Title: Geometry and topology for physicists 2021/2022 - Lecture 6

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Collection: Geometry and Topology for Physicists 2021/2022

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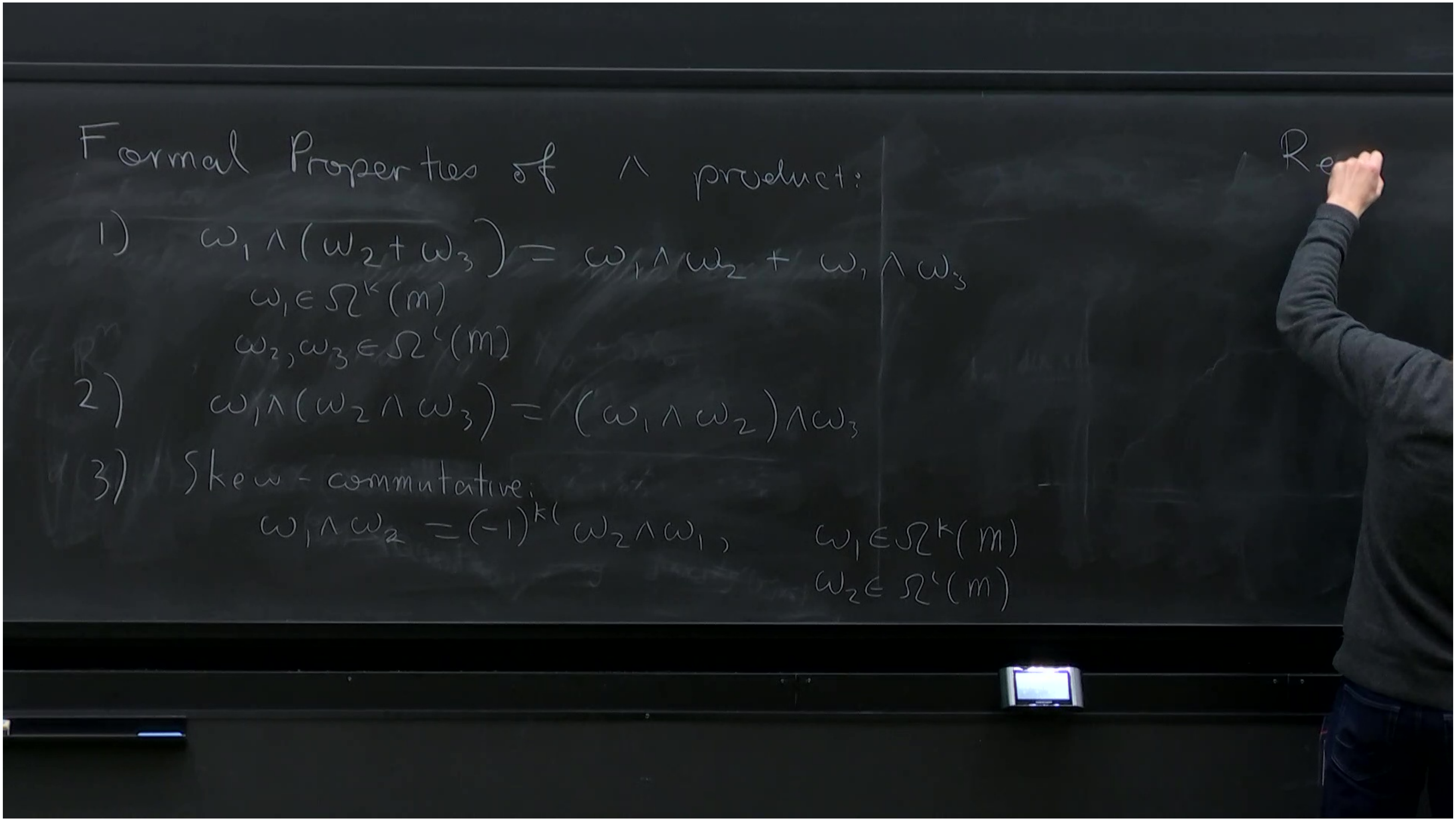
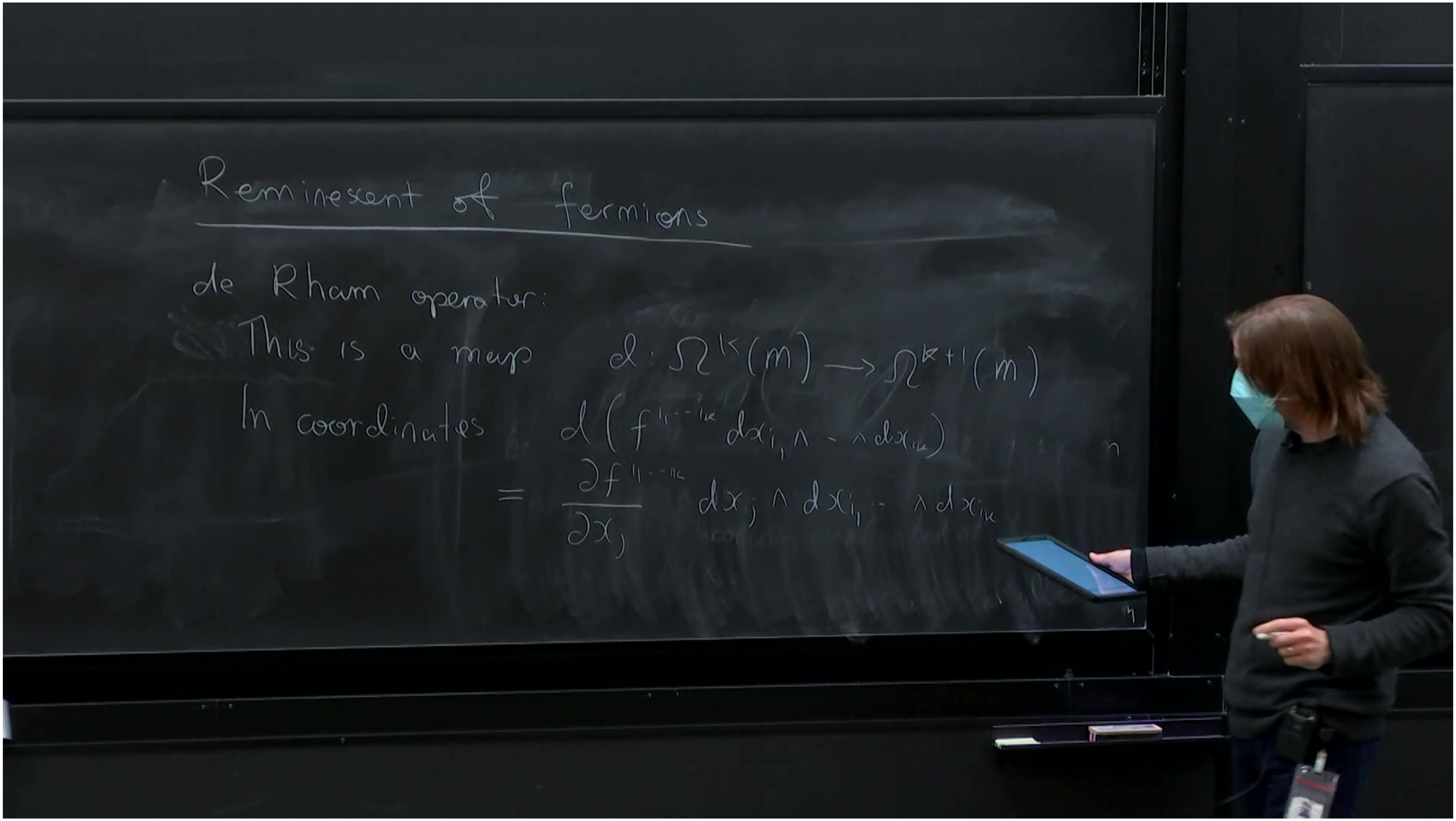

Formal Properties of $\wedge$ product:
1) $\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$
$\omega_1 \in \Omega^k(M)$
$\omega_2, \omega_3 \in \Omega^l(M)$
2) $\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3$
3) Skew-commutative:
$\omega_1 \wedge \omega_2 = (-1)^{kl} \omega_2 \wedge \omega_1,$
$\omega_1 \in \Omega^k(M)$
$\omega_2 \in \Omega^l(M)$
Re


Reminesent of fermions
de Rham operator:
This is a map $d : \Omega^k(m) \longrightarrow \Omega^{k+1}(m)$
In coordinates $d(f^{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k})$
$= \frac{\partial f^{i_1 \cdots i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \cdots \wedge dx_{i_k}$

$$\omega_1 \wedge \omega_2 = (-1)^{kl}\, \omega_2 \wedge \omega_1, \qquad \omega_1 \in \Omega^k(M)$$

$$\omega \wedge \omega = 0 \qquad\qquad \omega_2 \in \Omega^l(M)$$

On $\mathbb{R}^2$,

$$d(f^1 dx_1 + f^2 dx_2)$$

$$= -\frac{\partial f^1}{\partial x_2} dx_1 \wedge dx_2 + \frac{\partial f^2}{\partial x_1} dx_1 \wedge dx_2$$

Properties

1) $d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$

$\omega_1 \in \Omega^k$

$\omega_2 \in \Omega^l$

$\omega_1 \in \Omega^k(M)$

$\omega_2 \in \Omega^l(M)$

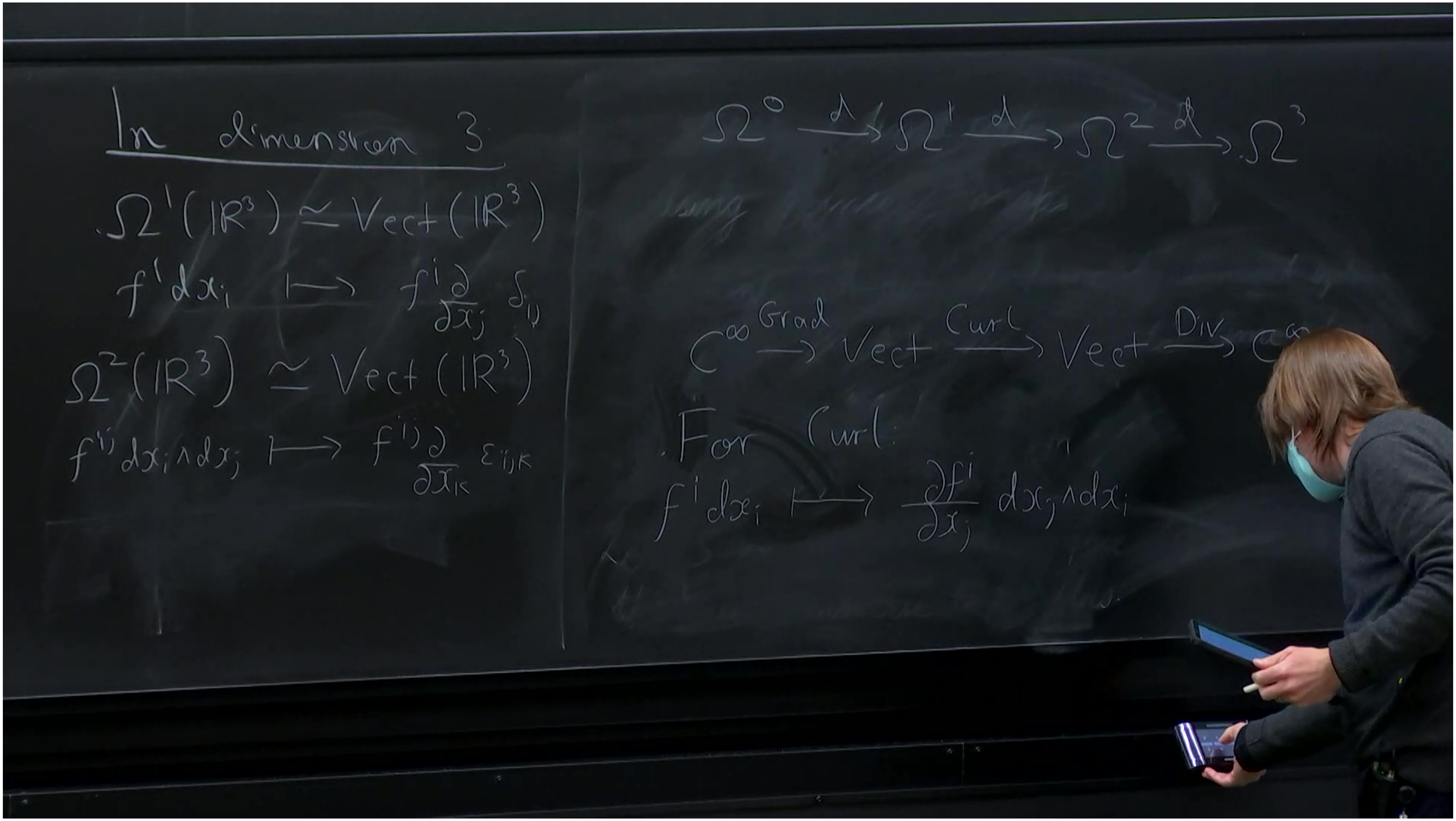
$d\omega_2$

2) $d(d\omega) = 0$

$$\omega = f \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$$d\omega = \frac{\partial f}{\partial x_{j_1}} dx_{j_1} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$$d(d\omega) = \underbrace{\frac{\partial^2 f}{\partial x_{j_2} \partial x_{j_1}}}_{\text{Symmetric in } j_1, j_2} \underbrace{dx_{j_2} \wedge dx_{j_1}}_{\text{anti-symmetric}} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} = 0$$


In dimension 3
$\Omega^1(\mathbb{R}^3) \simeq \mathrm{Vect}(\mathbb{R}^3)$
$f^i dx_i \longmapsto f^i \frac{\partial}{\partial x_j} \delta_{ij}$
$\Omega^2(\mathbb{R}^3) \simeq \mathrm{Vect}(\mathbb{R}^3)$
$f^{ij} dx_i \wedge dx_j \longmapsto f^{ij} \frac{\partial}{\partial x_k} \varepsilon_{ijk}$
$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3$
$C^\infty \xrightarrow{\mathrm{Grad}} \mathrm{Vect} \xrightarrow{\mathrm{Curl}} \mathrm{Vect} \xrightarrow{\mathrm{Div}} C^\infty$
For Curl:
$f^i dx_i \longmapsto \frac{\partial f^i}{\partial x_j} dx_j \wedge dx_i$

$$f^i \, dx_i \longleftrightarrow f^i \frac{\partial}{\partial x_j} \delta_{ij}$$

$$\frac{\partial f^i}{\partial x_j} dx_j \wedge dx_i \longleftrightarrow \varepsilon_{kji} \frac{\partial f^i}{\partial x_j} \frac{\partial}{\partial x_k}$$

de Rham sends

$$f^i \frac{\partial}{\partial x_j} \delta_{ij} \longrightarrow \varepsilon_{kji} \frac{\partial f^i}{\partial x_j} \frac{\partial}{\partial x_k}$$

This is curl.

$$\omega_1 \wedge \omega_2 = (-1)^{kl}\, \omega_2 \wedge \omega_1, \qquad \omega_1 \in \Omega^k(M)$$

$$\omega \wedge \omega = 0 \qquad\qquad \omega_2 \in \Omega^l(M)$$

A first look at cohomology

If M is a manifold,

$$H^k(M) = \left\{ \omega \in \Omega^k(M),\ d\omega = 0 \right\} \Big/ \quad \begin{array}{l} \omega \sim \omega' \\ \text{if } \omega - \omega' \\ = d\eta, \\ \eta \in \Omega^{k-1}(M) \end{array}$$

(ω is closed)

For this

$\Omega^k(m)$

$\Omega^l(m)$

For this to make sense, it's necessary that

$$d(d\eta) = 0$$

$\omega \sim \omega'$

$\omega - \omega'$

$= d\eta,$

$\eta \in \Omega^{k-1}(m)$

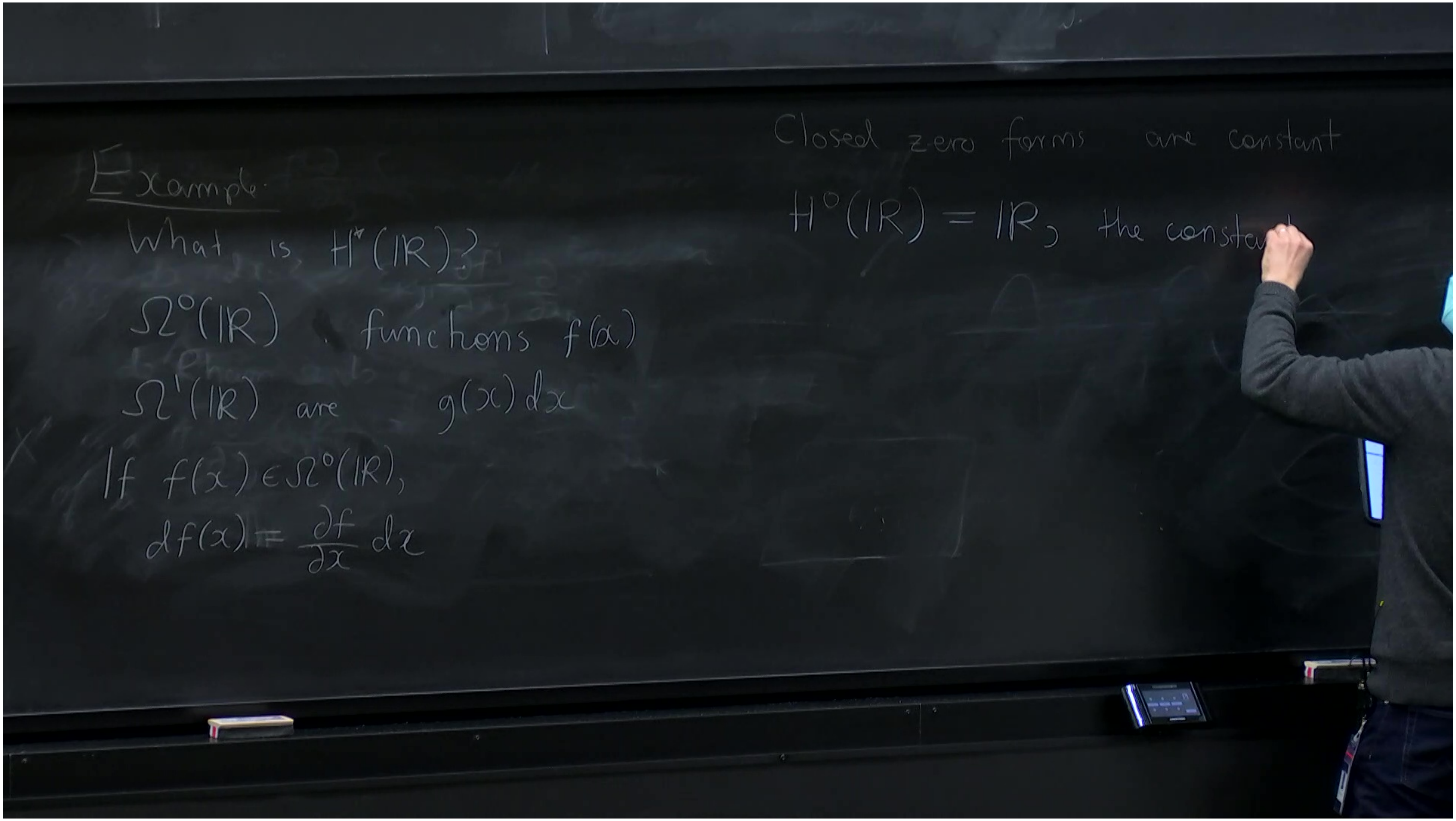
For this to make sense, it's necessary that

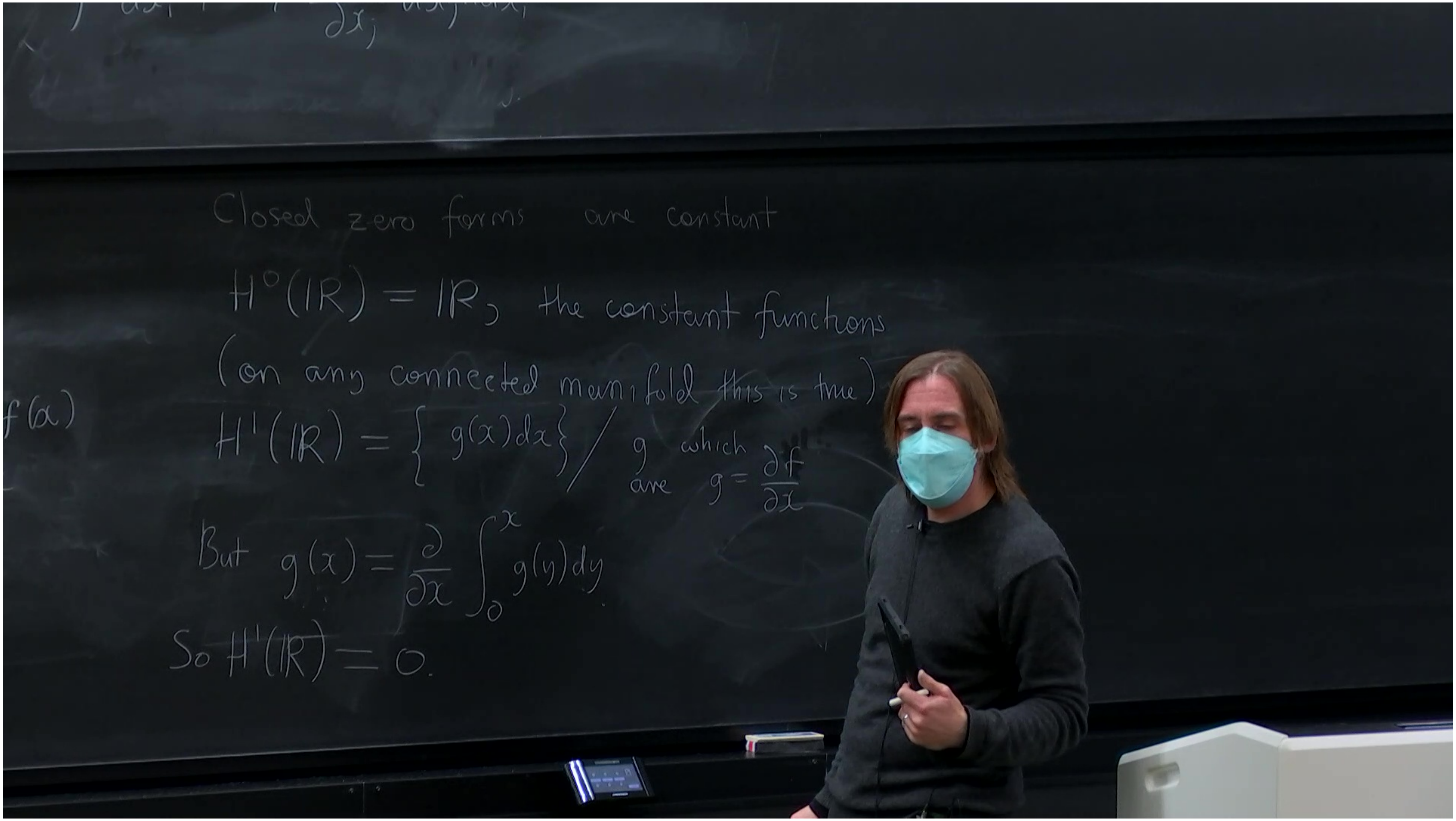
$$d(d\eta) = 0$$

$H^k(M,\mathbb{R})$ are topological invariants

$$b_k = \dim_{\mathbb{R}} H^k(M,\mathbb{R})$$

(typically finite)


Example.
What is H*(IR)?
Ω°(IR) functions f(x)
Ω¹(IR) are g(x) dx
If f(x) ∈ Ω°(IR),
df(x) = ∂f/∂x dx
Closed zero forms are constant
H°(IR) = IR, the consta


Closed zero forms are constant
$H^0(\mathbb{R}) = \mathbb{R}$, the constant functions
(on any connected manifold this is true)
$H^1(\mathbb{R}) = \{ g(x)dx \} / g$ which are $g = \frac{\partial f}{\partial x}$
But $g(x) = \frac{\partial}{\partial x} \int_0^x g(y)dy$
So $H^1(\mathbb{R}) = 0$.

So $H(\mathbb{R}) = 0$.

S^1

$\Omega^0 = \{ f(\theta),\ f(\theta + 2\pi) = f(\theta) \}$

$\Omega^1 = \{ g(\theta)\,d\theta,\ g(\theta + 2\pi) = g(\theta) \}$

If $f(\theta) \in \Omega^0$, $df(\theta) = 0$ iff f is constant

$H^0(S^1) = \mathbb{R}$

H^1: If $g(\theta)d\theta$, g is equivalent to 0 in cohomology if it has a periodic anti-derivative

$$g(\theta)d\theta = d\left(\int_{\varphi=0}^{\theta} g(\varphi)\,d\varphi\right)$$

But $\int_{\varphi=0}^{\theta} g(\varphi)\,d\varphi$ might not be periodic!

It's periodic iff $\int_0^{2\pi} g(\varphi)\,d\varphi = 0$

(typically finite)

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$-g(\varphi)d\varphi = 0$

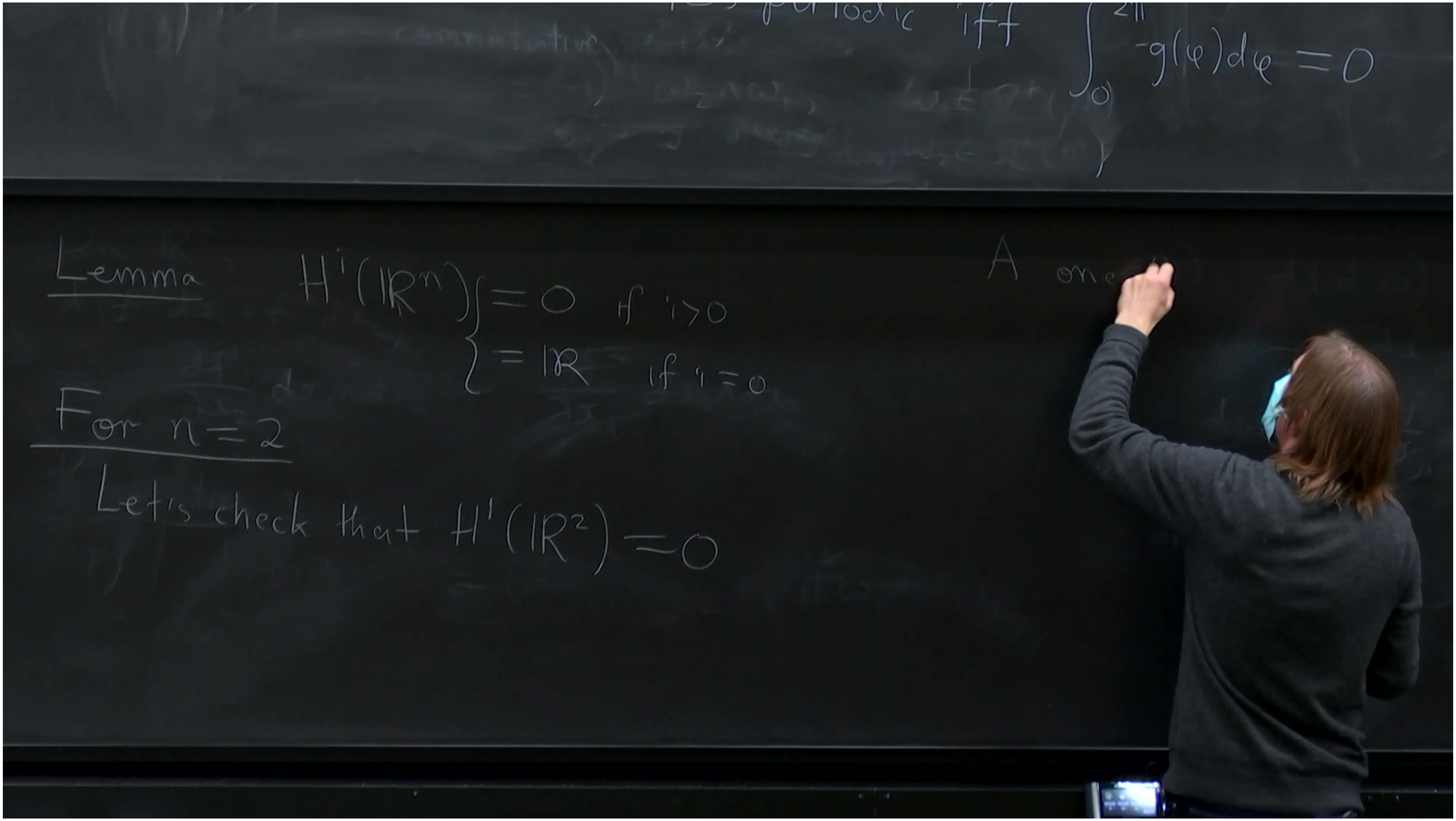
Exact 1-forms are $g(\theta)d\theta$ with 0 integral

Since we can add a constant $c\,d\theta$ to make anything have 0 integral,

$$H^1(S^1) = \mathbb{R}$$

The isomorphism is given by $\int_{S^1}$

Intuition This is true because $H^1(S^1)$ "measures" closed 1d manifolds


periodic iff $\int_0^{2\pi} g(\varphi)\,d\varphi = 0$
Lemma
$H^i(\mathbb{R}^n) = 0$ if $i>0$
$= \mathbb{R}$ if $i=0$
For $n=2$
Let's check that $H^1(\mathbb{R}^2)=0$
A one

iff $\int_0^{2\pi} g(\varphi)\, d\varphi = 0$

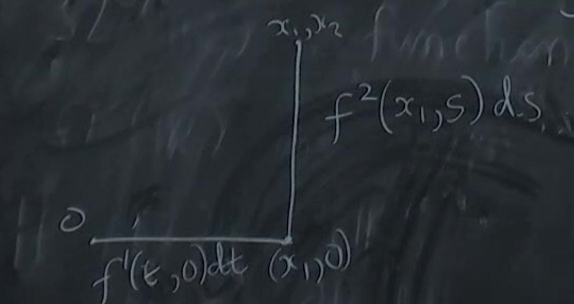
<u>Intuition</u> This is true because $H^1(S^1)$ "measures" closed 1d manifolds

A one-form is
$\omega = f^1 dx_1 + f^2 dx_2$
It's closed if
$$\frac{\partial f^1}{\partial x_2} = \frac{\partial f^2}{\partial x_1}$$

We want to show that there is $F(x_1, x_2)$ with
$$\frac{\partial F}{\partial x_1} = f^1$$
$$\frac{\partial F}{\partial x_2} = f^2$$

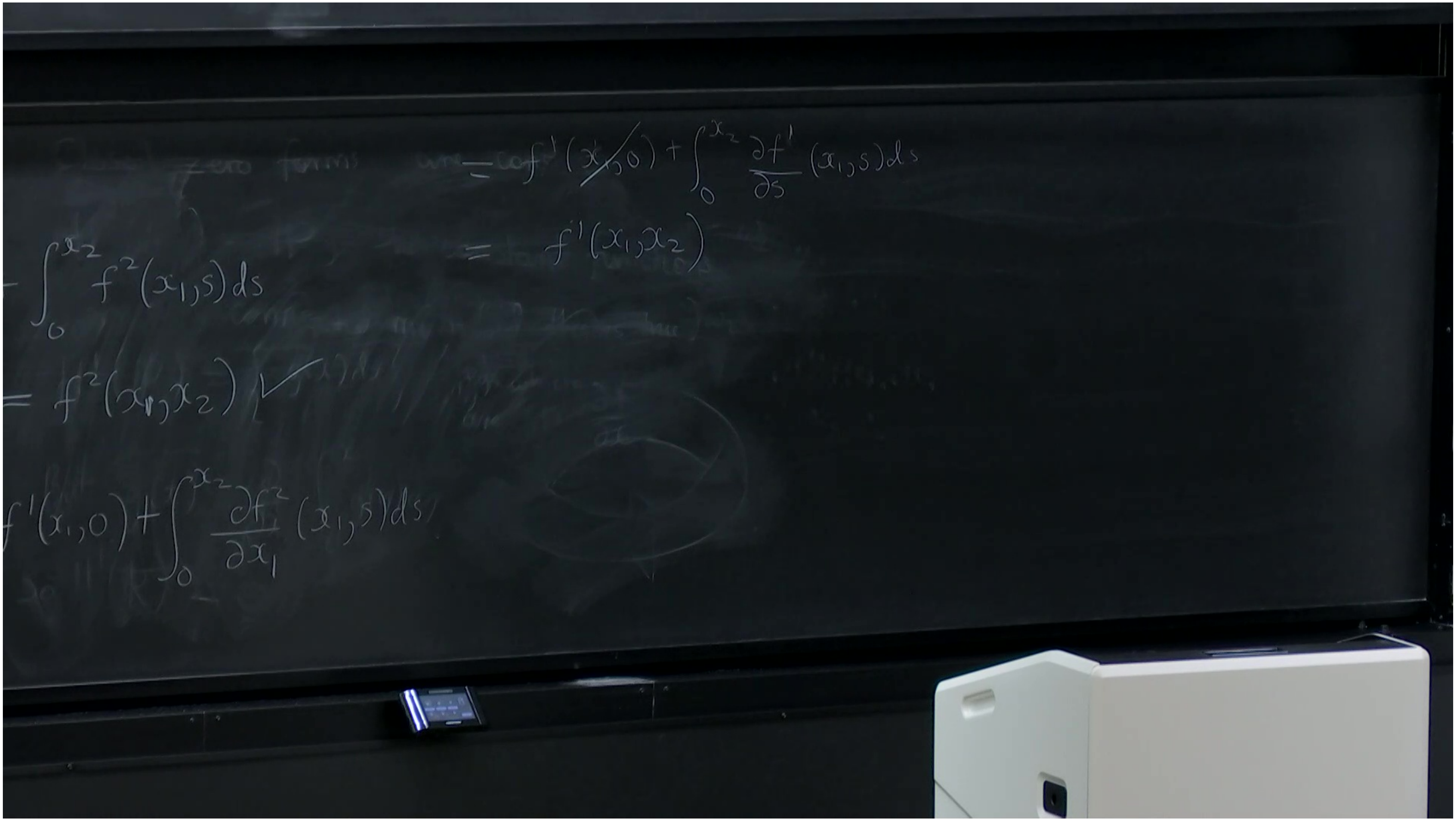
Define

$$F(x_1,x_2) = \int_0^{x_1} f^1(t,0)\,dt + \int_0^{x_2} f^2(x_1,s)\,ds$$



$$\frac{\partial F}{\partial x_2} = f^2(x_1,x_2)$$

$$\frac{\partial F}{\partial x_1} = f^1(x_1,0) + \int_0^{x_2} \frac{\partial f^2}{\partial x_1}(x_1,s)\,ds$$


$= f^1(\cancel{x_1},0) + \int_0^{x_2} \frac{\partial f^1}{\partial s}(x_1,s)\,ds$
$= f^1(x_1,x_2)$
$\int_0^{x_2} f^2(x_1,s)\,ds$
$= f^2(x_1,x_2)$
$f^1(x_1,0) + \int_0^{x_2} \frac{\partial f^2}{\partial x_1}(x_1,s)\,ds$

Integration

Recall, x_i, y_i are coords

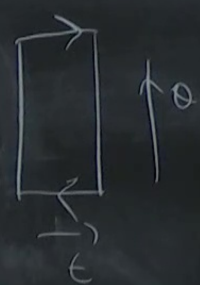
$$dy_1 \wedge \dots \wedge dy_n = \det\left(\frac{\partial y_i}{\partial x_j}\right) dx_1 \wedge \dots \wedge dx_n$$

But, when we change variables in integration,

we get $\left|\det\left(\frac{\partial y_i}{\partial x_j}\right)\right|$

Def^n

An ori



$\wedge dx_n$

$t \to -t$

$\theta \to \theta + 2\pi$

Jacobian is -ve

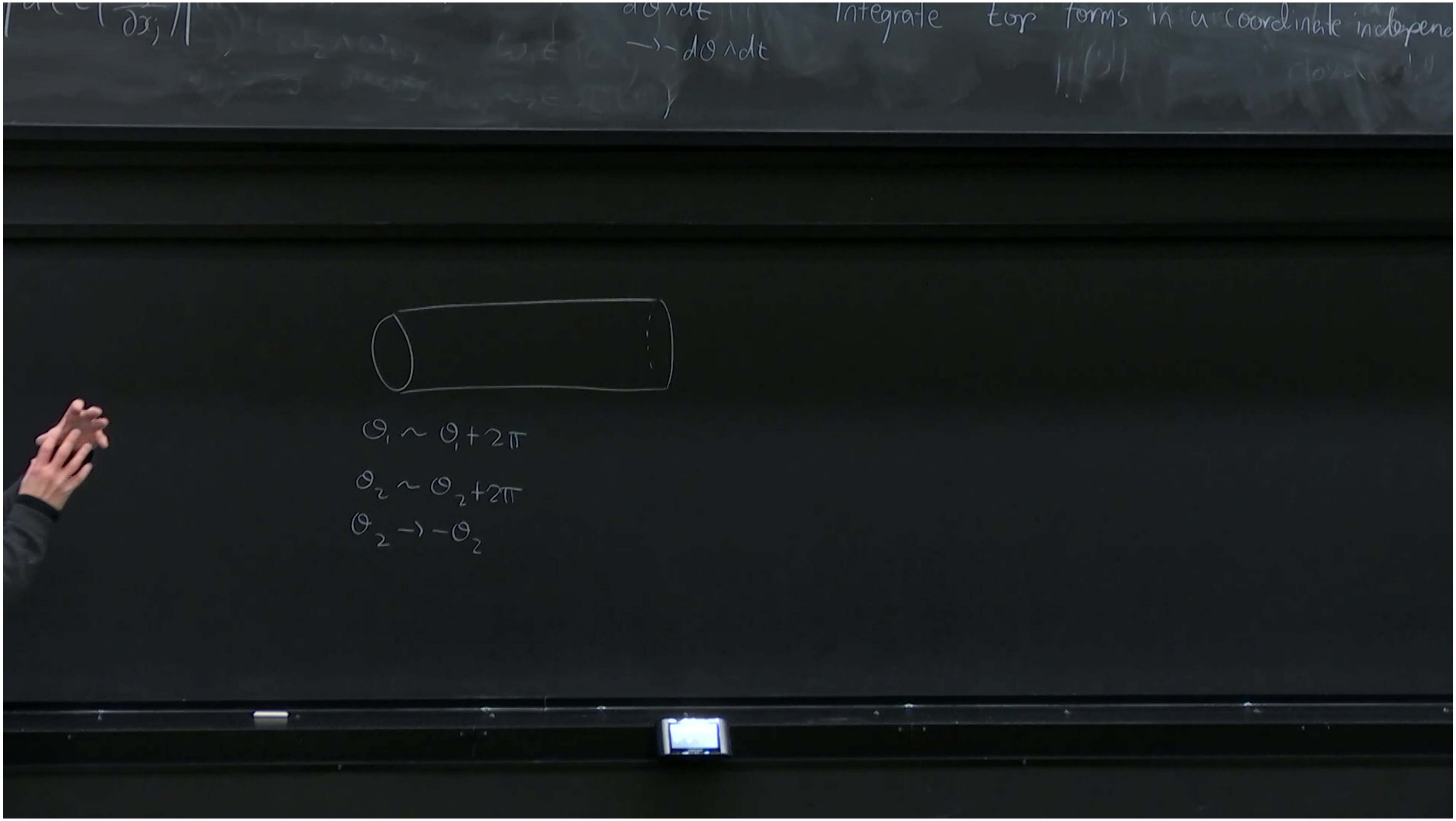
...tion,

<u>Def^n</u>

An orientation on a manifold is a coordinate atlas so that

$$\det\left(\frac{\partial y_i}{\partial x_j}\right) > 0 \text{ between any two patches.}$$

<u>Theorem</u> On any compact oriented manifold, we can integrate top forms in a coordinate independent way.


dθ∧dt
→ −dθ∧dt
Integrate top forms in a coordinate independe
θ1 ~ θ1 + 2π
θ2 ~ θ2 + 2π
θ2 → −θ2