Title: Geometry and topology for physicists 2021/2022 - Lecture 4

Speakers: Kevin Costello, Giuseppe Sellaroli

Collection: Geometry and Topology for Physicists 2021/2022

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Geometry and topology for physicists

Today's plan:

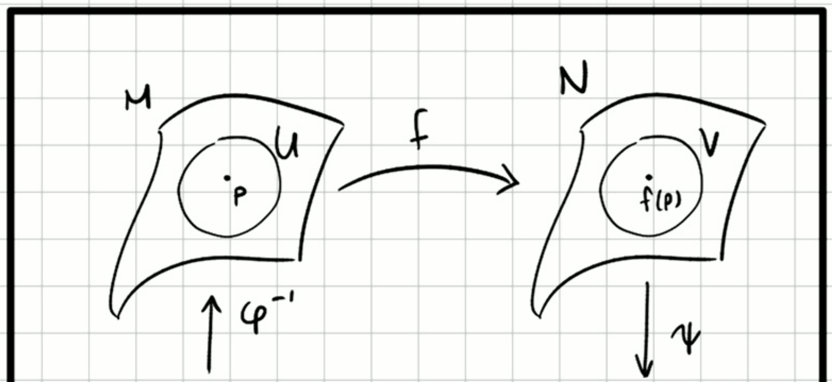
- continuous & smooth maps between manifolds
- tangent, cotangent, tensor bundle
- fields

$(M, \mathcal{A}_M)$ $(N, \mathcal{A}_N)$ manifolds

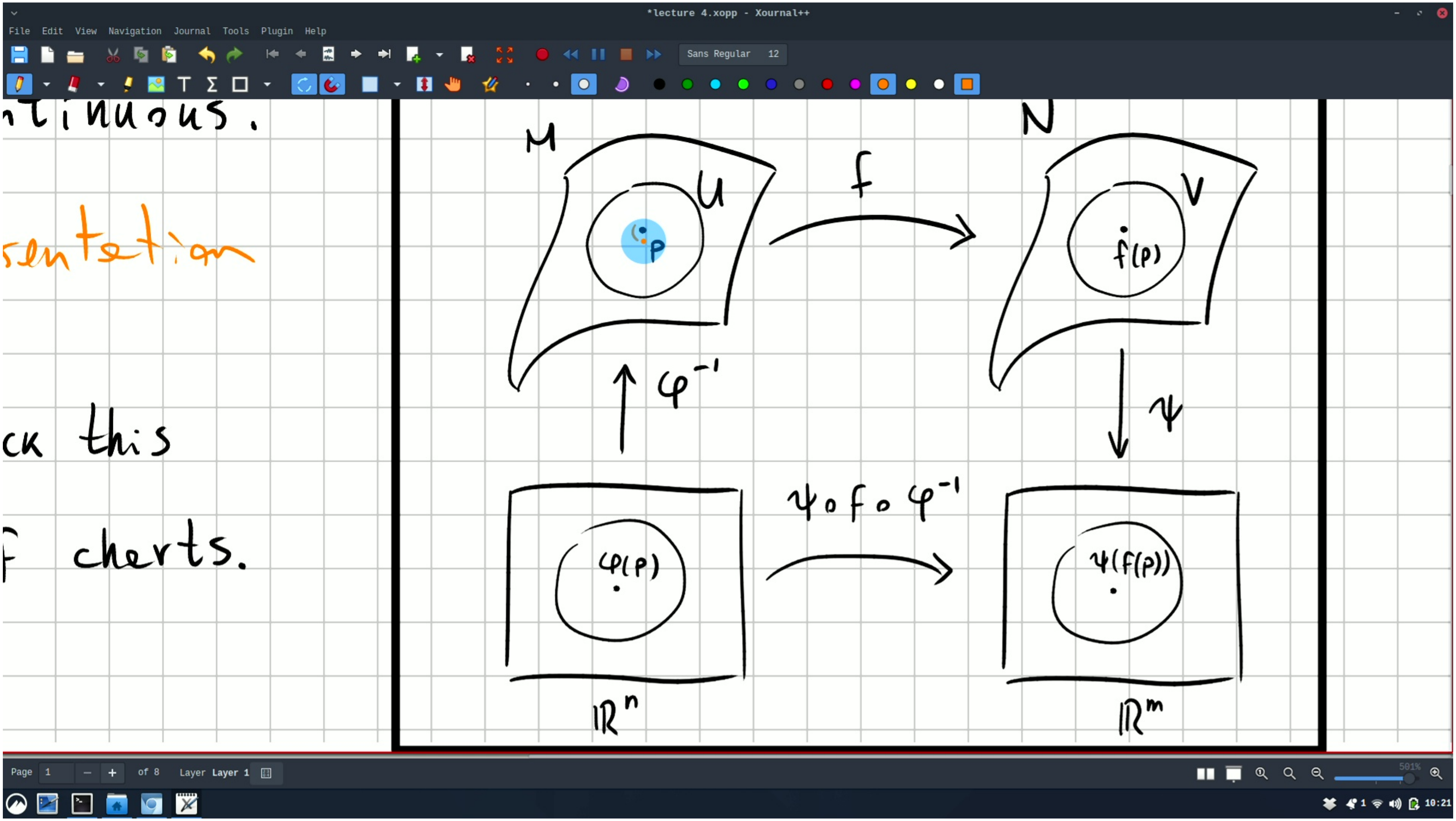
is $f: M \to N$ continuous/smooth?

- f continuous at $p \in M$ if for all charts $(U, \varphi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ with $p \in U$, $\varphi(p) \in V$, the function

$\psi \circ f \circ \varphi^{-1}$ is continuous.



Note: it's enough to check this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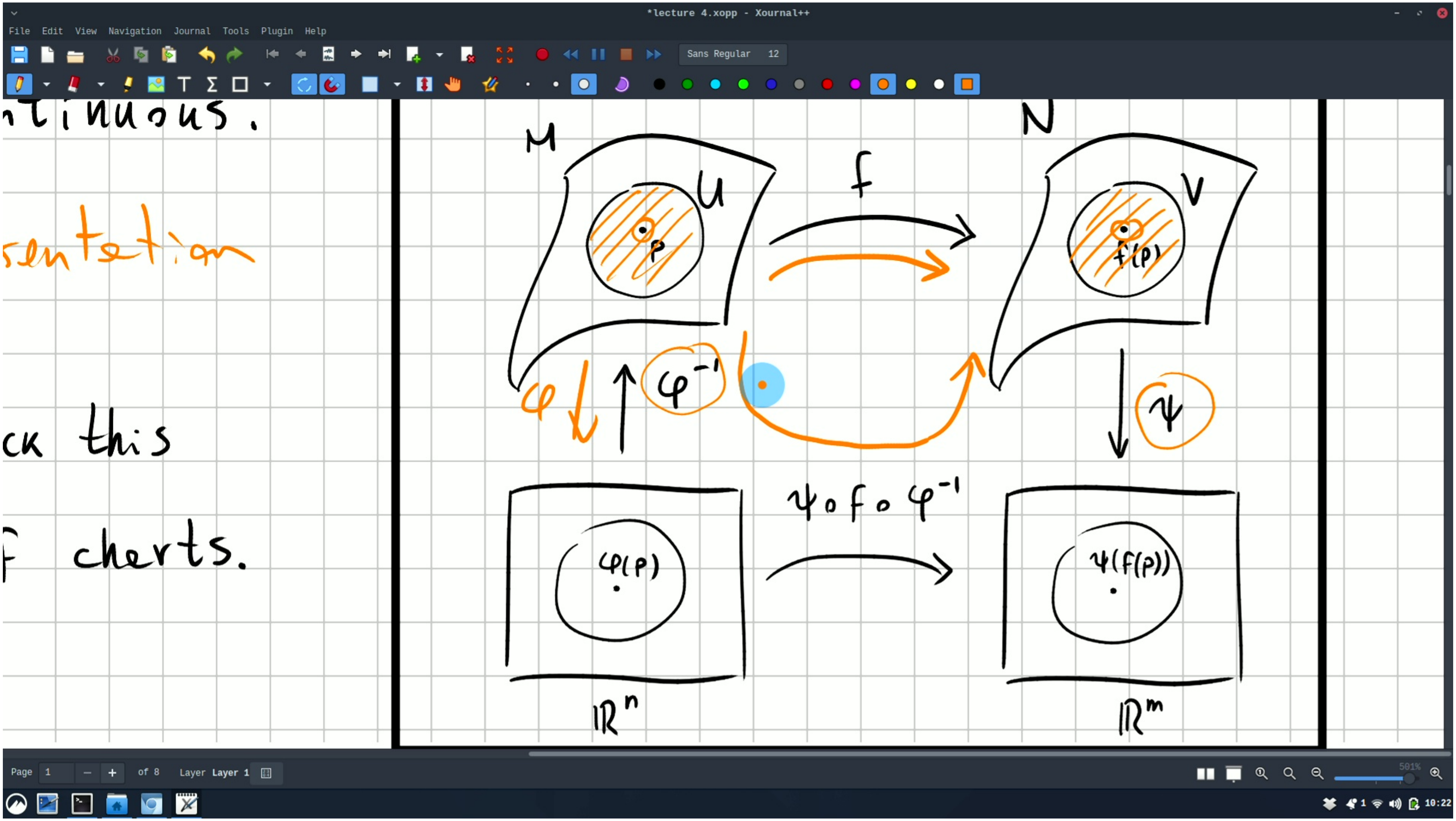

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$\mathbb{R}^n$ $\mathbb{R}^m$

$$\psi' \circ f \circ (\varphi')^{-1} = \psi' \circ \psi^{-1} \circ \psi \circ f \circ \varphi^{-1} \circ \varphi \circ (\varphi')^{-1}$$

$f : M \to N$

- f is continuous on M if it is continuous at each $p \in M$

- f smooth at $p \in M$ if for all charts $(U, \varphi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ with $p \in U$, $\varphi(p) \in V$, the function

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$f: M \to N$

- f is continuous on M if it is continuous at each $p \in M$

- f smooth at $p \in M$ if for all charts $(U, \varphi) \in \mathcal{A}_M$ and $(V, \psi) \in \mathcal{A}_N$ with $p \in U$, $f(p) \in V$, the function $\psi \circ f \circ \varphi^{-1}$ is of class $\mathcal{C}^{\infty}$.

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- f is smooth on M if it is smooth at each $p \in M$

Isomorphisms

- if $f: M \to N$ is invertible and both f and f^{-1} are continuous we say f is a <u>homeomorphism</u>. Two manifolds are homeomorphic if there is an homeomorphism between them.

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Isomorphisms

6: continuous

- if $f: M \to N$ is invertible and both f and f^{-1} are continuous we say f is a **homeomorphism**. Two manifolds are homeomorphic if there is an homeomorphism between them.

- if $f: M \to N$ is invertible and both f and f^{-1} are smooth

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- if $f: M \to N$ is invertible and both f and f^{-1} are continuous we say f is a homeomorphism. Two manifolds are homeomorphic if there is an homeomorphism between them.

- if $f: M \to N$ is invertible and both f and f^{-1} are smooth we say f is a diffeomorphism. Two manifolds are diffeomorphic if there is a diffeomorphism between them.

- if $f: M \to N$ is invertible and both f and f^{-1} are smooth we say f is a **diffeomorphism**. Two manifolds are diffeomorphic if there is a diffeomorphism between them.

Note: diffeomorphic $\Rightarrow$ homeomorphic, but homeomorphic $\not\Rightarrow$ diffeomorphic

Homeomorphisms preserve the topological structure
Diffeomorphisms preserve the differential structure

we say f is a diffeomorphism. Two manifolds are diffeomorphic if there is a diffeomorphism between them.

Note: diffeomorphic $\Rightarrow$ homeomorphic, but homeomorphic $\not\Rightarrow$ diffeomorphic

Homeomorphisms preserve the topological structure
Diffeomorphisms preserve the differential structure

Cannot distinguish isomorphic manifolds without any extra diffeomorphic if there is a diffeomorphism between them.

Note: diffeomorphic $\Rightarrow$ homeomorphic, but $\boxed{\text{homeomorphic} \not\Rightarrow \text{diffeomorphic}}$

Homeomorphisms preserve the topological structure

Diffeomorphisms preserve the differential structure

Cannot distinguish isomorphic manifolds without any extra structure

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Examples

- The cylinder $\boxed{C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}}$ is a manifold with the single chart $\varphi : (x,y,z) \in C \mapsto (e^z x, e^z y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

φ is a diffeomorphisim between C and $\mathbb{R}^2 \setminus \{(0,0)\}$ since

$id \circ \varphi \circ (\varphi)^{-1} = id$ and $\varphi \circ \varphi^{-1} \circ (id)^{-1} = id$

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φ is a diffeomorphisim between $\mathbb{C}$ and $\mathbb{R}^2 \setminus \{(0,0)\}$ since

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$\psi \circ f \circ \varphi^{-1}$ $\quad$ $\psi \circ f^{-1} \circ \varphi^{-1}$ $\quad$ $f^{-1} : N \to M$

- The open interval $(-1,1) \subset \mathbb{R}$ is diffeomorphic to $\mathbb{R}$, with a diffeomorphism given by $\operatorname{arctanh} : (-1,1) \to \mathbb{R}$

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The "intuitive" idea that $(-1,1)$ is geometrically smaller then $\mathbb{R}$ is incorrect!

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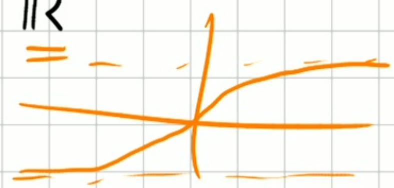

$[-1,1]$

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then $\mathbb{R}$ is incorrect!

Bundles

We want to talk about vector fields, which assign to each point $p \in M$ a vector $v \in T_pM$ in a smooth way.

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Bundles

We want to talk about vector fields, which assign to each point $p \in M$ a vector $v \in T_pM$ in a smooth way.

$\Rightarrow$ We need to put all the tangent vectors in one set, and make it into a manifold!

Tangent bundle

$$TM = \bigsqcup_{p\in M} T_pM := \bigcup_{p\in M} \{p\}\times T_pM = \{(p,v) \mid p\in M,\ v\in T_pM\}$$

$\bigsqcup_{p\in M} T_pM$: disjoint union

Atlas for TM: if $(U,\varphi)\in A_M$ we build $(\tilde U, \tilde\varphi)\in A_{TM}$

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disjoint union

$(p,v)\in TM \qquad v\in T_pM$

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Tangent bundle

$$T\mathbb{R}^n = \bigcup_{p \in \mathbb{R}^n} \mathbb{R}^n = \mathbb{R}^n$$

$$TM = \underbrace{\bigsqcup_{p \in M} T_pM}_{\text{disjoint union}} := \boxed{\bigcup_{p \in M} \{p\} \times T_pM} = \{(p,v) \mid p \in M, v \in T_pM\}$$

disjoint union

$(p,v) \in TM \quad v \in T_pM$

Atlas for TM: if $(U, \varphi) \in A_M$ we build $(\tilde{U}, \tilde{\varphi}) \in A_{TM}$

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→ We need to put all the tangent vectors in one set, and make it into a manifold!

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Tangent bundle

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$$(p,v)\in TM \qquad v\in T_pM$$

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$T\mathbb{R}^n = \bigcup_{p \in \mathbb{R}^n} \mathbb{R}^n = \mathbb{R}$

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$(p,v)\in TM \quad v\in T_pM$

Atlas for TM: if $(U,\varphi)\in \mathcal{A}_M$ we build $(\tilde{U},\tilde{\varphi})\in\mathcal{A}_{TM}$

$\tilde{U} = \bigsqcup T_pM = \{(p,v) \mid p\in U, v\in T_pM\}$

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$$TM = \bigsqcup_{p\in M} T_pM := \bigcup_{p\in M} \{p\}\times T_pM = \{(p,v) \mid p\in M,\ v\in T_pM\}$$

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$$\tilde{\varphi} : (p,v)\in \tilde{U} \mapsto (\varphi(p), d\varphi_p(v)) \in \mathbb{R}^n\times\mathbb{R}^n$$

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disjoint union

$(p,s) \in TM \quad s \in T_pM$

Atlas for TM: if $(U,\varphi) \in A_M$ we build $(\tilde{U},\tilde{\varphi}) \in A_{TM}$

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$\rightarrow$ isomorphism $T_pM \rightarrow \mathbb{R}^n$

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Atlas for TM: if $(U, \varphi) \in \mathcal{A}_M$ we build $(\tilde{U}, \tilde{\varphi}) \in \mathcal{A}_{TM}$

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$$\tilde{\varphi} : (p, v) \in \tilde{U} \mapsto \left(\varphi(p), d\varphi_p(v) \right) \in \mathbb{R}^n \times \mathbb{R}^n$$

$\hookrightarrow$ isomorphism $T_pM \to \mathbb{R}^n$

or $\tilde{\varphi} : \left(p, v^i \frac{\partial}{\partial x^i}\Big|_p \right) \in \tilde{U} \mapsto \left(\varphi(p), v^1, \dots, v^n \right) \in \mathbb{R}^n \times \mathbb{R}^n$

$\hookrightarrow \varphi = (x^1, x^2, \dots x$

?

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$$\varphi : (p, v) \in U \mapsto (\varphi(p), d\varphi_p(v)) \in \mathbb{R} \times \mathbb{R}$$

→ isomorphism $T_pM \to \mathbb{R}^n$

or $$\tilde{\varphi} : \left(p, v^i \frac{\partial}{\partial x^i}\Big|_p\right) \in \tilde{U} \mapsto (\varphi(p), v^1, \dots, v^n) \in \mathbb{R}^n \times \mathbb{R}^n$$

→ $\varphi = (x^1, x^2, \dots x^n)$

is this an atlas?

- covers TM

- $d\varphi_p$ is an invertible linear map with codomain $\mathbb{R}^n \underset{\text{open}}{\subseteq} \mathbb{R}^n$

$\rightarrow \varphi = (x^1, x^2, \dots x^n)$

is this an atlas?

- covers TM ✓
- $d\varphi_p$ is an invertible linear map with codomain $\mathbb{R}^n \underset{\text{open}}{\subseteq} \mathbb{R}^n$
- if φ, ψ are overlapping charts containing p then

$\tilde{\varphi} \circ \tilde{\psi}^{-1}$ smooth since $d\varphi_p \circ (d\psi_p)^{-1}$ is a linear map $\mathbb{R}^n \to \mathbb{R}^n$

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$\boxed{\text{Def}}$ The natural projection of TM on M is

$$\pi : (p, v) \in TM \mapsto p \in M$$

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Cotangent bundle

$$T^*M = \bigsqcup_{p \in M} T_p^*M$$

$$\tilde{U} = \bigsqcup_{p \in U} T_p^*M$$

$$\tilde{\varphi} : (p, \alpha_i \, dx^i_p) \in \tilde{U} \mapsto (\varphi(p), \alpha_1, \dots, \alpha_n) \in \mathbb{R}^n \times \mathbb{R}^n$$

$$\pi : (p, \alpha) \in T^*M \mapsto p \in M$$

Tensor bundles

$$\mathcal{T}^{r,s}M = \bigsqcup_{p\in M} T_p^{r,s}M$$

$$\tilde{U} = \bigsqcup_{p\in U} T_p^{r,s}M$$

$$\tilde{\varphi} : \left(p,\, T^{i_1\cdots i_r}{}_{j_1\cdots j_s}\, \frac{\partial}{\partial x^{i_1}}\Big|_p \otimes\cdots\otimes dx^{j_1}_p \otimes\cdots\otimes dx^{j_s}_p\right)\in\tilde{U} \mapsto \left(\varphi(p),\, \left(T^{i_1\cdots i_r}{}_{j_1\cdots j_s}\right)\right)\in\mathbb{R}^n\times\mathbb{R}^{nrs}$$

$$\pi : (p,T)\in\mathcal{T}^{r,s}M \to p\in M$$

Tensor bundles

$$\mathcal{T}^{r,s}M = \bigsqcup_{p\in M} T_p^{r,s}M$$ tensor bundle of type (r,s)

$$\tilde{U} = \bigsqcup_{p\in U} T_p^{r,s}M$$

$$\tilde{\varphi} : \left(p, T^{i_1\cdots i_r}{}_{j_1\cdots j_s} \frac{\partial}{\partial x^{i_1}}\Big|_p \otimes \cdots \otimes dx^{j_1}_p \otimes \cdots \otimes dx^{j_s}_p\right) \in \tilde{U} \mapsto \left(\varphi(p), \left(T^{i_1\cdots i_r}{}_{j_1\cdots j_s}\right)\right) \in \mathbb{R}^n \times \mathbb{R}^{n^{r+s}}$$

$$\pi : (p,T) \in \mathcal{T}^{r,s}M \mapsto p \in M$$

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$$\mathcal{T}^{r,s}M = \bigsqcup_{p\in M} T_p^{r,s}M \qquad \text{tensor bundle of type } (r,s)$$

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$$\tilde{\varphi} : \left(p, \boxed{T^{i_1\cdots i_r}{}_{j_1\cdots j_s} \frac{\partial}{\partial x^{i_1}}\Big|_p \otimes \cdots \otimes dx^{j_1}_p \otimes \cdots \otimes dx^{j_s}_p}\right) \in \tilde{U} \mapsto \left(\varphi(p), \left(T^{i_1\cdots i_r}{}_{j_1\cdots j_s}\right)\right) \in \mathbb{R}^n \times \mathbb{R}^{nrs}$$

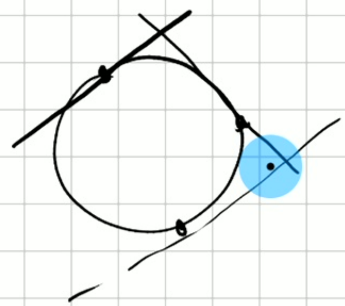
$$\pi : (p,T) \in \mathcal{T}^{r,s}M \mapsto p \in M \qquad TM \neq M \times \mathbb{R}^n$$

$$\tilde{U} = \bigsqcup_{p\in U} T_p M$$

$$\tilde{\varphi} : \left(p, T^{i_1\cdots i_r}{}_{j_1\cdots j_s} \frac{\partial}{\partial x^{i_1}}\Big|_p \otimes\cdots\otimes dx^{j_1}_p \otimes\cdots\otimes dx^{j_s}_p\right) \in \tilde{U} \mapsto \left(\varphi(p), \left(T^{i_1\cdots i_r}{}_{j_1\cdots j_s}\right)\right) \in \mathbb{R}^n \times \mathbb{R}^{nrs}$$

$$\pi : (p,T) \in T^{r,s}M \to p \in M$$

$$TM \neq M \times \mathbb{R}^n$$



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$$\pi : (p,T) \in \mathcal{T}^{r,s}M \mapsto p \in M$$

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Fields

- A vector field is a smooth map $X : U \underset{\text{open}}{\subseteq} M \to TM$ such that $\pi \circ X = id$

- A covector field is a smooth map $\alpha : U \underset{\text{open}}{\subseteq} M \to T^*M$ such that $\pi \circ \alpha = id$

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Fields

- A vector field is a smooth map $X : U \subseteq M \to TM$ (U open) such that $\pi \circ X = id$ $\rightarrow$ $X(p) = (p, v)$

$\pi(X(p)) = p$

- A covector field is a smooth map $\alpha : U \subseteq M \to T^*M$ (U open) such that $\pi \circ \alpha = id$

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such that $\pi \circ X = id$ → $X(p) = (p, v)$ || sections of TM

$\pi(X(p)) = p$

- A covector field is a smooth map $\alpha : U \subseteq M \to T^*M$ (open) such that $\pi \circ \alpha = id$

- A tensor field of type (r,s) is a smooth map $T : U \subseteq M \to T^{r,s}M$ (open) such that $\pi \circ T = id$

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- A covector field is a smooth map $\alpha : U \subseteq M \to T^*M$ (U open)

such that $\pi \circ \alpha = id$

manifolds

- A tensor field of type (r,s) is a smooth map $T : U \subseteq M \to T^{r,s}M$ (U open)

such that $\pi \circ T = id$

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such that $\pi \circ \Gamma = id$

Some notes:

- The set of vector / covector / tensor fields forms a vector space, with operations defined by

$$(aX + bY)(p) = (p, aX_p + bY_p) \quad \text{where} \quad \begin{cases} X(p) = (p, X_p) \\ Y(p) = (p, Y_p) \end{cases}$$

Some notes:

- The set of sector / covector / tensor fields forms a vector space, with operations defined by

$$(aX + bY)(p) = (p, aX_p + bY_p) \quad \text{where} \quad \begin{cases} X(p) = (p, X_p) \\ Y(p) = (p, Y_p) \end{cases}$$

- Fields can be given a coordinate basis. if $(U, \varphi) \in \mathcal{A}_M$ with $\varphi = (x^1, \dots, x^n)$ then

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$$\boxed{(a X + b Y)}(p) = (p, a X_p + b Y_p) \quad \text{where} \quad \begin{cases} X(p) = (p, X_p) \\ Y(p) = (p, Y_p) \end{cases}$$

- Fields can be given a coordinate basis. if $(U, \varphi) \in \mathcal{A}_M$ with $\varphi = (x^1, \dots, x^n)$ then

$$\boxed{\frac{\partial}{\partial x^i} : p \in U \mapsto \left(p, \left.\frac{\partial}{\partial x^i}\right|_p\right) \in TM}$$

$$dx^i : p \in U \mapsto (p, dx^i_p) \in T^*M$$

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$$\frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s} : p \in U \mapsto \left(p, \frac{\partial}{\partial x^{i_1}}\Big|_p \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}}\Big|_p \otimes dx^{j_1}{}_p \otimes \cdots \otimes dx^{j_s}{}_p\right) \in T^{r,s}M$$

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Some notes:

- The set of vector / covector / tensor fields forms a vector space, with operations defined by

$$\boxed{(a X + b Y)}(p) = (p, \underbrace{a X_p + b Y_p}) \qquad \text{where} \quad \begin{cases} X(p) = (p, X_p) \\ Y(p) = (p, Y_p) \end{cases}$$

- Fields can be given a coordinate basis. if $(U, \varphi) \in A_M$

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Examples

Pseudo-Riemannian metric tensor:

$$g : p \in M \mapsto (p, g_p) \in T^{0,2} M$$

with each $g_p \in T_p^{0,2} M$ symmetric and non-degenerate

In coordinates: $g(p) \; g_{\mu\nu}(p) \, dx^\mu \otimes dx^\nu$

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with each $g_p \in T_p^{0,2} M$ symmetric and non-degenerate

In coordinates: $g(p)\ \ g_{\mu\nu}(p)\, dx^\mu \otimes dx^\nu$

with $g_{\mu\nu}(p) = g_p \left(\frac{\partial}{\partial x^i}\Big|_p , \frac{\partial}{\partial x^j}\Big|_p \right)$

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↘ only works on U

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