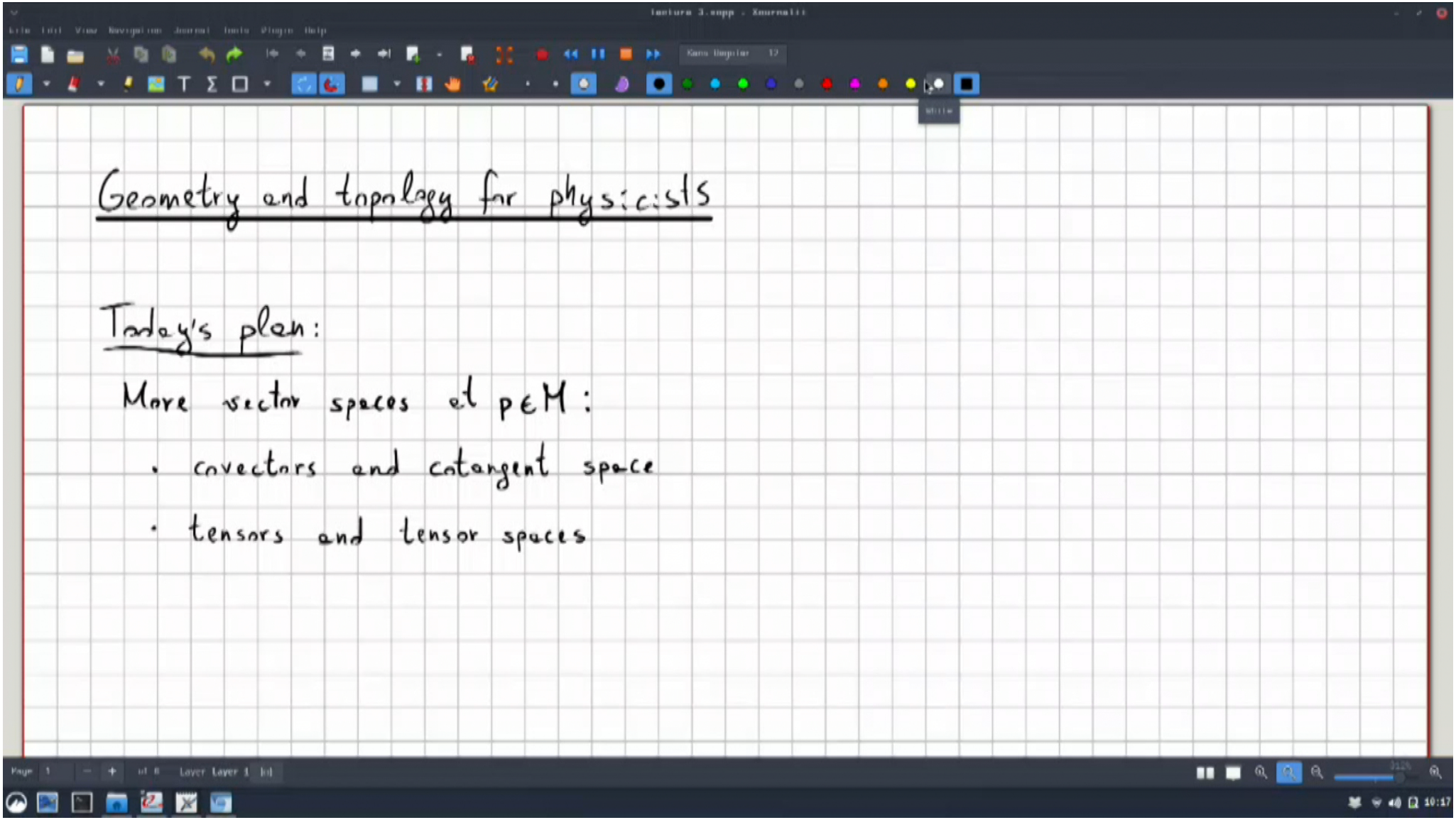
Title: Geometry and topology for physicists 2021/2022 - Lecture 3

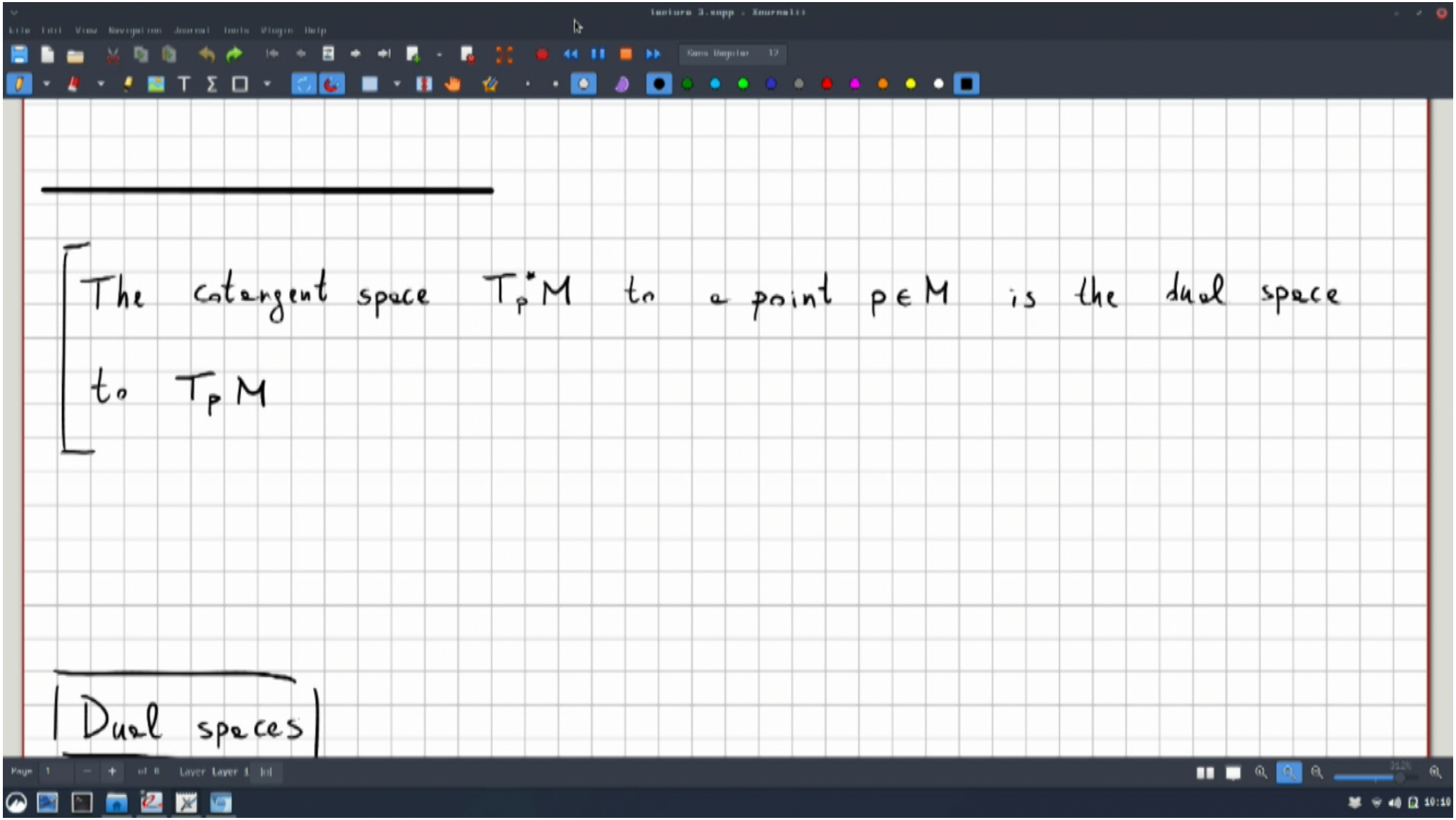
Speakers: Kevin Costello, Giuseppe Sellaroli

Collection: Geometry and Topology for Physicists 2021/2022

Date: March 04, 2022 - 10:15 AM

URL: https://pirsa.org/22030061


Geometry and topology for physicists
Today's plan:
More vector spaces at p∈M:
• covectors and cotangent space
• tensors and tensor spaces


The cotangent space T_p^*M to a point $p \in M$ is the dual space to T_pM
Dual spaces

Dual spaces

V finite-dimensional real vector space

Dual space: $V^* = \{ \alpha : V \to \mathbb{R} \mid \alpha \text{ linear} \}$

vector space structure?

Dual spaces

V finite-dimensional real vector space

Dual space: $V^* = \{ \alpha : V \to \mathbb{R} \mid \alpha \text{ linear} \}$

vector space structure? operations point-wise

$\alpha \in V^*$ $a \in \mathbb{R}$ $\quad (a\alpha)(v) = a \cdot \alpha(v)$

$\alpha, \beta \in V^*$ $\quad (\alpha + \beta)(v) = \alpha(v) + \beta(v)$

The elements of V^* are called covectors

Basis of V^* Let $\{e_i\}_{i=1}^n$ be a basis of V

Prop The linear maps $\varepsilon^i : V \to \mathbb{R}$ satisfying $\varepsilon^i(e_j) = \delta^i_j$ form a basis for V^*

- Linear maps are fully specified by their action on a basis $\rightarrow$ $\alpha(v^i e_i) = v^i \alpha(e_i)$

Prop The linear maps $\varepsilon^i : V \to \mathbb{R}$ satisfying $\varepsilon^i(e_j) = \delta^i_j$ form a basis for V^*

- Linear maps are fully specified by their action on a basis $\rightarrow$ $\alpha(v^i e_i) = v^i \alpha(e_i)$
- $\varepsilon^i(v^j e_j) = v^j \varepsilon^i(e_j) = v^j \delta^i_j = v^i$

Prop The linear maps $\varepsilon^i : V \to \mathbb{R}$ satisfying $\varepsilon^i(e_j) = \delta^i_j$ form a basis for V^*

- Linear maps are fully specified by their action on a basis $\rightarrow$ $\alpha(v^i e_i) = v^i \alpha(e_i)$

- $\varepsilon^i(v^j e_j) = v^j \varepsilon^i(e_j) = v^j \delta^i_j = v^i$

$\Rightarrow \alpha = \alpha(e_i)\varepsilon^i$

basis → $\alpha(v^i e_i) = v^i \alpha(e_i)$

- $\varepsilon^i(v^j e_j) = v^j \varepsilon^i(e_j) = v^j \delta^i_j = v^i$

$\Rightarrow \alpha = \alpha(e_i)\varepsilon^i$

- Only left to show that the ε^i are linearly independent

$\dim(V^*) = \dim(V) \Rightarrow V \cong V^*$

$\dim(V^*) = \dim(V) \Rightarrow V \cong V^*$

- The basis $\{\varepsilon^i\} \subset V^*$ is called the dual basis to $\{e_i\} \subset V$

Basis of cotangent space

$T_p^* M = (T_p M)^*$

$\varphi = (x^1, x^2, \ldots) : U \subseteq M \to \mathbb{R}^n$ chart

$\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}_{i=1}^n$ basis of $T_p M$

→ equivalent of ε^i

dual basis: $dx^i_p : T_p M \to \mathbb{R}$ with $dx^i_p \left(\frac{\partial}{\partial x^j}\Big|_p \right) = \delta^i_j$

$\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}_{i=1}^n$ basis of T_pM

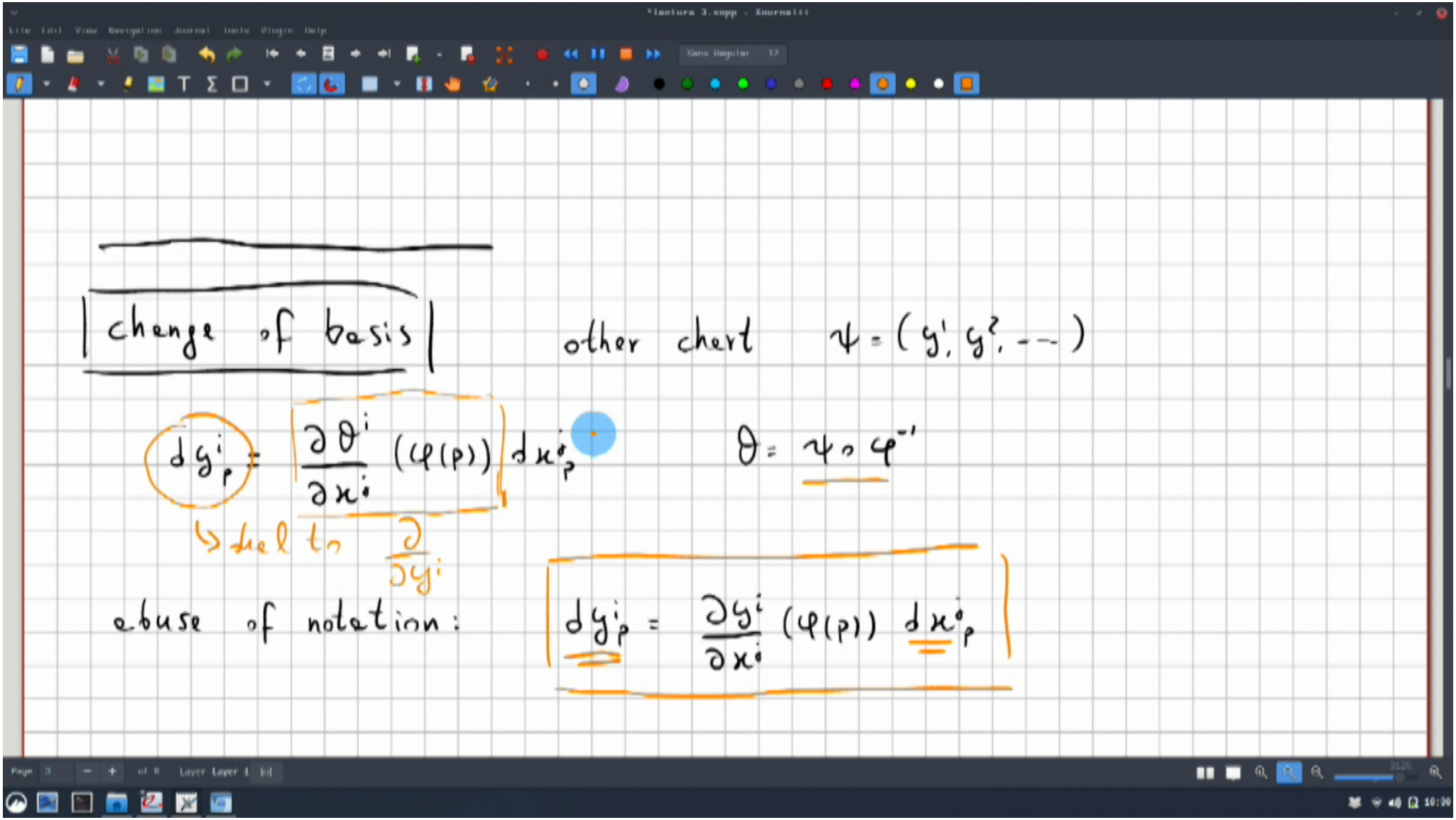
$\Rightarrow$ equivalent of ε^i

dual basis: $dx^i_p : T_pM \to \mathbb{R}$ with $dx^i_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \delta^i_j$

$d\varphi_p = (dx^1_p, dx^2_p, \dots) : T_pM \to \mathbb{R}^n$

change of basis

other chart $\psi = (y^1, y^2, \dots)$


change of basis
other chart $\psi = (y^1, y^2, \dots)$
$dy^i_p = \frac{\partial \theta^i}{\partial x^j}(\varphi(p))\, dx^j_p$
$\theta = \psi \circ \varphi^{-1}$
↳ def to $\frac{\partial}{\partial y^i}$
abuse of notation:
$dy^i_p = \frac{\partial y^i}{\partial x^j}(\varphi(p))\, dx^j_p$

Tensors

First, let's look at the **double dual space** of a vector space V

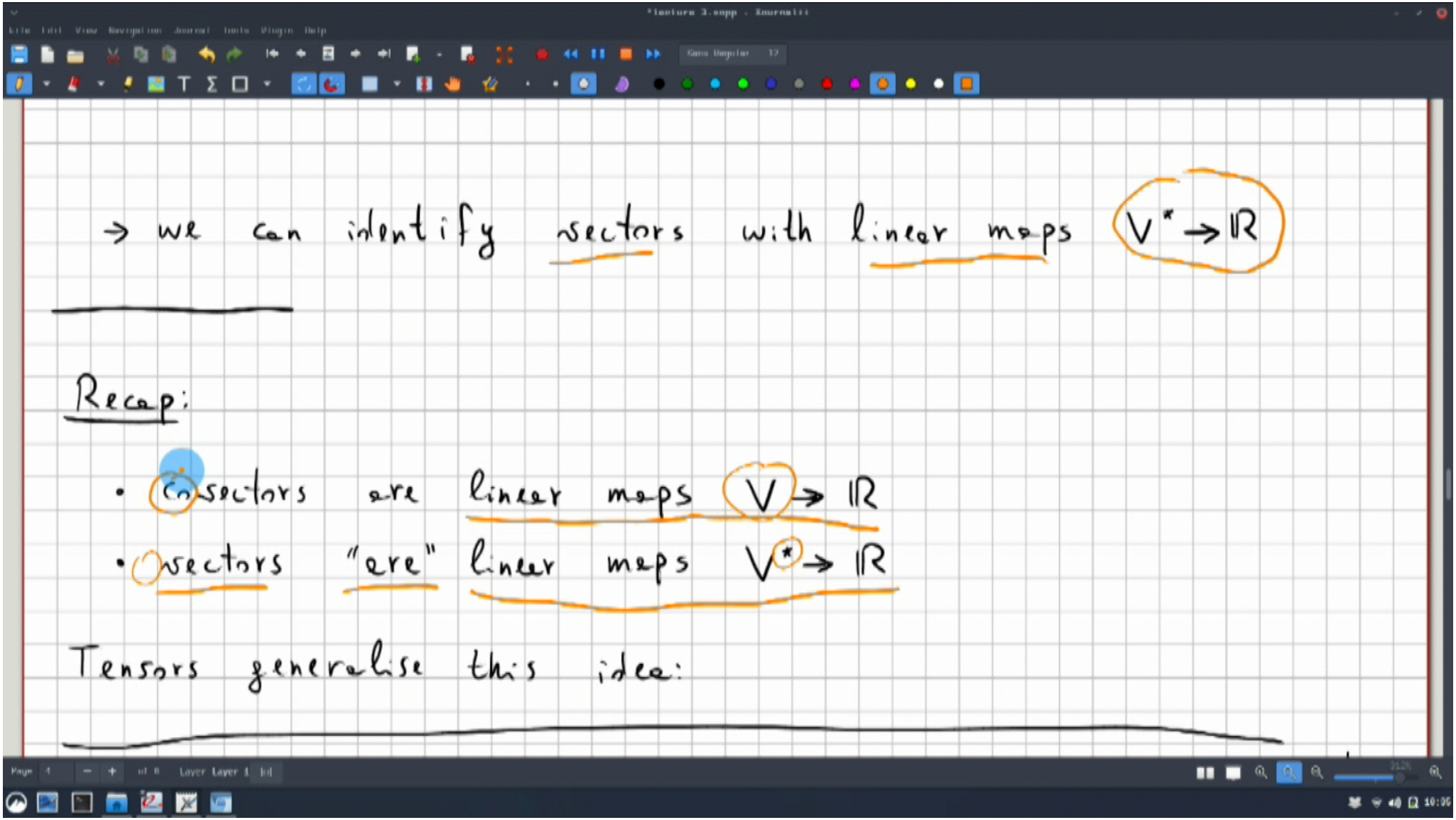
$$V \cong V^* \cong (V^*)^* = \{\alpha : V^* \to \mathbb{R} \text{ linear}\}$$

canonical isomorphism: $\Phi : V \to (V^*)^*$

$$v \mapsto \Phi_v$$

$$\Phi_v(\alpha) \equiv \alpha(v)$$

→ v act on α by "feeding itself" to α


→ we can identify vectors with linear maps $V^* \to \mathbb{R}$
Recap:
• covectors are linear maps $V \to \mathbb{R}$
• vectors "are" linear maps $V^* \to \mathbb{R}$
Tensors generalise this idea:

$V \cong V^* \cong (V^*)^* = \{ \alpha : V^* \to \mathbb{R} \text{ linear} \}$

canonical isomorphism: $\Phi : V \to (V^*)^*$

$v \mapsto \Phi_v$

$\Phi_v(\alpha) \equiv \alpha(v)$

→ v act on α by "feeding itself" to α

→ we can identify vectors with linear maps $V^* \to \mathbb{R}$

- vectors "are" linear maps $V^* \to \mathbb{R}$

Tensors generalise this idea:

Def A tensor of type (r, s) on V is a multi-linear map

$$T : \underbrace{V^* \times \cdots \times V^*}_{r \text{ copies}} \times \underbrace{V \times \cdots \times V}_{s \text{ copies}} \to \mathbb{R}$$

Tensors generalise this idea:

Def A tensor of type (r,s) on V is a multi-linear map

$$T : \underbrace{V^* \times \cdots \times V^*}_{r \text{ copies}} \times \underbrace{V \times \cdots \times V}_{s \text{ copies}} \to \mathbb{R}$$

$$T(as + bw, z) = aT(s,z) + bT(w,z)$$

$$T(z, as + bw) = aT(z,s) + bT(z,w)$$

- Covectors are linear maps $V \to \mathbb{R}$
- Vectors "are" linear maps $V^* \to \mathbb{R}$

Tensors generalise this idea:

Def A tensor of type (r, s) on V is a multi-linear map

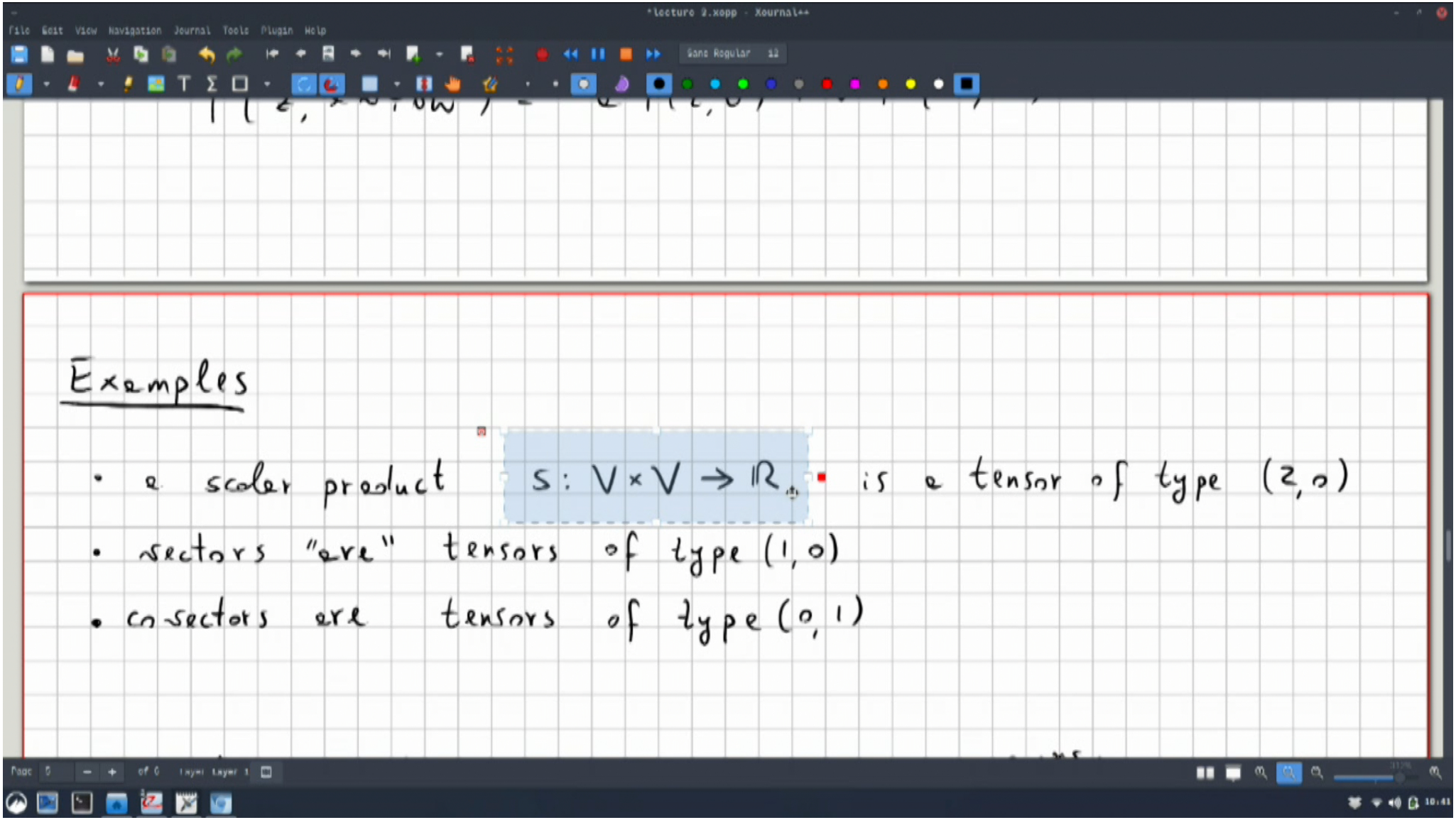
$$T : \underbrace{V^* \times \cdots \times V^*}_{r \text{ copies}} \times \underbrace{V \times \cdots \times V}_{s \text{ copies}} \to \mathbb{R}$$

$$T(as+bw, z) = aT(s,z) + bT(w,z)$$

$$T(z, as+bw) = aT(z,s) + bT(z,w)$$

Examples

- a scalar product $s: V \times V \to \mathbb{R}$ is a tensor of type $(2,0)$
- vectors "are" tensors of type $(1,0)$

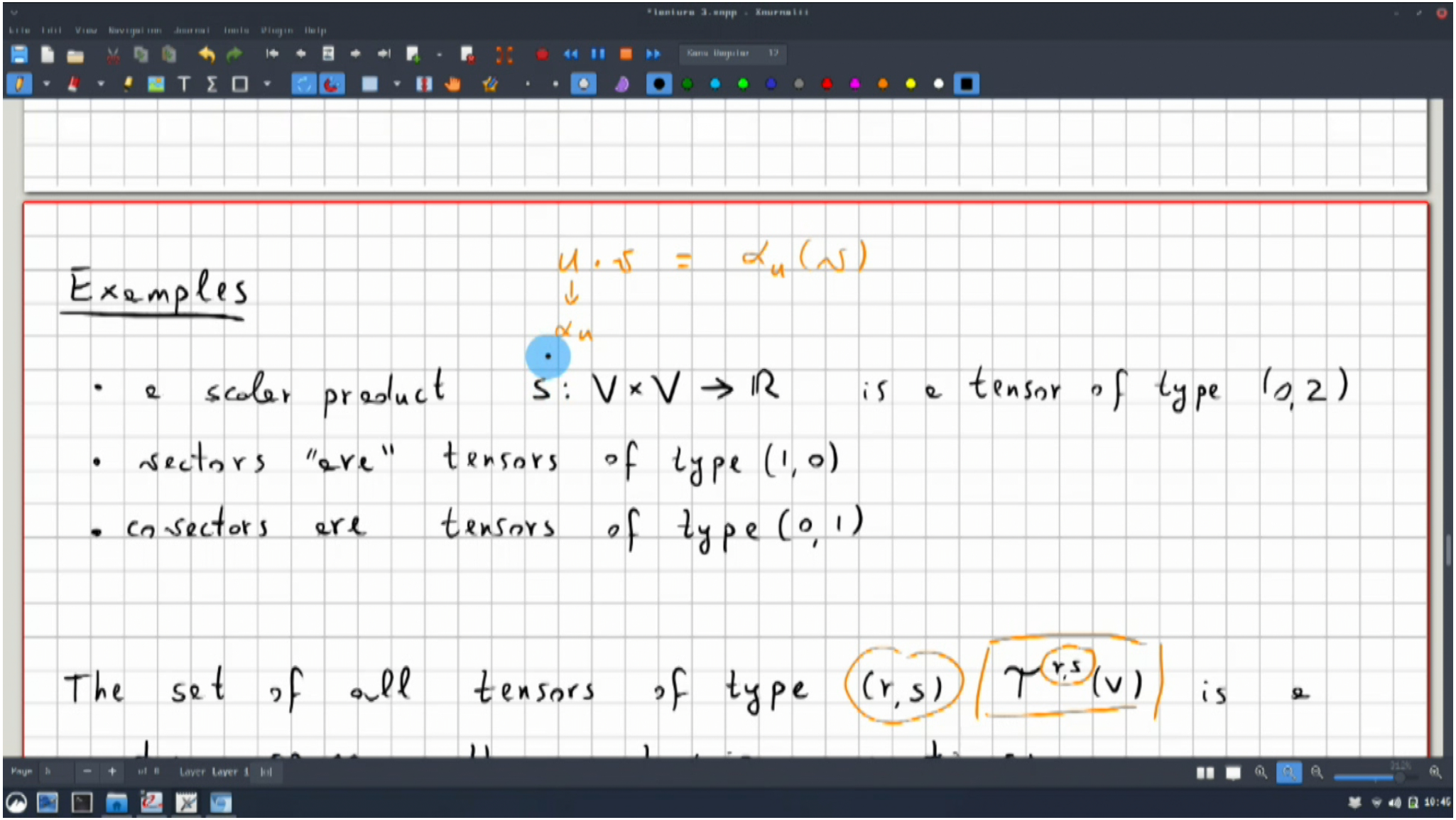

*lecture 2.xopp - Xournal++
File Edit View Navigation Journal Tools Plugin Help
Sans Regular 12
Examples
• a scaler product $s: V \times V \to \mathbb{R}$ is a tensor of type $(2,0)$
• vectors "are" tensors of type $(1,0)$
• covectors are tensors of type $(0,1)$

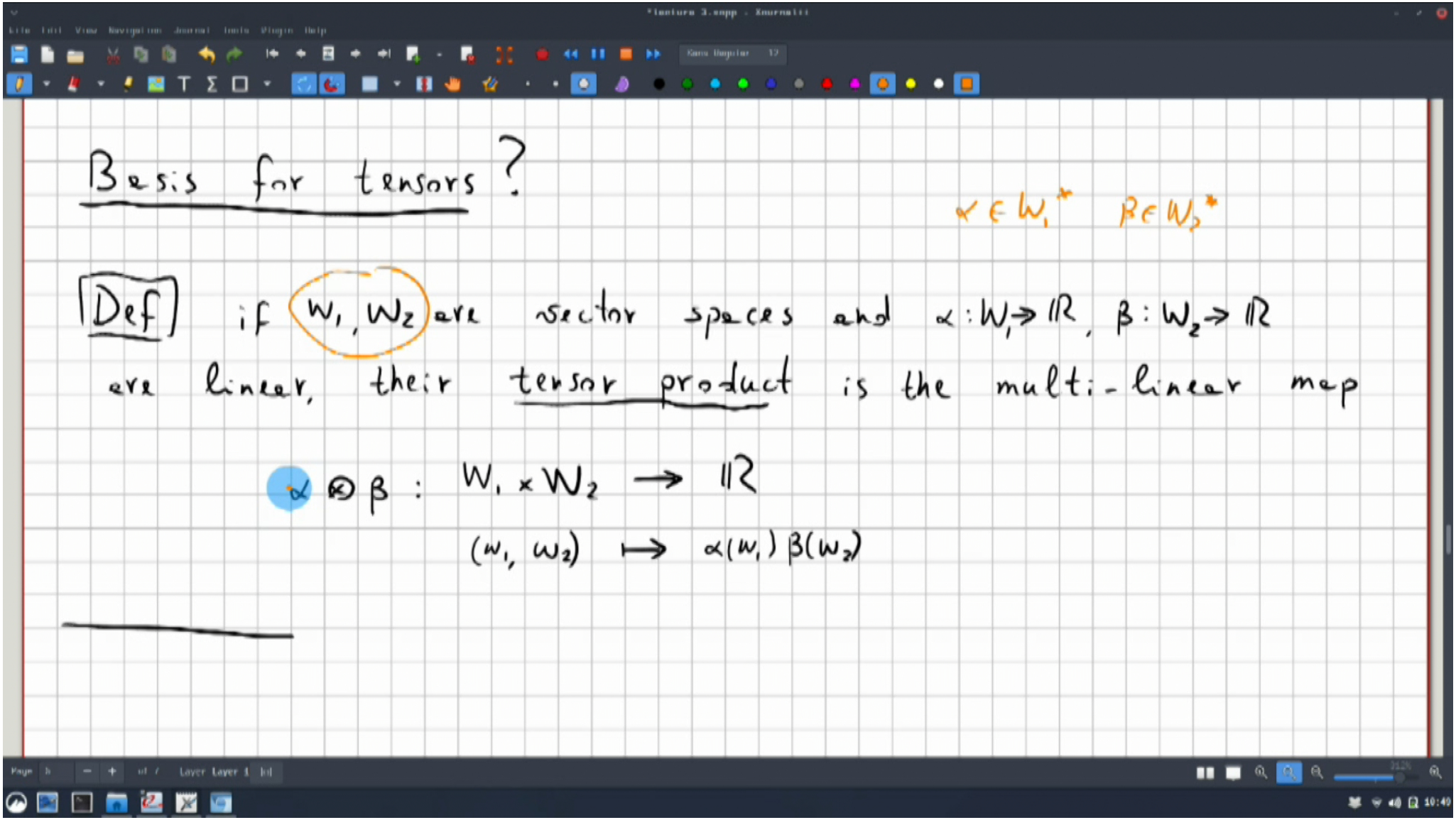
Examples

- a scaler product $s: V \times V \to \mathbb{R}$ is a tensor of type $(2,0)$
- vectors "are" tensors of type $(1,0)$
- covectors are tensors of type $(0,1)$

The set of all tensors of type (r,s) $T^{r,s}(V)$ is a vector space with point-wise operations:

$$(aT)(\dots) = a[T(\dots)]$$


Examples
$u \cdot v = \alpha_u(v)$
α_u
• a scalar product $s: V \times V \to \mathbb{R}$ is a tensor of type $(0,2)$
• vectors "are" tensors of type $(1,0)$
• covectors are tensors of type $(0,1)$
The set of all tensors of type (r,s) $\mathcal{T}^{(r,s)}(V)$ is a


Basis for tensors ?
$\alpha \in W_1^*$ $\beta \in W_2^*$
Def if W_1, W_2 are vector spaces and $\alpha : W_1 \to \mathbb{R}$, $\beta : W_2 \to \mathbb{R}$
are linear, their tensor product is the multi-linear map
$\alpha \otimes \beta : W_1 \times W_2 \to \mathbb{R}$
$(w_1, w_2) \mapsto \alpha(w_1)\beta(w_2)$

Basis for tensors ?

$\alpha \in W_1^*$ $\beta \in W_2^*$

Def if W_1, W_2 are vector spaces and $\alpha : W_1 \to \mathbb{R}$, $\beta : W_2 \to \mathbb{R}$ are linear, their tensor product is the multi-linear map

$$\alpha \otimes \beta : W_1 \times W_2 \to \mathbb{R}$$

$$(w_1, w_2) \mapsto \alpha(w_1)\beta(w_2)$$

$\alpha \otimes \beta$: tensor product

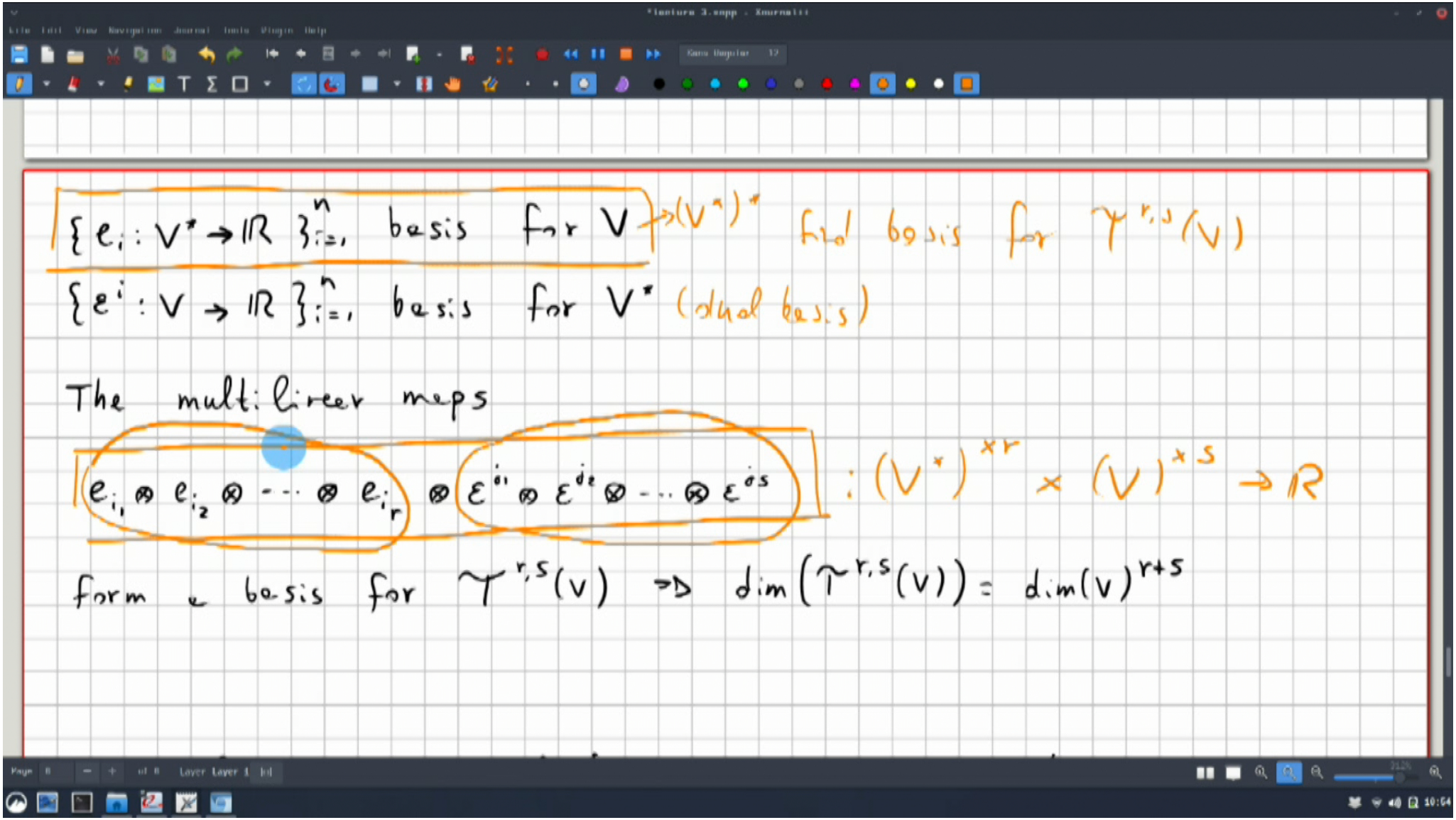
$\{e_i : V^* \to \mathbb{R}\}_{i=1}^n$ basis for $V \to (V^*)^*$ dual basis for $\mathcal{T}^{r,s}(V)$

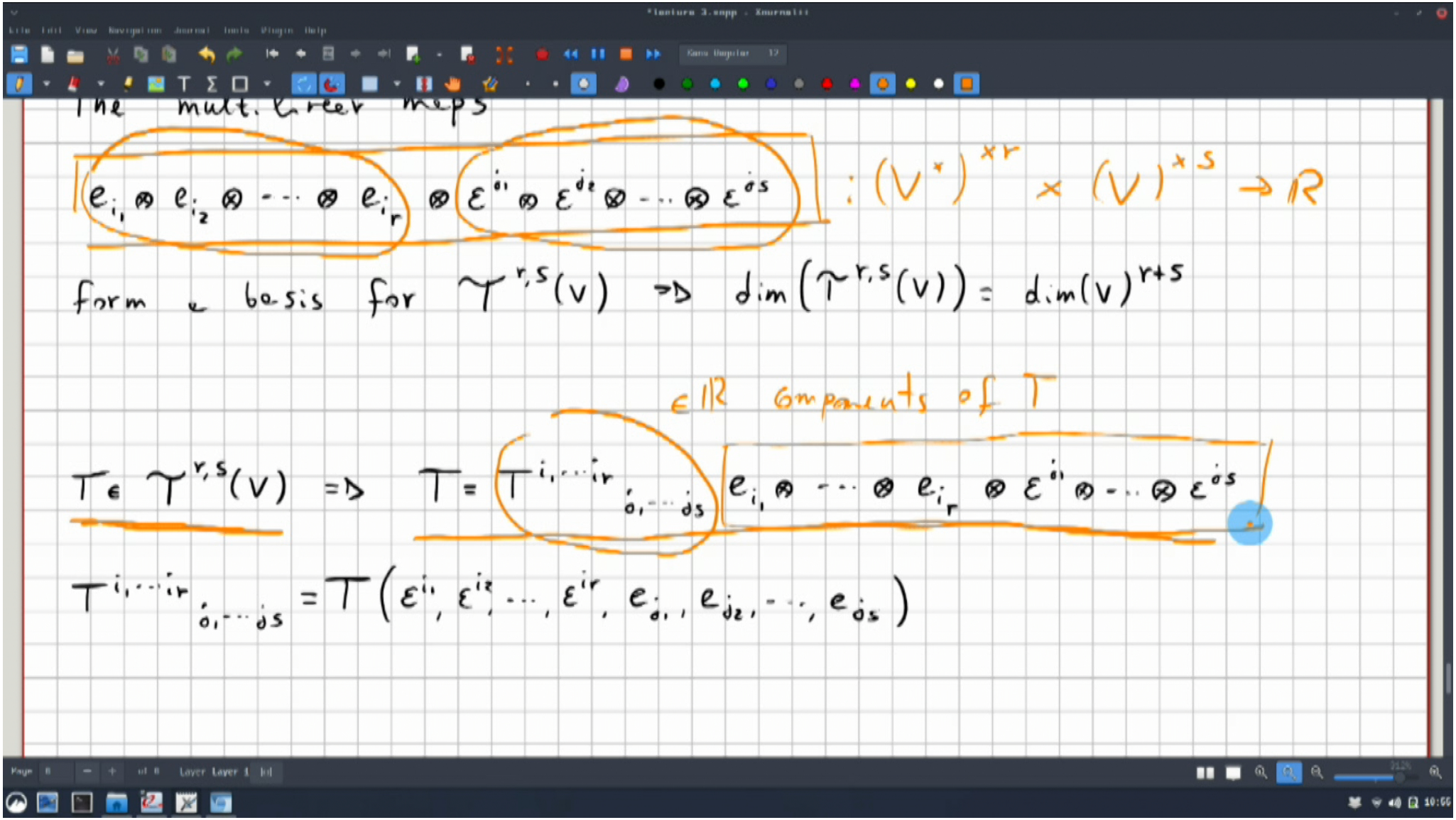
$\{e^i : V \to \mathbb{R}\}_{i=1}^n$ basis for V^* (dual basis)

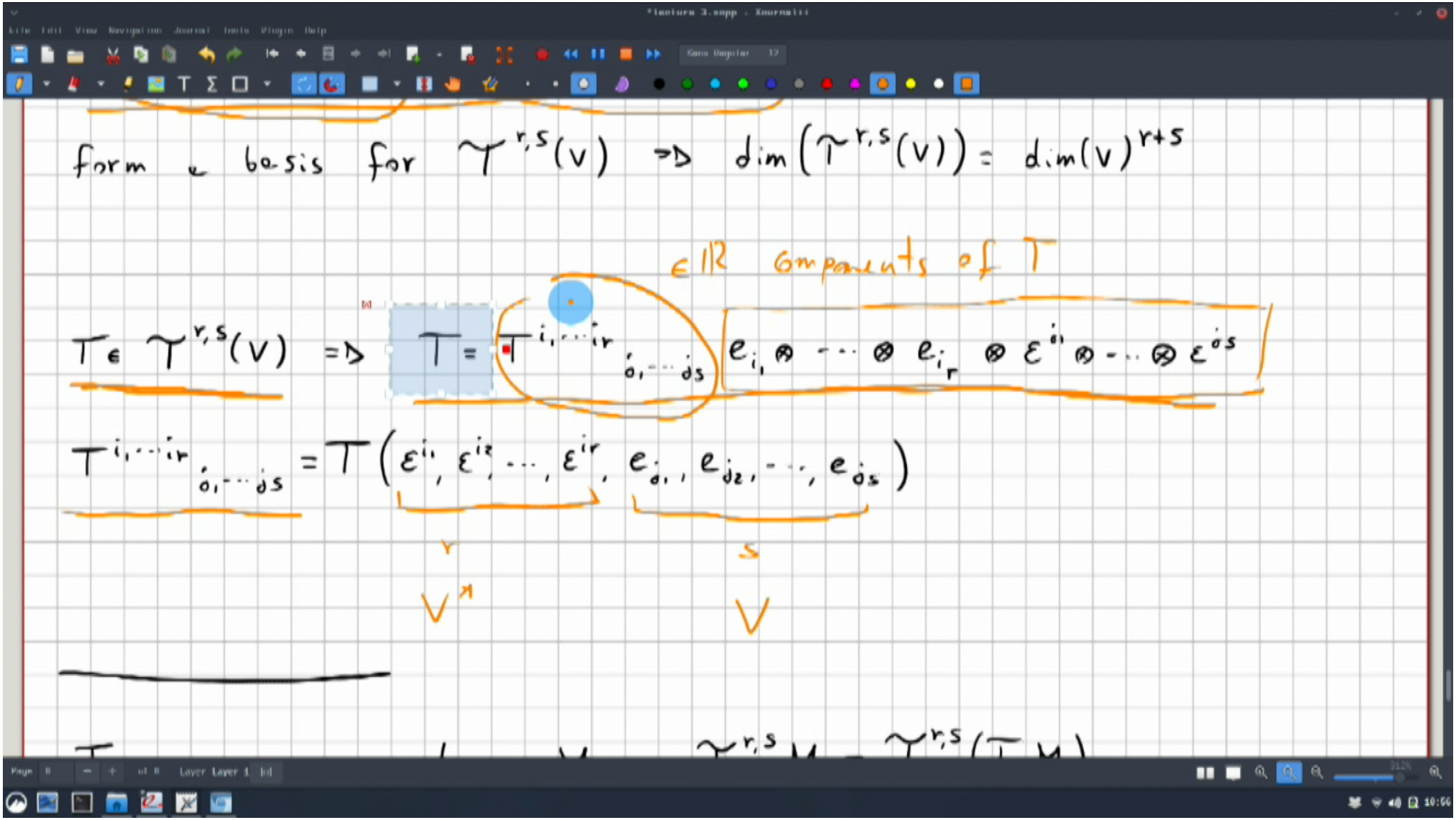
The multilinear maps

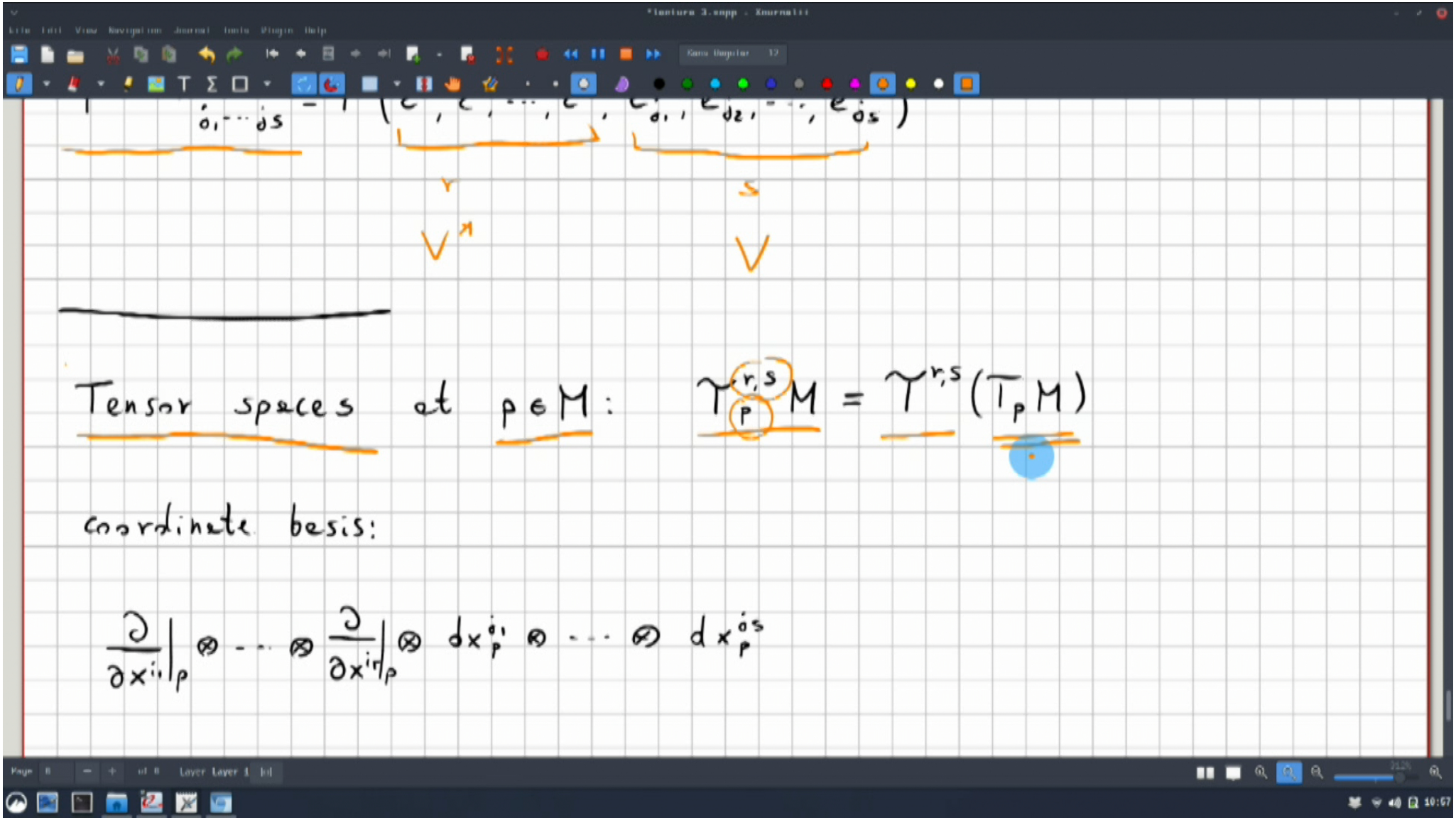
$$e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r} \otimes \varepsilon^{j_1} \otimes \varepsilon^{j_2} \otimes \cdots \otimes \varepsilon^{j_s}$$

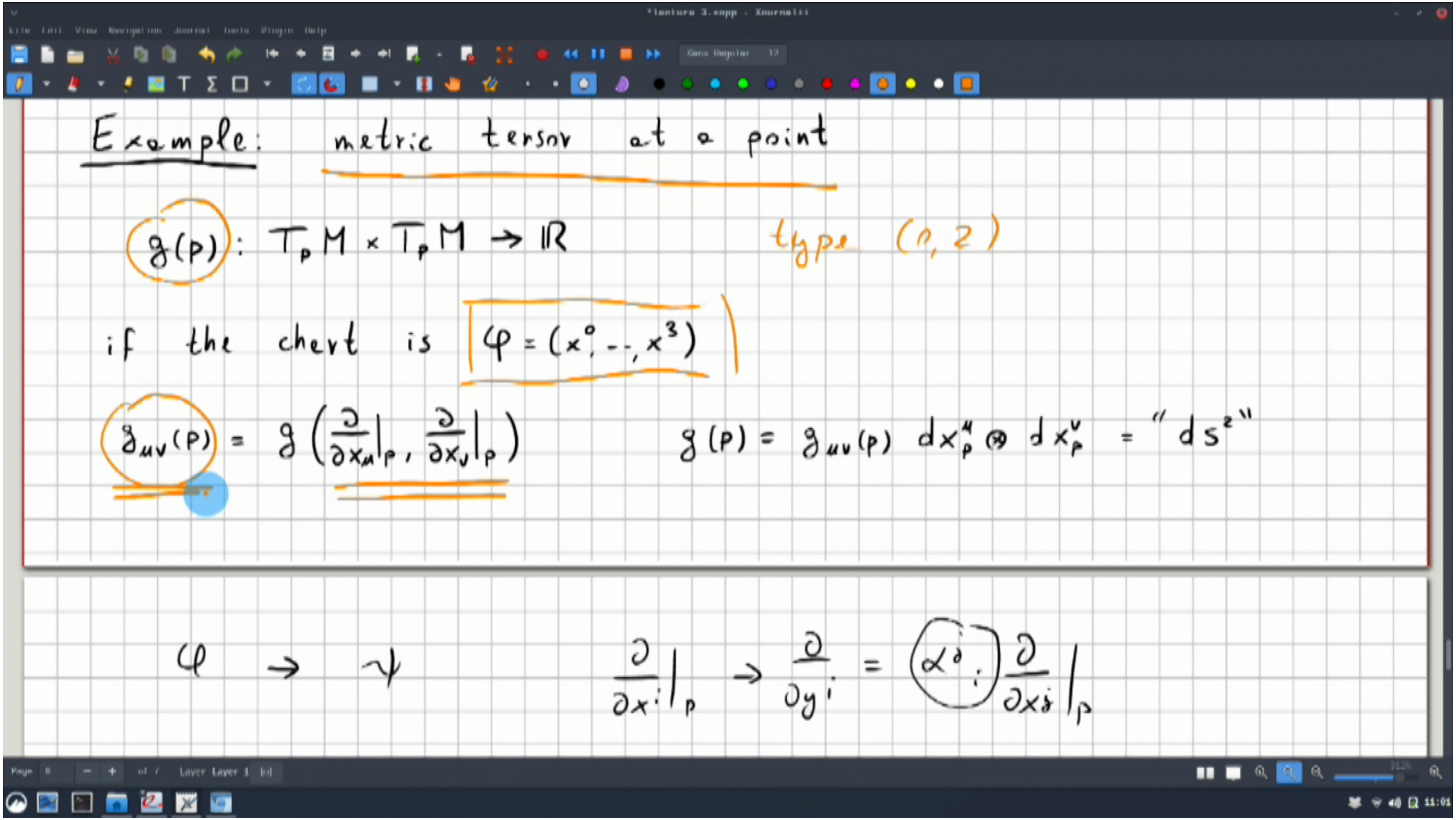
form a basis for $\mathcal{T}^{r,s}(V)$ $\Rightarrow$ $\dim\left(\mathcal{T}^{r,s}(V)\right) = \dim(V)^{r+s}$


$\{e_i : V^* \to \mathbb{R}\}_{i=1}^n$ basis for $V \to (V^*)^*$ dual basis for $\mathcal{T}^{r,s}(V)$
$\{e^i : V \to \mathbb{R}\}_{i=1}^n$ basis for V^* (dual basis)
The multilinear maps
$e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r} \otimes \varepsilon^{j_1} \otimes \varepsilon^{j_2} \otimes \cdots \otimes \varepsilon^{j_s} : (V^*)^{\times r} \times (V)^{\times s} \to \mathbb{R}$
form a basis for $\mathcal{T}^{r,s}(V)$ $\Rightarrow \dim(\mathcal{T}^{r,s}(V)) = \dim(V)^{r+s}$


The multilinear maps
$e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r} \otimes \varepsilon^{j_1} \otimes \varepsilon^{j_2} \otimes \cdots \otimes \varepsilon^{j_s} : (V^*)^{\times r} \times (V)^{\times s} \to \mathbb{R}$
form a basis for $\mathcal{T}^{r,s}(V)$ $\Rightarrow$ $\dim\left(\mathcal{T}^{r,s}(V)\right) = \dim(V)^{r+s}$
$\in \mathbb{R}$ components of T
$T \in \mathcal{T}^{r,s}(V)$ $\Rightarrow$ $T = T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_s}$
$T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s} = T\left(\varepsilon^{i_1}, \varepsilon^{i_2}, \ldots, \varepsilon^{i_r}, e_{j_1}, e_{j_2}, \ldots, e_{j_s}\right)$


form a basis for $\mathcal{T}^{r,s}(V)$ $\Rightarrow$ $\dim(\mathcal{T}^{r,s}(V)) = \dim(V)^{r+s}$
$\in \mathbb{R}$ components of T
$T \in \mathcal{T}^{r,s}(V)$ $\Rightarrow$ $T = T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}\, e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes \varepsilon^{j_1} \otimes \cdots \otimes \varepsilon^{j_s}$
$T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s} = T(\varepsilon^{i_1}, \varepsilon^{i_2}, \ldots, \varepsilon^{i_r}, e_{j_1}, e_{j_2}, \ldots, e_{j_s})$
r
s
V^*
V
$\mathcal{T}^{r,s}$


Tensor spaces at $p \in M$: $T^{r,s}_p M = T^{r,s}(T_p M)$
coordinate basis:
$\frac{\partial}{\partial x^{i_1}}\Big|_p \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}}\Big|_p \otimes dx^{j_1}_p \otimes \cdots \otimes dx^{j_s}_p$


Example: metric tensor at a point
$g(p): T_pM \times T_pM \to \mathbb{R}$
type (0, 2)
if the chart is $\varphi = (x^0, \dots, x^3)$
$g_{\mu\nu}(p) = g\left(\frac{\partial}{\partial x_\mu}\Big|_p, \frac{\partial}{\partial x_\nu}\Big|_p\right)$
$g(p) = g_{\mu\nu}(p)\, dx^\mu_p \otimes dx^\nu_p$ = "ds^2"
$\varphi \to \psi$
$\frac{\partial}{\partial x^i}\Big|_p \to \frac{\partial}{\partial y^i} = \alpha^j_i \frac{\partial}{\partial x^j}\Big|_p$