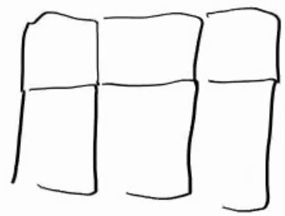
In y coords it is

$$\int_{h_i(y)=0}^{1} \cdots \int_{h_1(y)=0}^{1} \omega$$

Integrating over the same region.

If M is a global manifold
(compact, i.e. closed and bounded)

Cut M up into cubes:



each cube lives in a coord. patch

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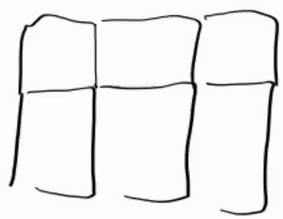

each cube lives in a coord. patch

Finitely many cubes.

Then, for each cube $C \subseteq M$

$\omega \in \Omega^n($

Finitely many cubes.

Then, for each cube $C \subseteq M$

$\omega \in \Omega^n(M)$,

$\int_C \omega$ makes senses

$\int_C \omega$ makes senses independent of words.

If $C_1 \dots C_n$ are the cubes,

$$\int_M \omega = \sum_i \int_{C_i} \omega$$

If

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If $C'_1, \ldots, C'_l$ is a different way of cutting M up then why is

$$\sum_i \int_{C_i} \omega \overset{?}{=} \sum_j \int_{C'_j} \omega$$

As, we can consider

$$\sum_i \int_{C_i} \omega \doteq \sum_j \int_{C'_j} \omega$$

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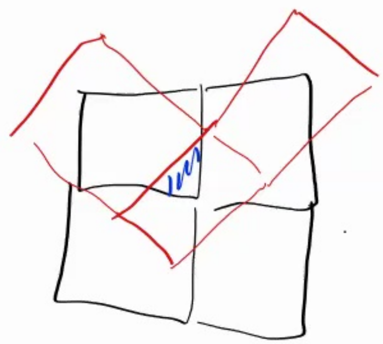
$$\sum_{i,j} \int_{C_i \cap C'_j} \omega$$



$$= \sum_i \left(\sum_j \int_{C_i \cap C'_j} \omega \right)$$

, is, we can consider

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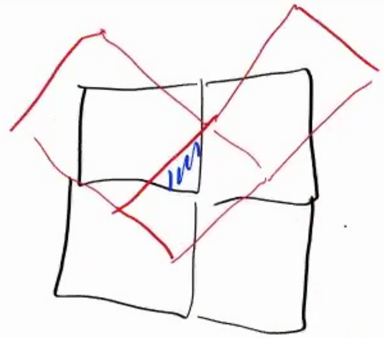

$$= \sum_i \left(\sum_j \int_{C_i \cap C_j'} \omega \right) = \sum_i \int_{C_i} \omega$$

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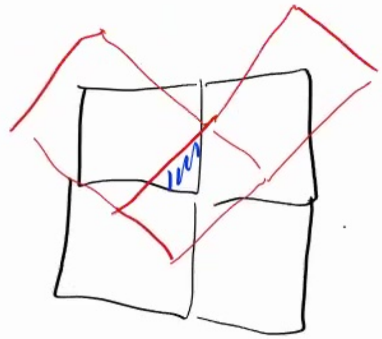

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$\omega \in \Omega^n(M)$

$\omega = f(x)\, dx_1 \wedge \dots \wedge dx_n$

We are doing usual $\int$ of f over some region.

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Choose 2 patches

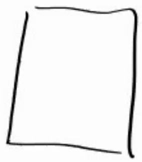
$x_3 \geq 0$

$x_3 \leq 0$

On each patch, coords are x_1, x_2 with $x_1^2 + x_2^2 \leq 1$

x_1, x_2 with $x_1^2 + x_2^2 \leq 1$

$$\iint f \, dx_1 \, dx_2$$

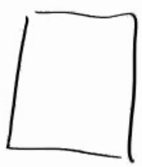


Oriented coords

- x_1, x_2
- $f \quad dx_1 \, dx_2$
- $\int$ <u>means</u> Lebesgue / Riemann integral:

$$\iint f \, dx_1 \, dx_2$$

Oriented coords



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$$\iint dx_1 \, dx_2 = \text{Fubini}$$

Different orientation

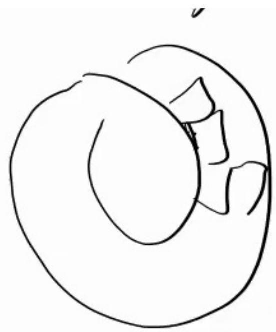
x_2, x_1 oriented coords

Rules: $\omega = g \, dx_2 \wedge dx_1$

perform Lebesgue integral

$\omega = f \, dx_1 \wedge dx_2$

$g = -f$



In general, naturally $\int$ a density:

transforms by $\left| \det \left(\frac{\partial y_i}{\partial x_j} \right) \right|$

Orientation is same as a way to identify n-forms with densities

Stokes Theorem

If M has boundary

$N = \partial M$

An orientation on M gives one on N

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In coords, $x_1 \ldots x_n$

$N = \{x_1 = 0\}$, $x_1 \geq 0$ on M

if $dx_1 \wedge \ldots \wedge dx_n$ is an oriented volume form on M,

declare $-dx_2 \wedge \ldots \wedge dx_n$ oriented on N

N

Then, if $\omega \in \Omega^{n-1}(M)$

$$\boxed{\int_M d\omega = \int_N \omega}$$

N

Then, if $\omega \in \Omega^{n-1}(M)$

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Generalizes Green's theorem.

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$$\boxed{\int_M d\omega = \int_N \omega}$$

Generalizes Green's theorem.

$M = [0,1]$ coords $0,1$

$\partial M = \{0, 1\}$

dx is an oriented 1 form

$x \geq 0$, 0 1

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$x \geq 0$, 0 has $-$ve orientation

1 : choose $y = 1 - x$

$-dy$ is oriented 1-form

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11/11

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$$\int_0^1 df = \int_0^1 \frac{\partial f}{\partial x}\, dx = f(1) - f(0)$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\mathrm{ve} \qquad\qquad -\mathrm{ve}$

Proof: 1) Work on a cube.

2

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Proof: 1) Work on a cube.

x_i, $\quad 0 \le x_i \le 1$

$$\omega = \sum f_i \, dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$$

$$d\omega = \sum \frac{\partial f}{\partial x_i} (-1)^{i+1} dx_1 \wedge \dots \wedge dx_n$$

Want:

$$\int_{x_1=0}^{1} \dots \int$$

Proof: 1) Work on a cube.

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Want:

$$\int_{x_1=0}^{1} \dots \int_{x_n=0}^{1} d\omega = \int_{\text{boundary}} \omega$$

$d\omega = \sum \frac{\quad}{\partial x_i} (\ \ )$

Want:

$$\int_{x_1=0}^{1} \cdots \int_{x_n=0}^{1} d\omega = \int_{\text{boundary}} \omega$$

$$= \sum_i \int_0^1 \cdots \widehat{\int_{x_i}} \cdots \int_0^1 \left(\omega\big|_{x_i=1} - \omega\big|_{x_i=0} \right)$$

$$= \sum_i \int_0^1 \cdots \widehat{\int_{x_i}} \cdots \int_0^1 \left(\omega|_{x_i=1} - \omega|_{x_i=0} \right)$$

$\square$

Contribution from $(-1)^i \frac{\partial f}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n$

by IBP, $(-1)^i \int_{x_1 \cdots \hat{x}_i \cdots x_n} \left(\omega|_{x_i=1} - \omega|_{x_i=0} \right)$

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Why $(-1)^{i+1}$?

On the boundary where $x_i = 0$

$dx_1 \wedge \dots \wedge dx_n$ is oriented volume form

$(-1)^{i+1} dx_i \wedge dx_1 \wedge \dots \widehat{dx_i} \dots dx_n$

oriented volume fo

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Why $(-1)^{\dots}$

On the boundary where $x_i = 0$

$dx_1 \wedge \dots \wedge dx_n$ is oriented volume form

$(-1)^{i+1} \, dx_i \wedge dx_1 \wedge \dots \widehat{dx_i} \dots dx_n$

oriented volume form

$\Rightarrow$ on boundary

$(-1)^i \, dx_1 \wedge \dots \wedge \widehat{dx_i} \dots dx_n$

is an oriented form.