Title: Geometry and topology for physicists 2021/2022 - Lecture 5

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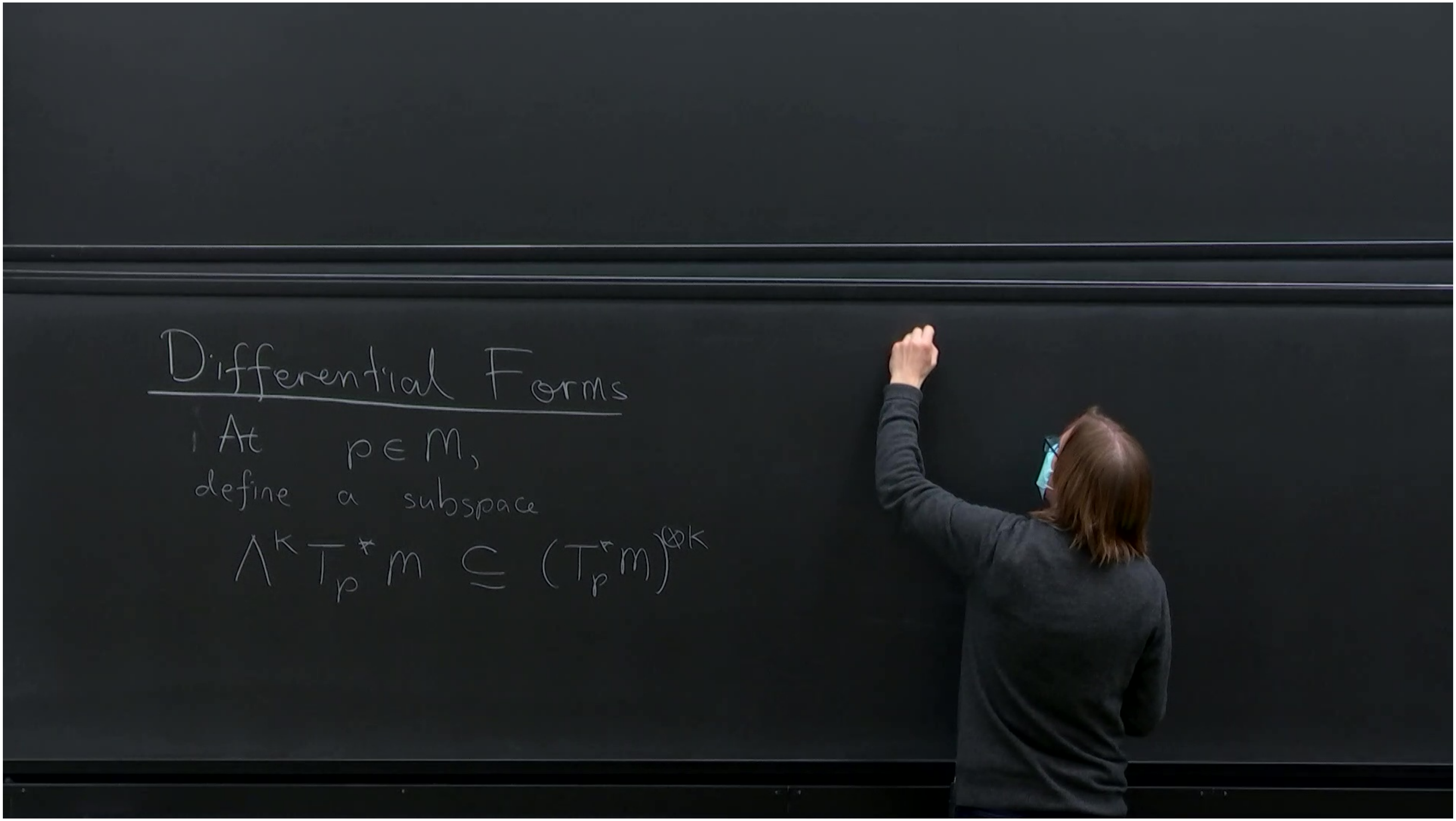
Collection: Geometry and Topology for Physicists 2021/2022

Date: March 10, 2022 - 10:15 AM

URL: https://pirsa.org/22030063


Differential Forms
At $p \in M$,
define a subspace
$\Lambda^k T_p^* M \subseteq (T_p^* M)^{\otimes k}$

as the totally anti-symmetric tensors.

$$(T_p^* M)^{\otimes k} = \left\{ \begin{array}{l} \text{multi-linear maps} \\ \omega : T_p M \times \cdots \times T_p M \to \mathbb{R} \end{array} \right\}$$

$$\cup\!\!|$$

$$\Lambda^k T_p^* M = \left\{ \begin{array}{l} \omega \text{ s.t. if } V_1, \dots, V_k \text{ are vectors} \\ \sigma \in S_k, \quad \omega(V_1, \dots, V_k) = (-1)^\sigma \omega(V_{\sigma_1}, \dots, V_{\sigma_k}) \end{array} \right\}$$

On a manifold M, a k-form is a tensor field of type $(0,k)$ which is totally anti-symmetric.

In c

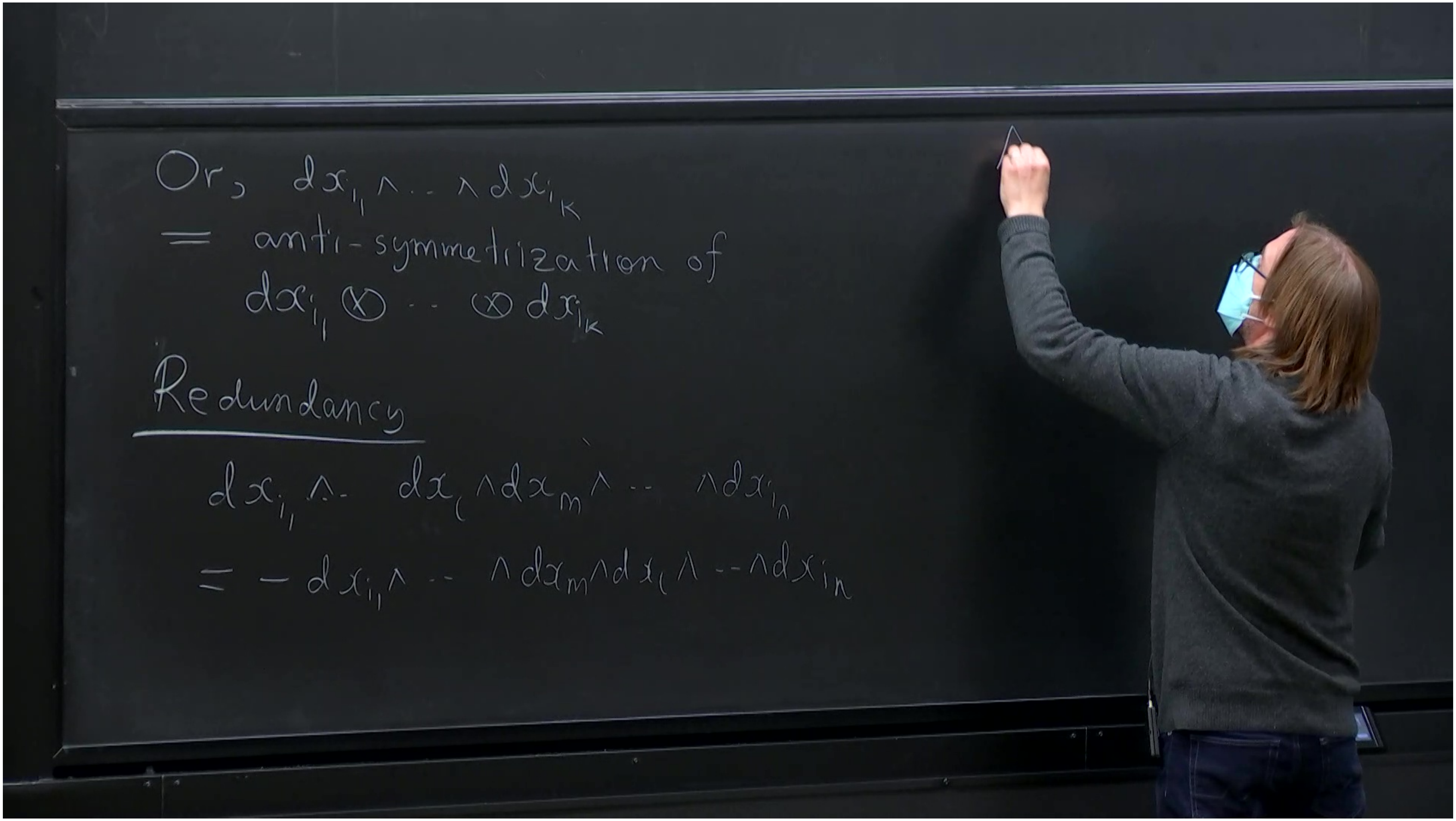
In coordinates $x_1, \dots, x_n$

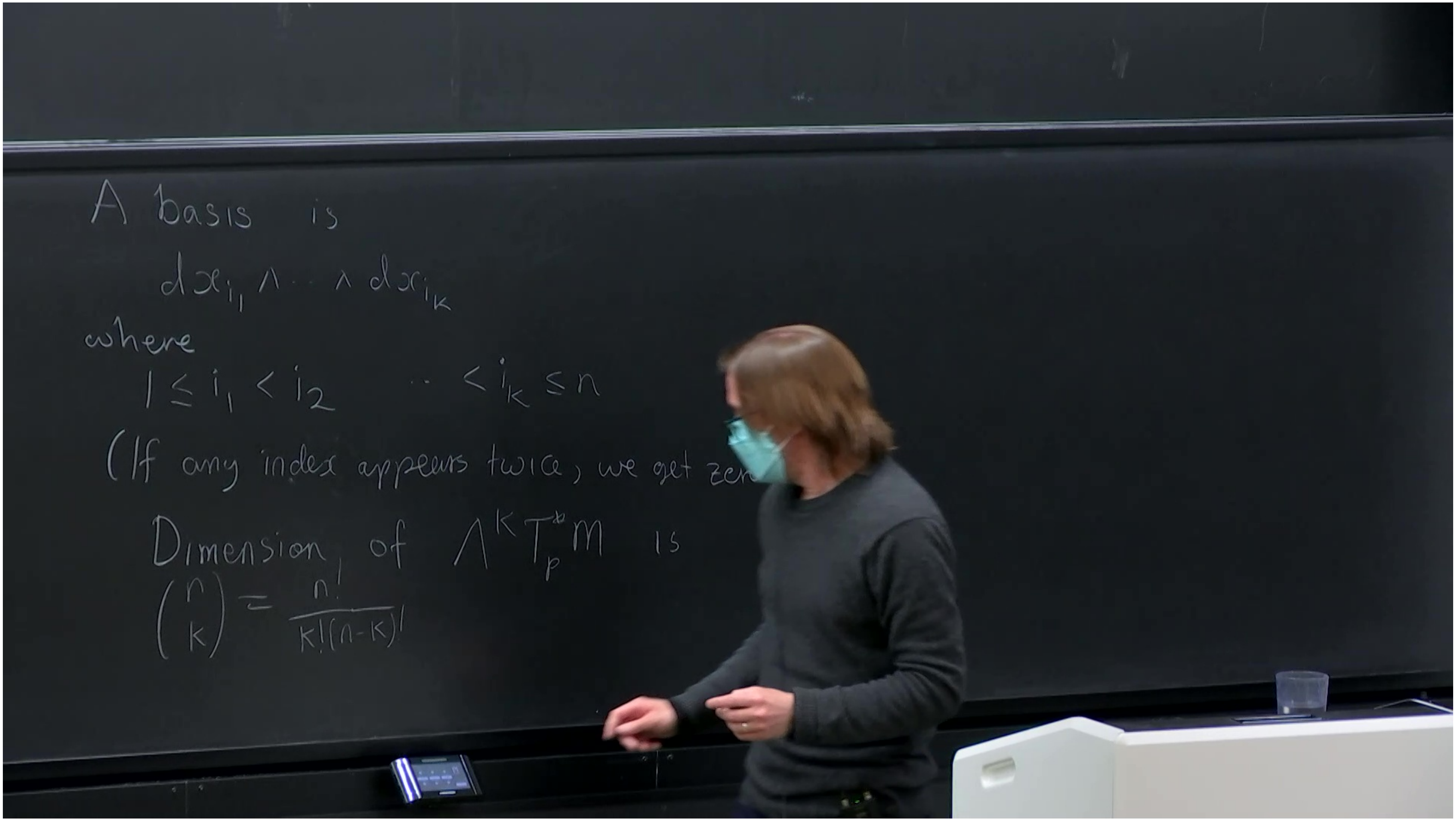
the expressions dx_i form a basis for $T_p^* M$

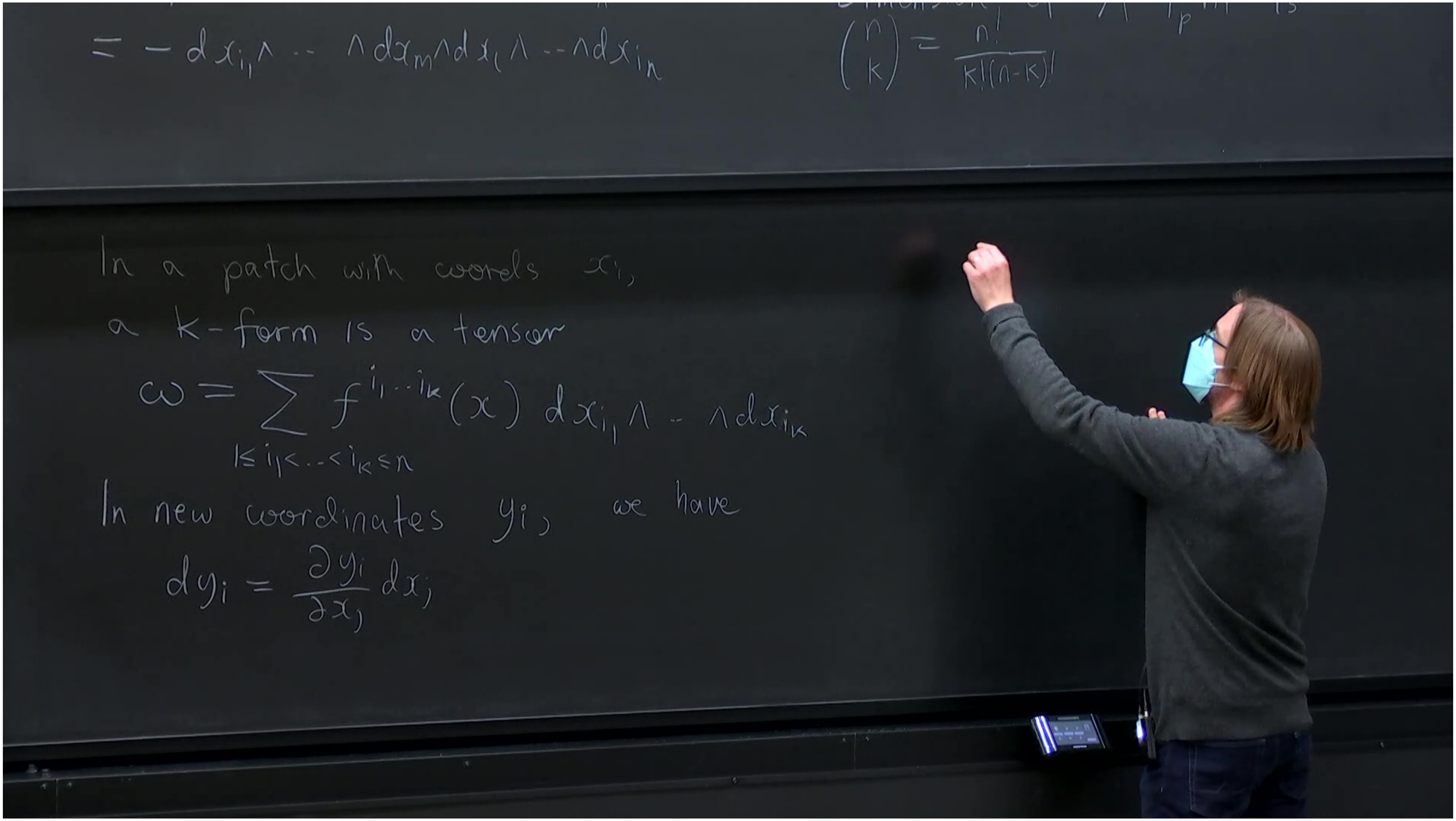
We will use the notation

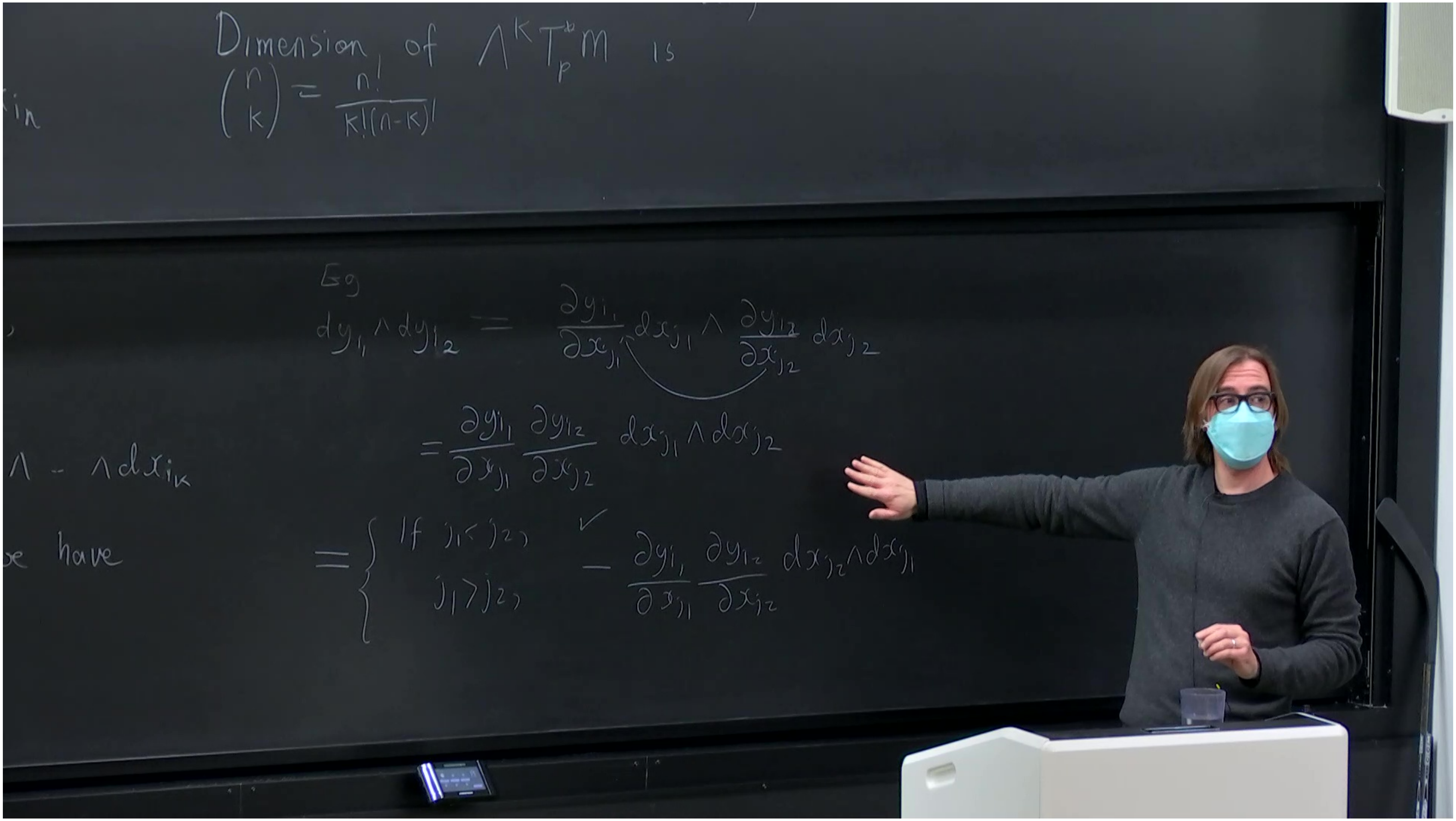
$dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ for the linear map

$$\frac{\partial}{\partial x_{j_1}}, \dots, \frac{\partial}{\partial x_{j_k}} \longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{\sigma} \delta_{i_1}^{j_{\sigma_1}} \cdots \delta_{i_k}^{j_{\sigma_k}}$$


Or, $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$
$=$ anti-symmetrization of
$dx_{i_1} \otimes \cdots \otimes dx_{i_k}$
Redundancy
$dx_{i_1} \wedge \cdots dx_l \wedge dx_m \wedge \cdots \wedge dx_{i_n}$
$= - dx_{i_1} \wedge \cdots \wedge dx_m \wedge dx_l \wedge \cdots \wedge dx_{i_n}$


A basis is
$dx_{i_1} \wedge \cdots \wedge dx_{i_k}$
where
$1 \le i_1 < i_2 \cdots < i_k \le n$
(If any index appears twice, we get zer
Dimension of $\Lambda^k T_p^* M$ is
$\binom{n}{k} = \frac{n!}{k!(n-k)!}$


$= - dx_{i_1} \wedge \cdots \wedge dx_m \wedge dx_l \wedge \cdots \wedge dx_{i_n}$
$\binom{n}{k} = \frac{n!}{k!(n-k)!}$
In a patch with coords x_i,
a k-form is a tensor
$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} f^{i_1 \cdots i_k}(x)\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$
In new coordinates y_i, we have
$dy_i = \frac{\partial y_i}{\partial x_j} dx_j$


Dimension of $\Lambda^k T_p^* M$ is
$\binom{n}{k} = \frac{n!}{k!(n-k)!}$
i_n
Eg
$dy_{i_1} \wedge dy_{i_2} = \frac{\partial y_{i_1}}{\partial x_{j_1}} dx_{j_1} \wedge \frac{\partial y_{i_2}}{\partial x_{j_2}} dx_{j_2}$
$= \frac{\partial y_{i_1}}{\partial x_{j_1}} \frac{\partial y_{i_2}}{\partial x_{j_2}} dx_{j_1} \wedge dx_{j_2}$
$\wedge \cdots \wedge dx_{i_k}$
we have
If $j_1 < j_2$, ✓
$j_1 > j_2$, $- \frac{\partial y_{i_1}}{\partial x_{j_1}} \frac{\partial y_{i_2}}{\partial x_{j_2}} dx_{j_2} \wedge dx_{j_1}$

$\frac{\partial}{\partial x_{j_1}}, \ldots, \frac{\partial}{\partial x_{j_k}} \longmapsto$

$k = n$

form is $f(y)\, dy_1 \wedge \ldots \wedge dy_n$

In x coords,

$$dy_1 \wedge \ldots \wedge dy_n = (\text{Some function}) \times dx_1 \wedge \ldots \wedge dx_n$$

Lemma The function is

$$\det\left(\frac{\partial y_i}{\partial x_j}\right) \rightarrow \text{viewed as an } n \times n \text{ matrix}$$

$$\ldots, \frac{\partial}{\partial x_{j_k}} \longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma \, \delta_{i_1}^{j_{\sigma_1}} \cdots \delta_{i_k}^{j_{\sigma_k}}$$

Proof

$$dy_1 \wedge \cdots \wedge dy_n = \left(\frac{\partial y_1}{\partial x_{j_1}} dx_{j_1} \right) \wedge \cdots \wedge \left(\frac{\partial y_n}{\partial x_{j_n}} dx_{j_n} \right)$$

$j_1, \ldots, j_n$ are all distinct. $j_k = \sigma_k$, a permutation

$$= \sum_{\sigma \in S_n} (-1)^\sigma \frac{\partial y_1}{\partial x_{\sigma_1}} \cdots \frac{\partial y_n}{\partial x_{\sigma_n}} \, dx_1 \wedge \cdots \wedge dx_n$$

$$= \det\left(\frac{\partial y_i}{\partial x_j} \right) dx_1 \wedge \cdots \wedge dx_n$$

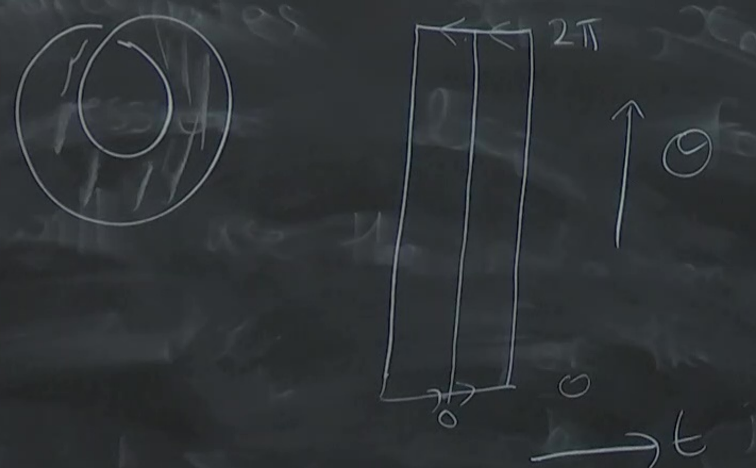
S^1

coordinate θ, $\theta \sim \theta + 2\pi$

$$d(\theta + 2\pi) = d\theta$$

So, 1-forms on S^1 are expressions $f(\theta)\,d\theta$, where $f(\theta + 2\pi) = f(\theta)$

Möbius band



$\theta \sim \theta + 2\pi$

$t \sim -t$

$d\theta = d\theta$

$dt = -dt$

A one form on Möbius band is

$$f(\theta,t)\,d\theta + g(\theta,t)\,dt$$

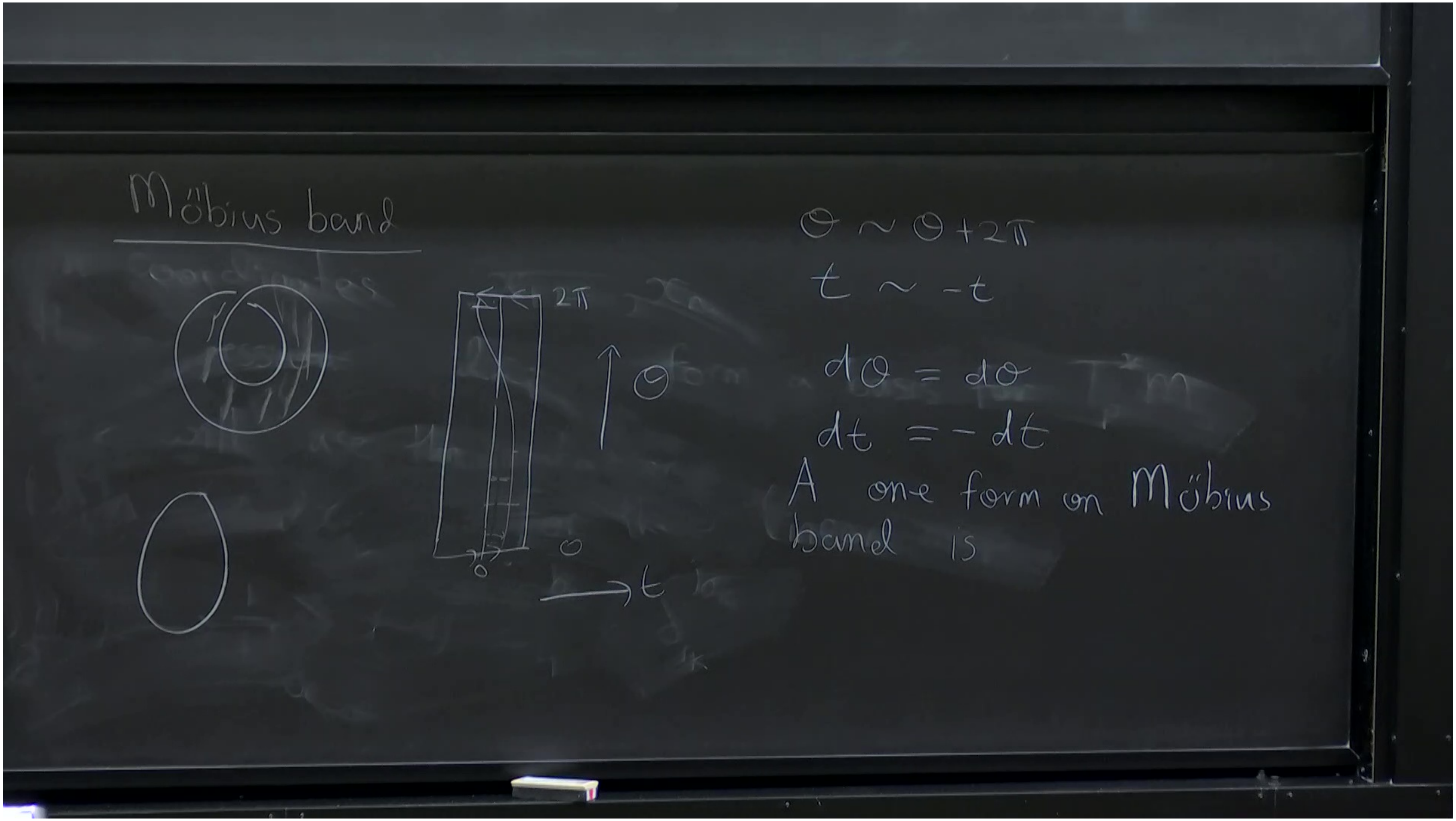
where $f(\theta+2\pi,-t) = f(\theta,t)$

$$g(\theta+2\pi,-t) = -g(\theta,t)$$

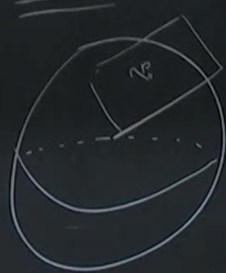
$$g(\theta+2\pi,0) = -g(\theta,0)$$

anti-periodic function.

g has a zero.


Möbius band
2π
θ
0
t
θ ∼ θ + 2π
t ∼ −t
dθ = dθ
dt = −dt
A one form on Möbius band is

S^2



Tangent vector:

at $p \in S^2$

a vector $v \in \mathbb{R}^3$

with $\langle v, p \rangle = 0$

(v is tangent)

Theorem (Lefschetz?)

Every vector field (or 1-form) on S^2 has a zero.

anti-periodic function.
g has a zero.

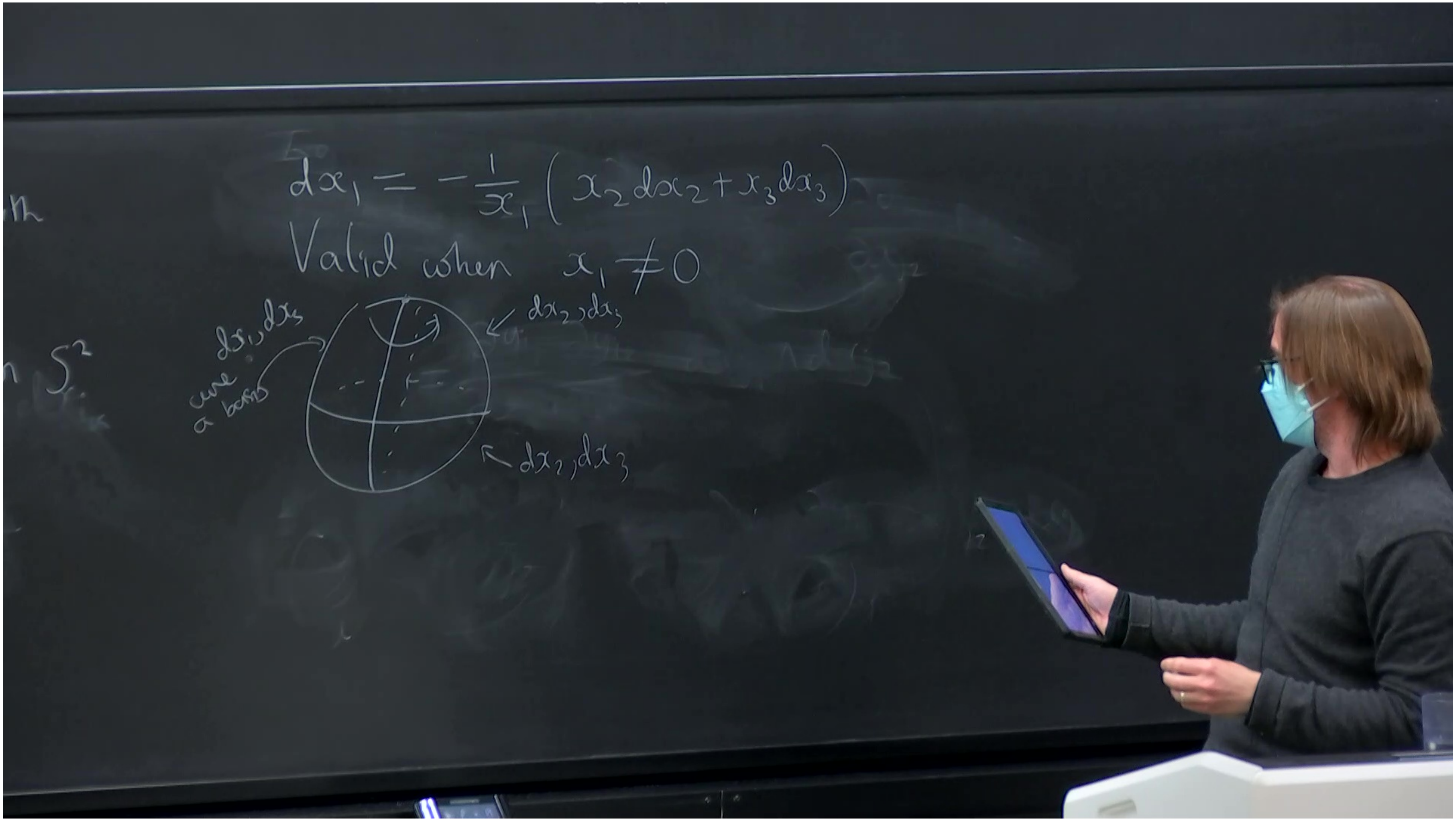
Every vector field (or 1-form) on S^2 has a zero.

S^2 has coords x_1, x_2, x_3 with
$$\sum x_i^2 = 1$$

dx_1, dx_2, dx_3 are 1-forms on S^2
but
$$x_1 dx_1 + x_2 dx_2 + x_3 dx_2 = 0$$

Near north pole, $(1,0,0)$, dx_2, dx_3 are a basis of $T_p^* S^2$

$$dx_1 = -\frac{1}{x_1}\left(x_2 dx_2 + x_3 dx_3\right.$$

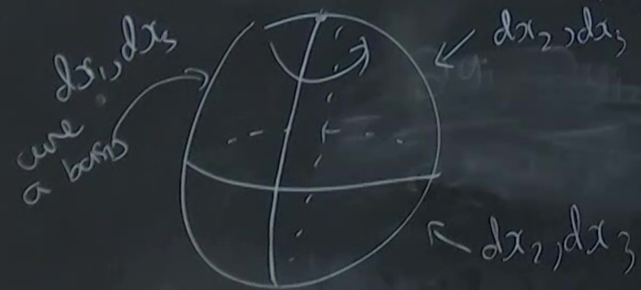

$dx_1 = -\frac{1}{x_1}(x_2 dx_2 + x_3 dx_3)$
Valid when $x_1 \neq 0$
dx_1, dx_3
are a basis
dx_2, dx_3
dx_2, dx_3

th

n S^2

$$dx_1 = -\frac{1}{x_1}\left(x_2\,dx_2 + x_3\,dx_3\right)$$

Valid when $x_1 \neq 0$

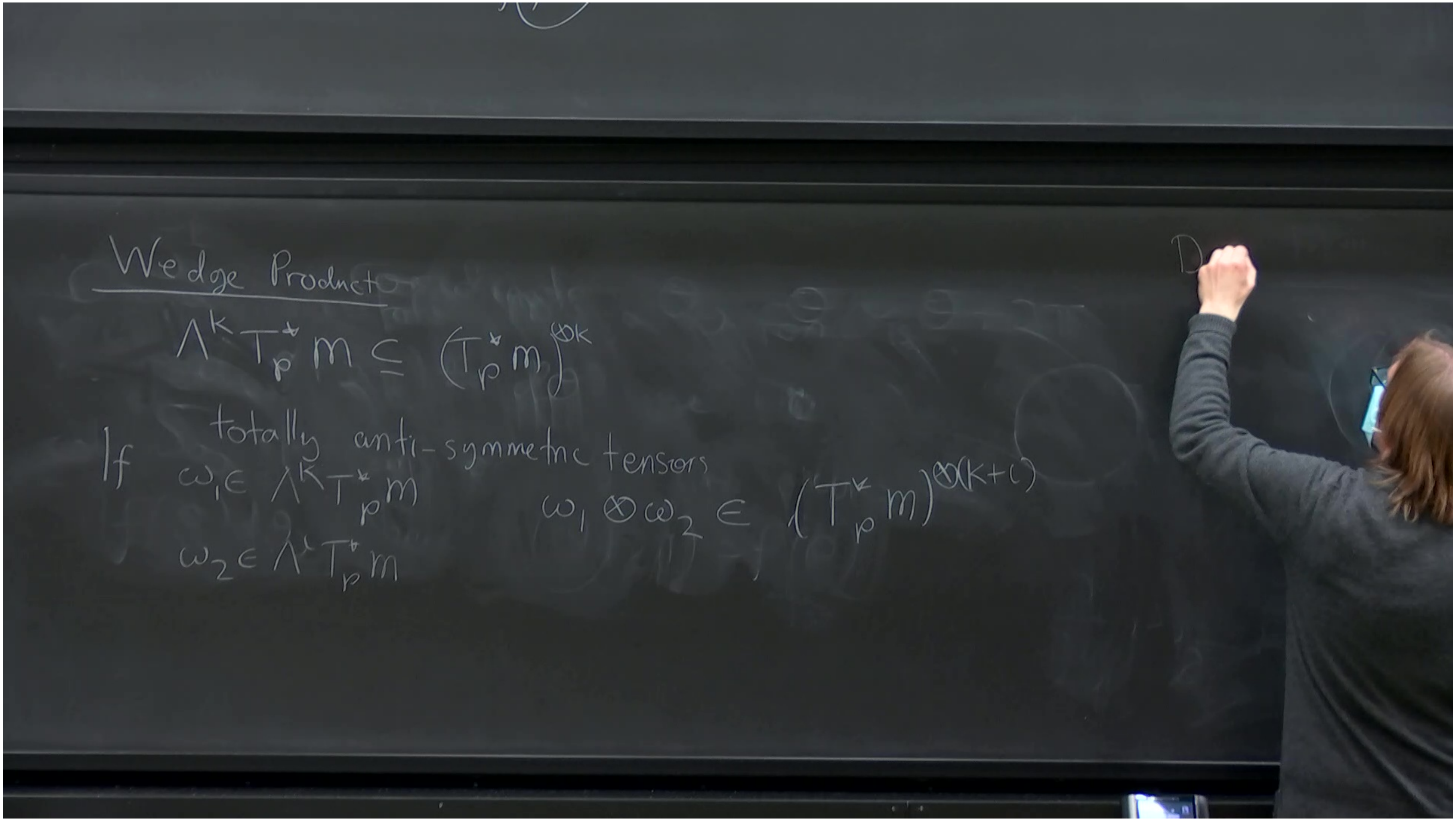


Not possible to find $f^i dx_i$ which, mod relation, is non-zero everywhere.

dx_1 near $(1,0,0)$ is

$$-\frac{1}{x_1}\left(x_2\,dx_2 + x_3\,dx_3\right)$$

$= 0$ at North Pole (and South pole)


Wedge Product
$\Lambda^k T_p^* M \subseteq (T_p^* M)^{\otimes k}$
totally anti-symmetric tensors
If $\omega_1 \in \Lambda^k T_p^* M$
$\omega_1 \otimes \omega_2 \in (T_p^* M)^{\otimes (k+l)}$
$\omega_2 \in \Lambda^l T_p^* M$

$$= \det\left(\frac{\partial}{\partial x_j}\right) dx_1 \wedge \cdots \wedge dx_n.$$

$m)^{\otimes(k+l)}$

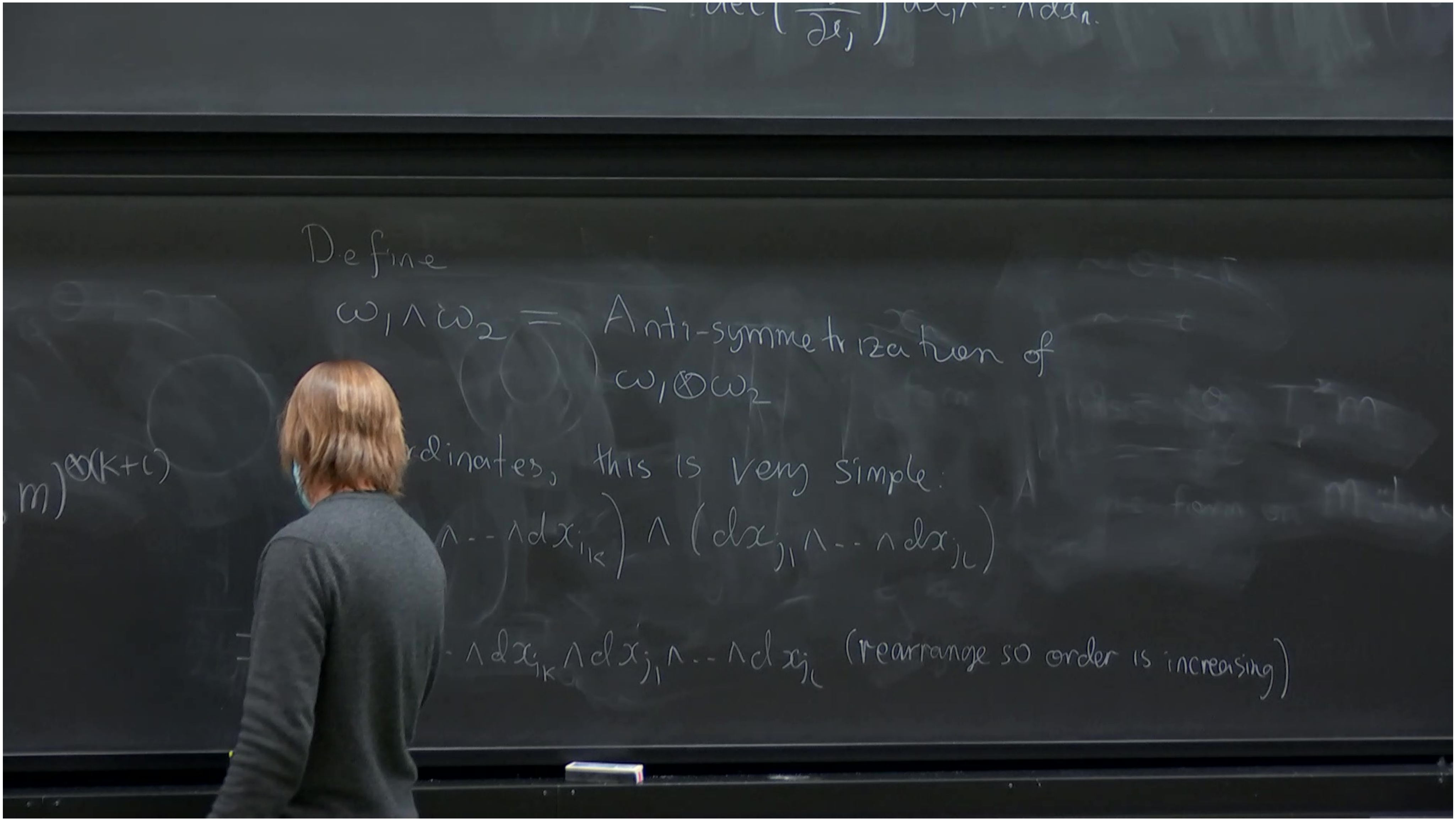
Define

$\omega_1 \wedge \omega_2 =$ Anti-symmetrization of $\omega_1 \otimes \omega_2$

In coordinates, this is very simple:

$$(dx_{i_1} \wedge \cdots \wedge dx_{i_k}) \wedge (dx_{j_1} \wedge \cdots \wedge dx_{j_l})$$

$$= dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}$$ (rearrange so orde


$dx_1 \wedge \dots \wedge dx_n$.
Define
$\omega_1 \wedge \omega_2$ = Anti-symmetrization of $\omega_1 \otimes \omega_2$
$m)^{\otimes(k+l)}$
this is very simple:
$\wedge \dots \wedge dx_{i_k}) \wedge (dx_{j_1} \wedge \dots \wedge dx_{j_l})$
$\wedge dx_{i_k} \wedge dx_{j_1} \wedge \dots \wedge dx_{j_l}$ (rearrange so order is increasing)

If $\omega_1 = f^{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$

$\omega_2 = g^{j_1 \cdots j_l} dx_{j_1} \cdots dx_{j_l}$

then, $\omega_1 \wedge \omega_2 = f^{i_1 \cdots i_k} g^{j_1 \cdots j_l} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}$

$\text{Dim}^n\ 2$

$$\omega_1 = f^i\, dx_i$$

$$\omega_2 = g^j\, dx_j$$

$$\omega_1 \wedge \omega_2 = (f^1 dx_1 + f^2 dx_2)(g^1 dx_1 + g^2 dx_2)$$

$$= (f^1 g^2 - f^2 g^1)\, dx_1 \wedge dx_2$$

$$= \varepsilon_{ij} f^i g^j\, dx_1 \wedge dx_2$$

of $T_p^* S^2$

(and Sou

Or, Dim^n 3

$\omega \in \Omega^2$

$\omega = f^{ij}\, dx_i \wedge dx_j$

$\Omega^2(\mathbb{R}^3) =$ Vectors

$\omega \mapsto f^{ij}\varepsilon_{ijk}\dfrac{\partial}{\partial x_k}$

If we take

$\omega_1, \omega_2 \in \Omega^1(\mathbb{R}^3) =$ Vector field

$\omega_1 \wedge \omega_2 \in \Omega^2(\mathbb{R}^3) =$ Vector field

$\Omega^1 \times \Omega^1 \longrightarrow \Omega^2$

is same as cross product

Vect $\times$ Vect $\longrightarrow$ Vect