Title: Geometry and topology for physicists 2021/2022 - Lecture 2

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Collection: Geometry and Topology for Physicists 2021/2022

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Geometry and topology for physicists

Today's plan:

- One more example
- Tangent vectors and tangent space

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Example

we can use the implicit function theorem to construct manifolds by imposing constraints on $\mathbb{R}^n$:

if $F : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \to \mathbb{R}^m$ with $m < n$ is a C^∞ function, then

$$M = F^{-1}(0) = \{ x \in U \mid F(x) \}$$

is a smooth manifold with topology and differential structure inherited from $\mathbb{R}^n$.

$$M = F^{-1}(0) = \{ x \in U \mid F(x) = 0 \}$$

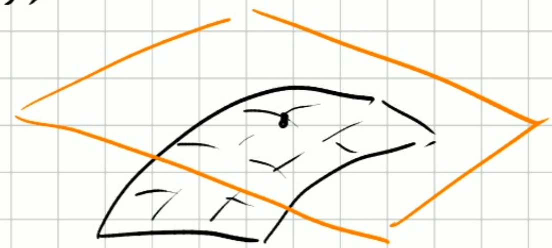
is a smooth manifold with topology and differential structure inherited from $\mathbb{R}^n$.

$$F(x,y) = x^2 + y^2 - 1 \quad \text{on } \mathbb{R}^2 \qquad F^{-1}(0) = S^2$$

$F(x,y) = 0$ then locally we can write $y = g(x)$ or $x = h(y)$

$$y = \sqrt{1-x^2} \qquad y = -\sqrt{1-x^2} \qquad x = \pm\sqrt{1-y^2}$$

explicit

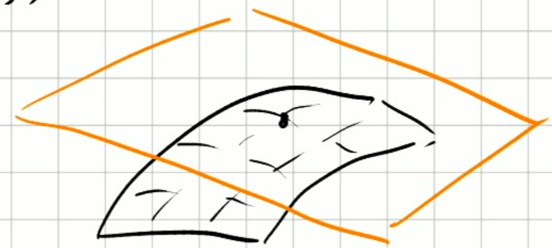


$\boxed{\mathbb{R}^n \text{ case}}$

- $M = \mathbb{R}^n$
- Tangent space at p: "$T_p\mathbb{R}^n$" $= \mathbb{R}^n$

Let's look at curves:

explicit



$\boxed{\mathbb{R}^n \text{ case}}$

- $M = \mathbb{R}^n$ → manifold

- Tangent space at p : "$T_p \overbrace{\mathbb{R}^n}^{M}$" $= \mathbb{R}^n$ → vector space

Let's look at curves:

- $M = \mathbb{R}^n$

- Tangent space at p: "$T_p \overbrace{\mathbb{R}^n}^{M}$" $= \mathbb{R}^n \rightarrow$ vector space

Let's look at curves: $\varepsilon > 0$

- if $\boxed{\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n}$ is a differentiable curve with $\gamma(0) = p$

then $\gamma'(0) = \lim_{t \to 0} \frac{\gamma(t) - \gamma(0)}{t}$

is a vector tangent to $\mathbb{R}^n$ at p

- $M = \mathbb{R}^n$

- Tangent space at p: "$T_p \overbrace{\mathbb{R}^n}^{M}$" $= \mathbb{R}^n \to$ vector space

Let's look at curves: $\varepsilon > 0$

- if $\boxed{\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n}$ is a differentiable curve with $\gamma(0) = p$

Can always do this

then $\gamma'(0) = \lim_{t \to 0} \frac{\gamma(t) - \gamma(0)}{t} \in \mathbb{R}^n$ tangent vector to curve

is a vector tangent to $\mathbb{R}^n$ at p

$\gamma(0) = p$

- if $v \in T_p \mathbb{R}^n$ then we can use $\gamma(t) = p + tv$ to get $\boxed{\gamma'(0) = v}$

$\rightarrow$ we can identify tangent vectors with curves through p

$[\gamma$

- Note: many curves have the same tangent vector, so we need identify vectors with equivalence classes of curves

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$[\gamma] \Leftrightarrow$ tangent vector to γ

- Note: many curves have the same tangent vector, so we need identify vectors with equivalence classes of curves

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Let's look at curves:

$\varepsilon > 0$

can always do this

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is a vector tangent to $\mathbb{R}^n$ at p

$\gamma(0) = p$

• if $v \in T_p \mathbb{R}^n$ then we can use $\gamma(t) = p + tv$

Generic manifolds

M smooth manifold with $\dim(M) = n$

$\boxed{\text{Def}}$ A curve at $p \in M$ is a map $\gamma : I \underset{\text{open}}{\subseteq} \mathbb{R} \to M$

with $0 \in I$ and $\gamma(0) = p$.

γ is of class C^1 if there is a chart $(U, \varphi) \in \mathcal{A}_M$ with

$p \in U$ such that $\varphi \circ \gamma$ is continuously differentiable

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open intervall

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Def Two C^1-curves at p γ_1, γ_2 are equivalent if

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$\varphi \circ \gamma : I \to \mathbb{R}^m$

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$p \in U$ such that $\varphi \circ \gamma$ is continuously differentiable

$\varphi \circ \gamma : I \to \mathbb{R}^m$

Def Two C^1-curves at p γ_1, γ_2 are equivalent if

$$(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$$

$\longrightarrow \mathbb{R} \to \mathbb{R}^n$

in a chart (U, φ) containing p.

$\to$ if it works in one chart it works in all of them

---

in a chart (U,φ) containing p.

→ if it works in one chart it works in all of them

Def The tangent space at $p \in M$ is the set

$$\boxed{T_pM = \{[\gamma] \mid \gamma \text{ is a } C^1\text{-curve at } p\}}$$

Is this a vector space?

Isomorphism with $\mathbb{R}^n$

Consider the function $$\boxed{d\varphi_p : [\gamma] \in T_pM \to (\varphi \circ \gamma)'(0) \in \mathbb{R}^n}$$

- $d\varphi_p$ well-defined
- $d\varphi_p$ surjective
- $d\varphi_p$ injective

$\to$ invertible

is it an isomorphism?

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Consider the function $\boxed{d\varphi_p : [\gamma] \in T_pM \to (\varphi \circ \gamma)'(0) \in \mathbb{R}^n}$

- $d\varphi_p$ well-defined
- $d\varphi_p$ surjective
- $d\varphi_p$ injective

$\to$ invertible

$v \in \mathbb{R}^n$ $\quad (\varphi \circ \gamma)(t) = \varphi(p) + tv$

$\gamma(t) = \varphi^{-1}(\varphi(p) + tv)$

$\gamma(0) = p$

$\boxed{\text{is it an isomorphism?}}$

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is it an isomorphism?

$a, b \in \mathbb{R} \qquad [\gamma_1], [\gamma_2] \in T_pM$

$$a[\gamma_1] + b[\gamma_2] = (d\varphi_p)^{-1}\left(a\, d\varphi_p([\gamma_1]) + b\, d\varphi_p([\gamma_2])\right)$$

Notes:

$$a[\gamma_1] + b[\gamma_2] = (d\varphi_p)^{-1}\left(a\, d\varphi_p([\gamma_1]) + b\, d\varphi_p([\gamma_2])\right)$$

$$d\varphi_p\left(a[\gamma_1] + b[\gamma_2]\right) = a\, d\varphi_p([\gamma_1]) + b\, d\varphi_p([\gamma_2])$$

Notes:

- $\boxed{\dim(T_pM) = \dim(M) = n}$

- $d\varphi_p$ and $d\psi_p$ are different isomorphisms

Coordinate-induced basis

$\boxed{d\varphi_p}$ isomorphism $\rightarrow$ we can use it to steal a basis from $\mathbb{R}^n$

if $\varphi = (x^1, x^2, \ldots)$ $\qquad x^i : U \rightarrow \mathbb{R}^n$

$$\boxed{\left.\frac{\partial}{\partial x^i}\right|_p = (d\varphi_p)^{-1}(e_i)}$$

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Coordinate-induced basis

$\{e_i\}_{i=1}^{n}$ standard basis in $\mathbb{R}^n$

$d\varphi_p$ isomorphism $\rightarrow$ we can use it to steal a basis from $\mathbb{R}^n$

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$\varphi : U \to \mathbb{R}^n$

$$\boxed{\left.\frac{\partial}{\partial x^i}\right|_p = (d\varphi_p)^{-1}(e_i)}$$

change of basis ?

$$\boxed{\frac{\partial}{\partial x^i}\Big|_p = (d\varphi_p)^{-1}(e_i)} \in T_pM$$

change of basis?

$$\psi = (y^1, y^2, \ldots)$$

$$\frac{\partial}{\partial y^i}\Big|_p = (d\psi_p)^{-1}(e_i) = \alpha^j{}_i \frac{\partial}{\partial x^j}\Big|_p$$

?

$$\frac{\partial}{\ }\Big| = [\gamma] \quad \text{with} \quad (\psi \circ \gamma)'(0) = e_i$$

$$\psi = (y^1, y^2, \dots)$$

$$\left.\frac{\partial}{\partial y^i}\right|_p = (d\psi_p)^{-1}(e_i) = \alpha^j{}_i \left.\frac{\partial}{\partial x^j}\right|_p$$

$$\left.\frac{\partial}{\partial y^i}\right|_p = [\gamma] \quad \text{with} \quad (\psi \circ \gamma)'(0) = \boxed{e_i}$$

$$\gamma(t) = ?$$

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$$\frac{\partial}{\partial y^i}\Big|_p = (d\psi_p)(e_i) = a \quad \frac{\partial}{\partial x^i}\Big|_p$$

$$\frac{\partial}{\partial y^i}\Big|_p = [\gamma] \quad \text{with} \quad (\psi \circ \gamma)'(0) = \boxed{e_i}$$

$$\gamma(t) = \psi^{-1}\left(\psi(p) + t\, e_i\right)$$

$$d\varphi_p\left(\frac{\partial}{\partial y^i}\Big|_p\right) = d\varphi_p([\gamma]) = (\varphi \circ \gamma)'(0)$$

$$= \frac{d}{dt}\Big|_{t=0} (\varphi \circ \psi^{-1})(\psi(p) +$$

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$$\gamma(t) = \psi^{-1}(\psi(p) + t e_i)$$

$$d\varphi_p\left(\frac{\partial}{\partial y^i}\Big|_p\right) = d\varphi_p([\gamma]) = (\varphi \circ \gamma)'(0)$$

$$= \frac{d}{dt}\Big|_{t=0} \underbrace{(\varphi \circ \psi^{-1})}_{\Theta : \mathbb{R}^n \to \mathbb{R}^n}\overbrace{(\psi(p) + t e_i)}^{\mathbb{R} \to \mathbb{R}^n}$$

$$= \partial$$

$$= \frac{\partial \theta^j}{\partial y^k}(\psi(p)) \; \delta^k{}_i \; e_j$$

$$= \frac{\partial \theta^j}{\partial y^i}(\psi(p)) \; e_j$$

$$d\varphi_p \left(\frac{\partial}{\partial y^i}\Big|_p \right) = d\varphi_p \left(\alpha^j{}_i \frac{\partial}{\partial x^j}\Big|_p \right)$$

$$= \alpha^j$$

$$= \alpha^j_i \, d\varphi_p \left(\frac{\partial}{\partial x^j}\Big|_p \right) = \boxed{\alpha^j_i \, e_j}$$

$$\alpha^j_i = \frac{\partial \theta^j}{\partial y^i}(\psi(p))$$

$$\boxed{\frac{\partial}{\partial y^i}\Big|_p = \frac{\partial \theta^j}{\partial y^i}(\psi(p)) \, \frac{\partial}{\partial x^j}\Big|_p}$$

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$$= \alpha^j_i \, \ldots_p \left(\frac{\partial}{\partial x^j}\Big|_p \right) = \alpha^j_i \, e_j$$

$$\alpha^j_i = \frac{\partial \theta^j}{\partial y^i}(\psi(p))$$

$$\boxed{\frac{\partial}{\partial y^i}\Big|_p = \frac{\partial \theta^j}{\partial y^i}(\psi(p)) \, \frac{\partial}{\partial x^j}\Big|_p}$$

abuse of notation $\qquad \theta^j = \varphi^j \circ \psi^{-1} = x^j \circ \psi^{-1}$

$$\boxed{\frac{\partial}{\partial y^i}\Big|_p = \frac{\partial x^j}{\partial y^i}(\psi(p)) \, \frac{\partial}{\partial x^j}\Big|_p}$$

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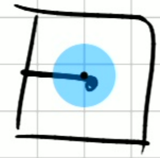
Example



$$M = \{ (r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \mid r > 0,\ \theta \in (-\pi, \pi) \} \equiv \mathbb{R}^2 \setminus \{ (x,0) \mid x \leq 0 \}$$

charts:

$$\varphi = (r, \theta) : (a,b) \in M \mapsto \left(\sqrt{a^2+b^2},\ \mathrm{atan2}(b,a) \right) \in \mathbb{R}^2$$

$$id = (x,y) : (a,b) \in M \mapsto (a,b) \in \mathbb{R}^2$$

$$\varphi^{-1} : (r,\theta) \in \mathbb{R}_+ \times (-\pi,\pi) \mapsto (r\cos\theta,\ r\sin\theta)$$

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$$\varphi = (r,\theta) : (a,b) \in M \mapsto \left(\sqrt{a^2+b^2},\ \operatorname{atan2}(b,a)\right) \in \mathbb{R}^2$$

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$$\left.\frac{\partial}{\partial r}\right|_p \quad \left.\frac{\partial}{\partial \theta}\right|_p \quad \Big\| \quad \left.\frac{\partial}{\partial x}\right|_p \quad \left.\frac{\partial}{\partial y}\right|_p$$

$id \circ \varphi^{-1}$

$$\left.\frac{\partial}{\partial r}\right|_p = \frac{\partial x}{\partial r}(\varphi(p)) \left.\frac{\partial}{\partial x}\right|_p + \frac{\partial y}{\partial r}(\varphi(p)) \left.\frac{\partial}{\partial y}\right|_p$$

$$= \cos(\theta(p)) \left.\frac{\partial}{\partial x}\right|_p + \sin(\theta(p)) \left.\frac{\partial}{\partial y}\right|_p$$

$$\varphi^{-1} : (r,\theta) \in \mathbb{R}_+ \times (-\pi,\pi) \mapsto (r\cos\theta,\, r\sin\theta)$$

$$\frac{\partial}{\partial r}\Big|_p \qquad \frac{\partial}{\partial \theta}\Big|_p \qquad \Big\| \qquad \frac{\partial}{\partial x}\Big|_p \qquad \frac{\partial}{\partial y}\Big|_p$$

$\nearrow \; id \circ \varphi^{-1}$

$$\frac{\partial}{\partial r}\Big|_p = \frac{\partial x}{\partial r}(\varphi(p))\,\frac{\partial}{\partial x}\Big|_p + \frac{\partial y}{\partial r}(\varphi(p))\,\frac{\partial}{\partial y}\Big|_p$$

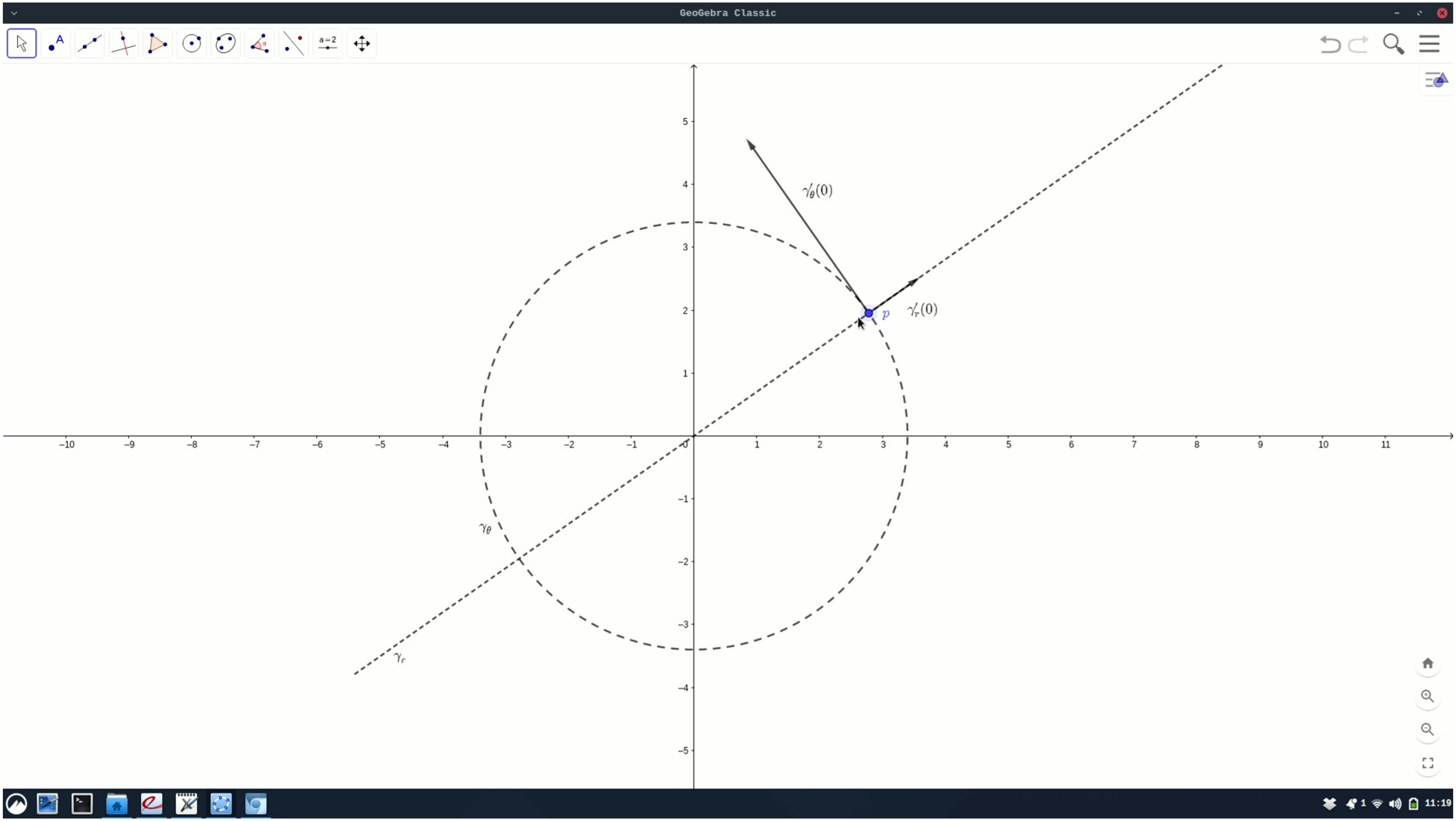
$$= \boxed{\cos(\theta(p))\,\frac{\partial}{\partial x}\Big|_p + \sin(\theta(p))\,\frac{\partial}{\partial y}\Big|_p}$$

Visualise basis vectors:

$$= \boxed{\cos(\theta(p)) \frac{\partial}{\partial x}\Big|_p + \sin(\theta(p)) \frac{\partial}{\partial y}\Big|_p}$$

Visualise basis vectors: $\frac{\partial}{\partial r}\Big|_p = [\gamma_r]$ $\frac{\partial}{\partial \theta}\Big|_p = [\gamma_\theta]$

$$\gamma_r(t) = \varphi^{-1}(\varphi(p) + t e_1) \qquad \gamma_\theta(t) = \varphi^{-1}(\varphi(p) + t e_2)$$


GeoGebra Classic
$\dot\gamma_\theta(0)$
$\dot\gamma_r(0)$
p
γ_θ
γ_r

l1.pdf — Geometry and topology for physicists (part 1)

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Lecture 1 1
Review of… 1
Manifolds 2

$$U_\pm = S^2 \setminus \{(0,0,\pm 1)\} \tag{1.2.2}$$

and

$$\varphi_\pm : (x,y,z) \in U_\pm \mapsto \left(\frac{x}{1 \mp z}, \frac{y}{1 \mp z} \right) \in \mathbb{R}^2. \tag{1.2.3}$$

You can see how stereographic projections work in this GeoGebra applet, as well as this one. The topology induced by the atlas is the same as the subspace topology induced from $\mathbb{R}^3$.

- The torus

$$T = \{(\cos\phi\,(R - r\cos\theta), \sin\phi\,(R - r\cos\theta), r\sin\theta) \in \mathbb{R}^3 \,|\, \theta, \phi \in (-\pi, \pi))\}, \quad R > r \tag{1.2.4}$$

can be made into a smooth 2-manifold using four charts (U_{ab}, φ_{ab}), $a, b = \pm 1$ with

$$U_{+\pm} = T \setminus \left(\{(x,y,0) \in \mathbb{R}^3 \,|\, x^2 + y^2 = (R+r)^2\} \cup \{(x,0,z) \in \mathbb{R}^3 \,|\, \pm x < 0\}\right) \tag{1.2.5}$$

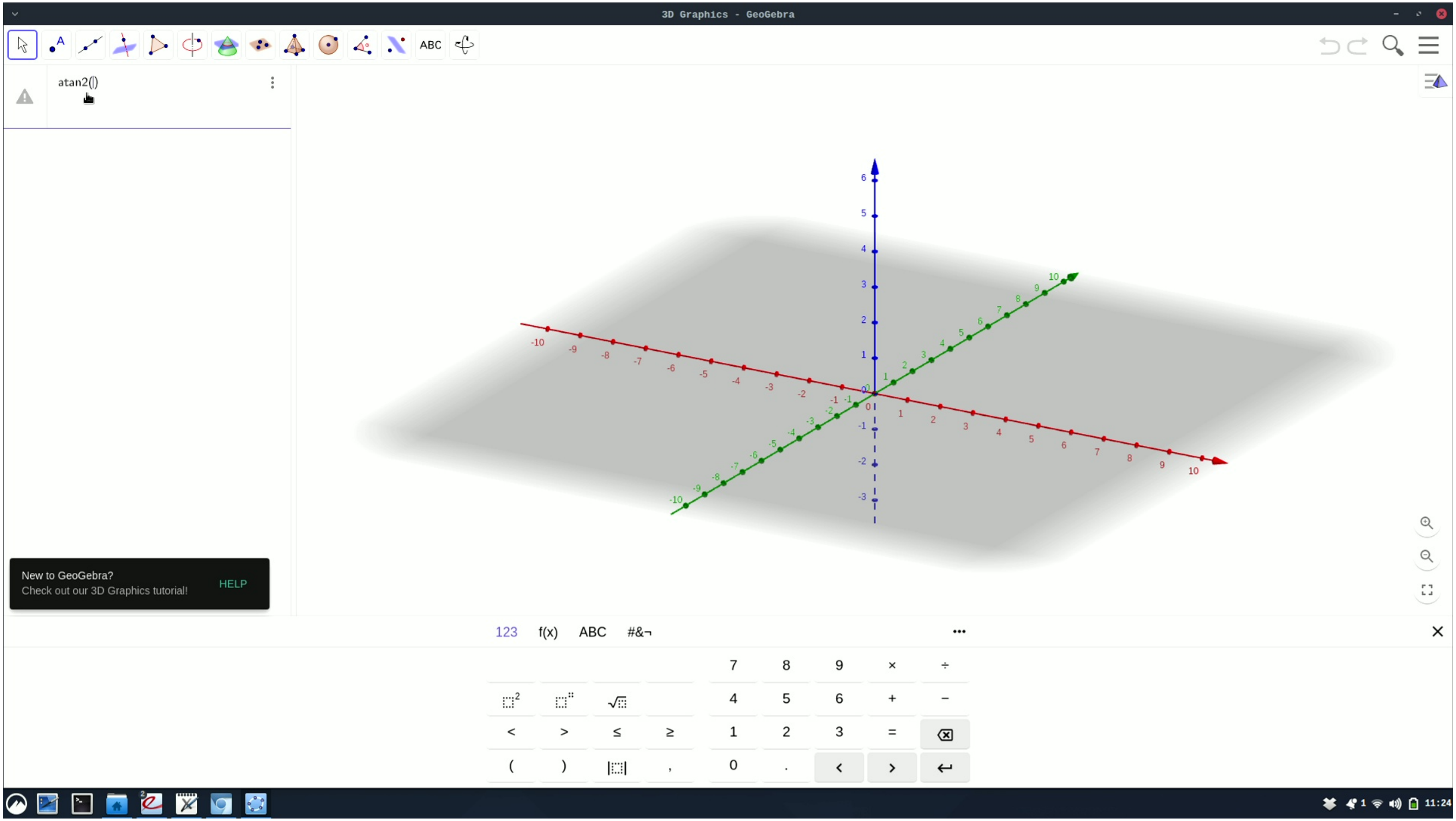
$$U_{-\pm} = T \setminus \left(\{(x,y,0) \in \mathbb{R}^3 \,|\, x^2 + y^2 = (R-r)^2\} \cup \{(x,0,z) \in \mathbb{R}^3 \,|\, \pm x < 0\}\right) \tag{1.2.6}$$

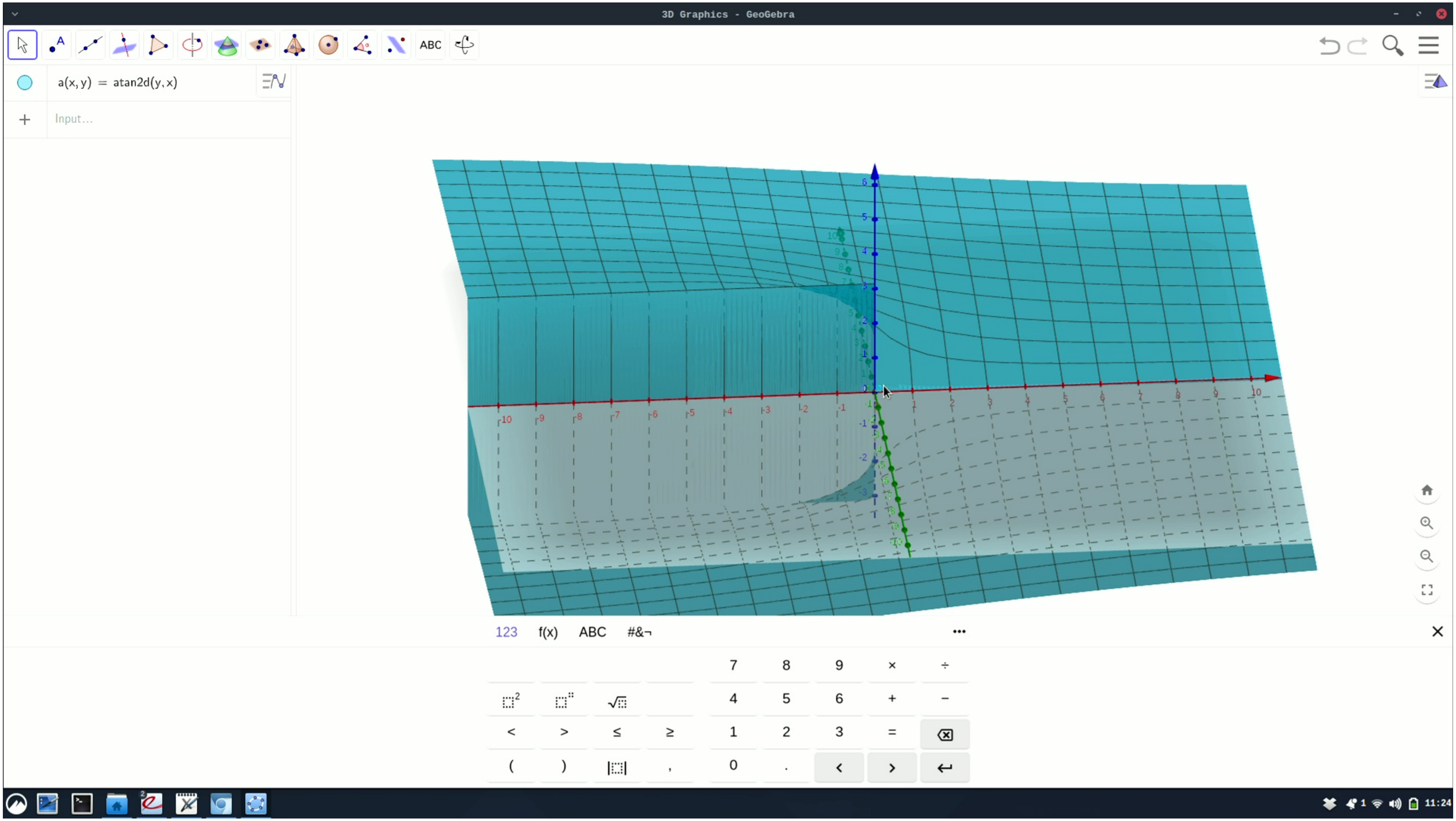
and

$$\varphi_{+\pm} : (x,y,z) \in U_{+\pm} \mapsto \left(\mathrm{atan2}(z, R - \sqrt{x^2+y^2}), \mathrm{atan2}(\pm y, \pm x) \right) \in \mathbb{R}^2 \tag{1.2.7}$$

$$\varphi_{-\pm} : (x,y,z) \in U_{-\pm} \mapsto \left(\mathrm{atan2}(-z, -R + \sqrt{x^2+y^2}), \mathrm{atan2}(\pm y, \pm x) \right) \in \mathbb{R}^2. \tag{1.2.8}$$

You can see what the regions U_{ab} look like in this GeoGebra applet. The topology induced by the atlas is the same as the subspace topology induced from $\mathbb{R}^3$.


3D Graphics - GeoGebra
ABC
atan2()
New to GeoGebra?
Check out our 3D Graphics tutorial!
HELP
123
f(x)
ABC
#&¬
11:24


3D Graphics - GeoGebra
ABC
a(x, y) = atan2d(y, x)
Input...
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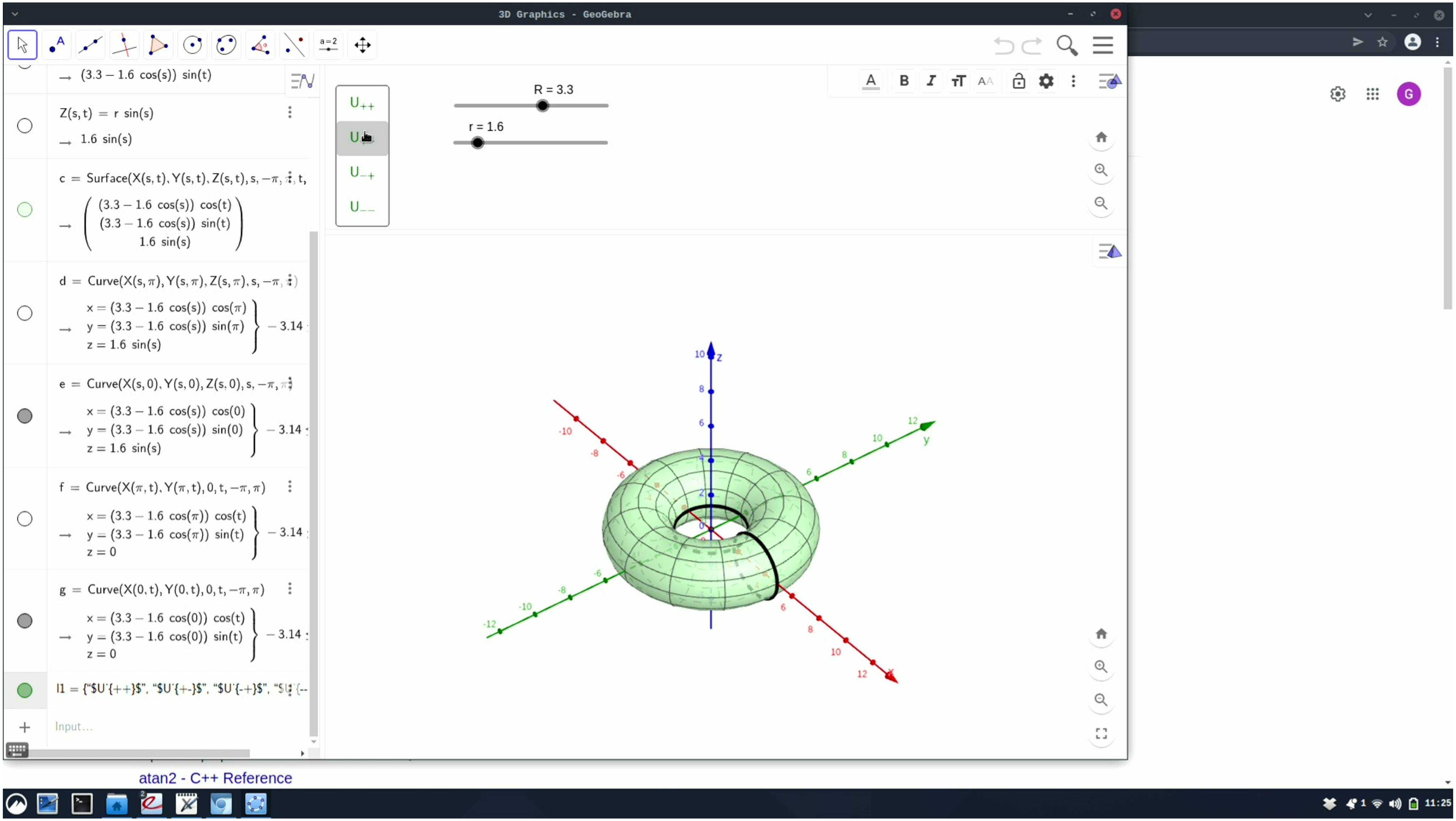
3D Graphics - GeoGebra
→ (3.3 − 1.6 cos(s)) sin(t)
Z(s, t) = r sin(s)
→ 1.6 sin(s)
c = Surface(X(s, t), Y(s, t), Z(s, t), s, −π,
(3.3 − 1.6 cos(s)) cos(t)
(3.3 − 1.6 cos(s)) sin(t)
1.6 sin(s)
d = Curve(X(s, π), Y(s, π), Z(s, π), s, −π,
x = (3.3 − 1.6 cos(s)) cos(π)
y = (3.3 − 1.6 cos(s)) sin(π)
z = 1.6 sin(s)
− 3.14
e = Curve(X(s, 0), Y(s, 0), Z(s, 0), s, −π,
x = (3.3 − 1.6 cos(s)) cos(0)
y = (3.3 − 1.6 cos(s)) sin(0)
z = 1.6 sin(s)
− 3.14
f = Curve(X(π, t), Y(π, t), 0, t, −π, π)
x = (3.3 − 1.6 cos(π)) cos(t)
y = (3.3 − 1.6 cos(π)) sin(t)
z = 0
− 3.14
g = Curve(X(0, t), Y(0, t), 0, t, −π, π)
x = (3.3 − 1.6 cos(0)) cos(t)
y = (3.3 − 1.6 cos(0)) sin(t)
z = 0
− 3.14
l1 = {"$U'{++}$", "$U'{+-}$", "$U'{-+}$",
Input...
U++
U−+
U−−
R = 3.3
r = 1.6
z
y
atan2 - C++ Reference
11:25