

Title: Geometry and topology for physicists 2021/2022 - Lecture 1

Speakers: Kevin Costello, Giuseppe Sellaroli

Collection: Geometry and Topology for Physicists 2021/2022

Date: March 01, 2022 - 10:15 AM

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## Geometry and topology for physicists

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- Course information 0 / 2 ▲
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- Course outline (PDF version)  PDF
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## Course outline

# Geometry and topology for physicists

PSI 2021/2022

Course outline

### Instructors

Name: Kevin Costello  
Email: [kcostello@perimeterinstitute.ca](mailto:kcostello@perimeterinstitute.ca)

Name: Giuseppe Sellaroli (he/him)  
Email: [gsellaroli@perimeterinstitute.ca](mailto:gsellaroli@perimeterinstitute.ca)

### Academic staff

Name: Giuseppe Sellaroli (he/him)  
Email: [gsellaroli@perimeterinstitute.ca](mailto:gsellaroli@perimeterinstitute.ca)  
Office: 264

**How to get in touch with me:** You can always reach me by email. I should be in my office for at least part of the day when we have lectures or tutorials, just knock on the door if you want to talk to me. If I'm not in the office you can email me and we can discuss by email or set up a Zoom meeting.

### Teaching assistant

Name: Yehao Zhou   
Email: [yzhou3@perimeterinstitute.ca](mailto:yzhou3@perimeterinstitute.ca)

## Land acknowledgement

As PI residents, it is important that we are aware that Perimeter Institute is situated on the traditional territory of the Anishinaabe, Haudenosaunee, and Neutral peoples. Moreover, Perimeter Institute is located on the Haldimand Tract, which was granted to the Six Nations of the Grand River and the Mississaugas of the Credit First Nation. We encourage you to learn about the history of the [Haldimand proclamation](#) and of the people whose land we are on.

## Course description

The aim of this course is to introduce concepts in topology and geometry for applications in theoretical physics. The course will be divided in four parts, covering the following topics:

Topics	Instructor
Review of topology, topological and smooth manifolds, tensor bundles	Giuseppe
Differential forms, de Rham cohomology, integration on manifolds	Kevin
Riemann surfaces	Kevin

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### Course outline

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Riemann surfaces	Kevin
Brief review of Lie groups and Lie algebras, principal bundles, introduction to gauge theory	Kevin

#### Course components

The course will run in a synchronous format from March 1st to March 31st. The course will have:

- Lectures on Tuesdays, Thursdays, and Fridays from 10:15 to 11:15, for a total of 14 lectures.
- Six tutorials distributed through the five weeks of the course. The tutorial schedule will be available on the [PSI calendar](#).

#### Learning outcomes

By the end of this course students should be able to:

- Describe manifolds and tensor fields in a coordinate-free manner
- Integrate a  $k$ -form on a manifold of dimension  $k$
- Compute the de Rham cohomology for common examples of manifolds, and use it to extract topological information
- Describe gauge invariance in terms of a connection on a principal bundle

#### Assessments

In order to complete the course, students will have to:

- Participate in all six tutorials or complete and submit the relevant work if unable to attend.

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Lecture notes (Giuseppe's part) 0/1 ▾

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### Assessments

In order to complete the course, students will have to:

- Participate in all six tutorials or complete and submit the relevant work if unable to attend.
- Complete two homeworks.

### Accommodations

Accommodations for the various course components will be made according to PI's [Accommodation Policy](#). Students can contact the instructor or People and Culture if accommodations are required.

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The image shows a digital whiteboard application with a grid background. At the top, there is a toolbar with various icons for file operations, drawing tools, and color selection. The main area contains handwritten text and a list.

Geometry and topology for physicists

Today's plan:

- Brief introduction to topology
- Manifolds + examples

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Topology

Page 1 of 6 Layer Layer 1

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$\{a_n | n \in \mathbb{N}\}$

limits in  $\mathbb{R}$ :  $\lim_{n \rightarrow \infty} a_n = L \Leftrightarrow \forall \varepsilon > 0, \exists N > 0$  s.t.  $n > N \Rightarrow a_n \in (L - \varepsilon, L + \varepsilon)$

We want to generalise this to sets other than  $\mathbb{R}$  by replacing the open intervals with the more general notion of "open sets"

Page 1 of 6 Layer Layer 1

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Topology

$\{a_n \mid n \in \mathbb{N}\}$ .

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## Topology

$$\{a_n \mid n \in \mathbb{N}\}$$

Limits in  $\mathbb{R}$ :

$$\lim_{n \rightarrow \infty} a_n = L \iff \forall \varepsilon > 0, \exists N > 0 \text{ s.t. } n > N \Rightarrow a_n \in (L - \varepsilon, L + \varepsilon)$$

We want to generalise this to sets other than  $\mathbb{R}$  by replacing the open intervals with the more general notion of "open sets"

$$\lim_{n \rightarrow \infty} a_n = L \in X \iff \forall \text{"open sets"} U \text{ containing } L$$

$$\exists N > 0 \text{ s.t. } n > N \Rightarrow a_n \in U$$

 $n > \infty$ 

$$\exists N > 0 \text{ s.t. } n > N \Rightarrow a_n \in U$$

| Def) A topological space is a set  $X$  together with a collection  $\tau_X$  of subsets of  $X$  (the "open sets") satisfying:

- $\emptyset, X \in \tau_X$
- $U, V \in \tau_X \Rightarrow U \cap V \in \tau_X$
- $U_i \in \tau_X, i \in I \Rightarrow \bigcup_{i \in I} U_i \in \tau_X$

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topology on  $X$ .

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in  $\mathbb{R}$   $C = \{(-\varepsilon, \varepsilon) \mid \varepsilon > 0\}$   $\bigcap C = \{0\}$

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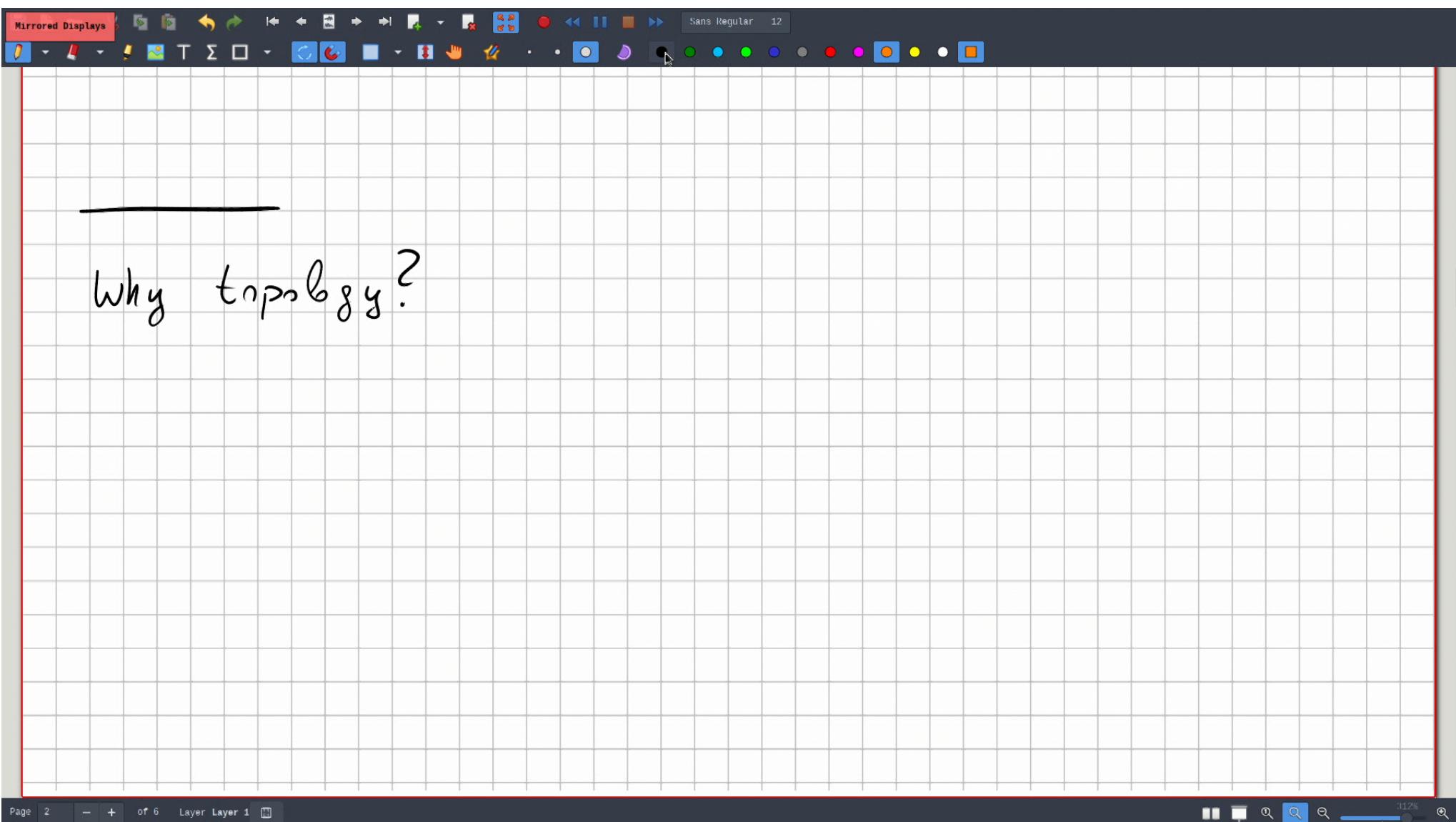
Examples

- Standard topology in  $\mathbb{R}$ : open sets are all possible unions of open intervals  $(a, b)$  with  $a < b$ .
- Standard topology in  $\mathbb{R}^n$ : open sets are all possible unions of open balls  $B_r(x) = \{y \in \mathbb{R}^n \mid \|y-x\| < r\}$  with  $r \geq 0, x \in \mathbb{R}^n$ .

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- Standard topology in  $\mathbb{R}^n$ : open sets are all possible unions of open balls  $B_r(x) = \{y \in \mathbb{R}^n \mid \|y-x\| < r\}$  with  $r \geq 0, x \in \mathbb{R}^n$ .
- if  $X$  is a topological space and  $S \subseteq X$   
 we can make  $S$  into a topological space with the  
induced topology  $\tau_S = \{U \cap S \mid U \in \tau_X\}$

Page 2 - of 6 Layer Layer 1 312%



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Why topology?

topology encodes geometrical information!

- connectedness (if  $X$  is made of disjoint pieces)
- presence of holes
- compactness (generalisation of "closed and bounded".)

Page 2 - of 6 Layer Layer 1

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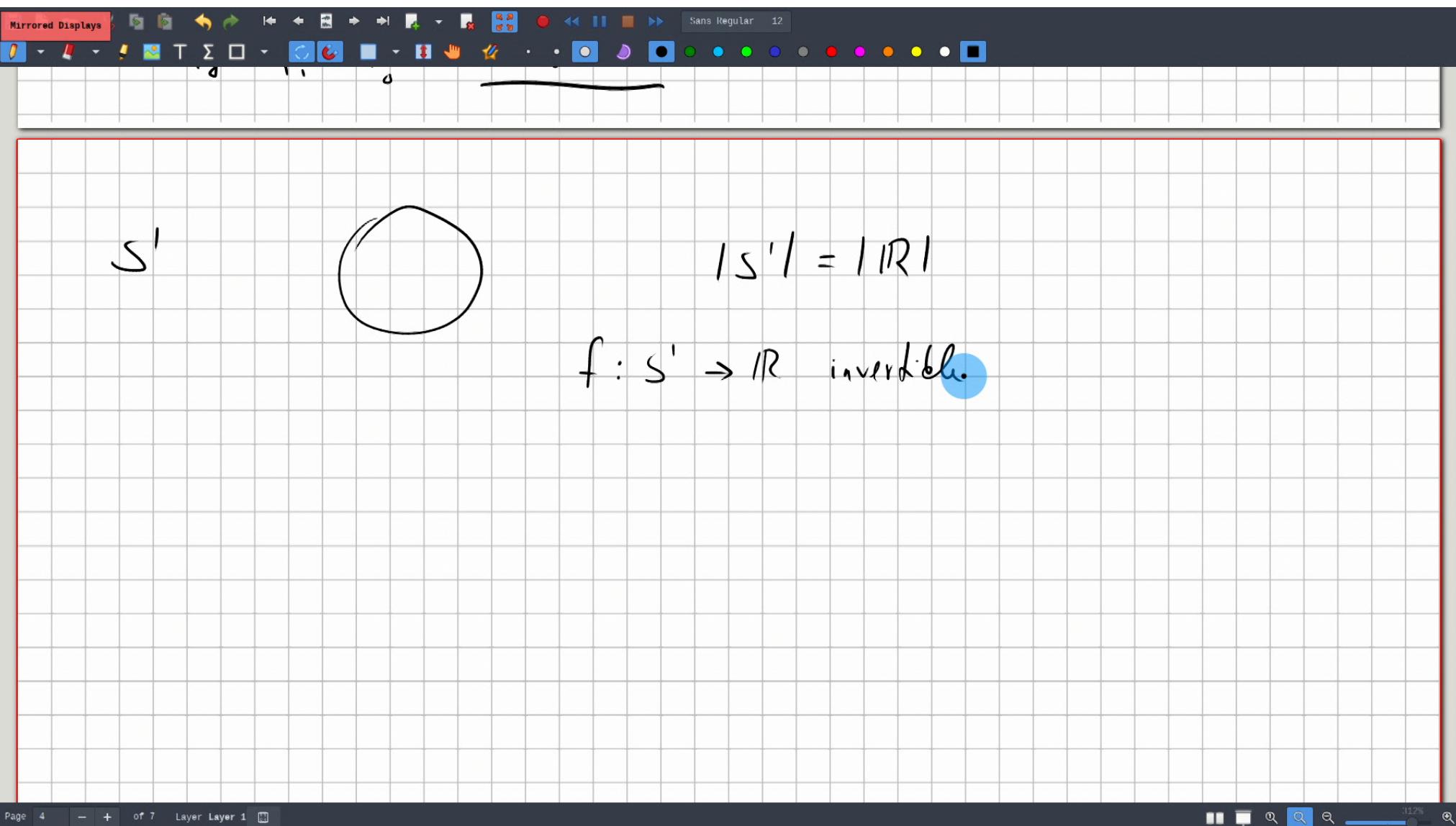
Why topology?

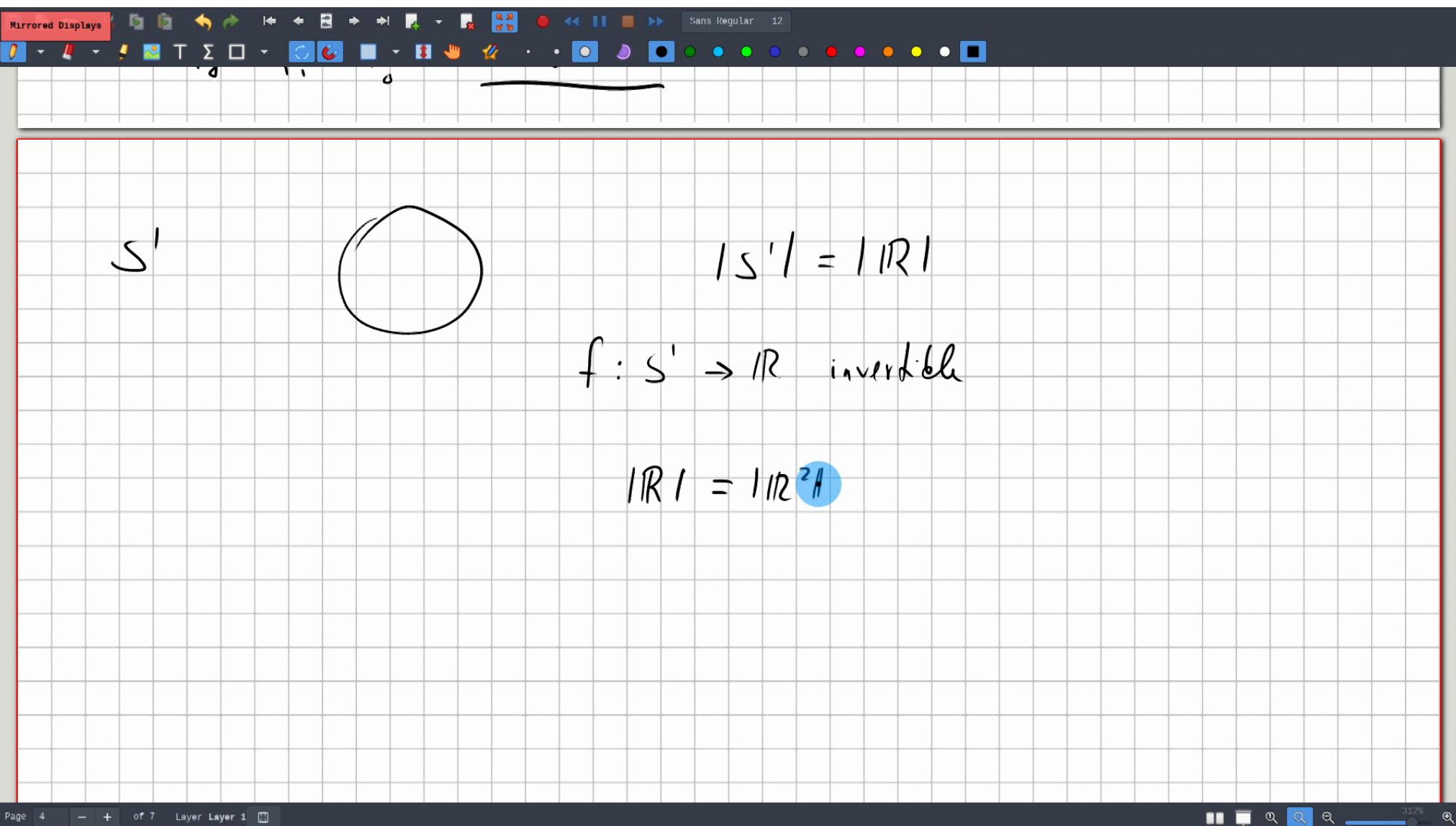
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- boundary

Page 2 of 6 Layer Layer 1

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- compactness (generalisation of "closed and bounded")
- boundary

Manifolds

we are interested in topological spaces that "look like  $\mathbb{R}^n$ " at least locally.

Page 3 of 7 Layer Layer 1

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at least locally.

[Def] Given a set  $M$ , a chart is a way to assign coordinates in  $\mathbb{R}^n$  to a subset  $U \subseteq M$ .

→ A chart is a pair  $(U, \varphi)$  with

- $U \subseteq M$
- $\varphi: U \rightarrow \mathbb{R}^n$  injective



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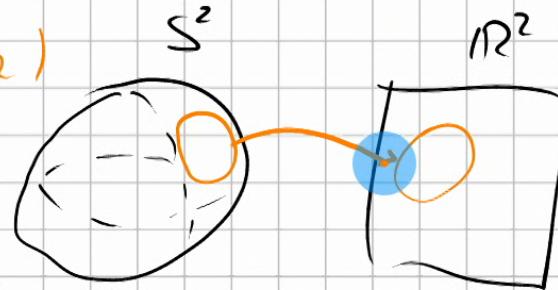
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|Def| An  $n$ -dimensional atlas  $\mathcal{A}$  on a set  $M$  is a collection of charts  $\{(U_i, \varphi_i) | i \in I\}$  with  $\varphi_i : U_i \subseteq M \rightarrow \mathbb{R}^n$  satisfying

- $\bigcup_{i \in I} U_i = M$
- $\varphi_i(U_i \cap U_j)$  open  $\mathbb{R}^n \quad \forall i, j \in I$
- $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$  continuous



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transition maps



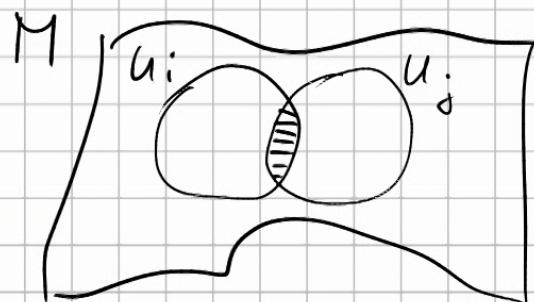
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transition maps



on  $U_i \cap U_j$  both  $\varphi_i$  and  $\varphi_j$  work

$$\varphi_i \circ \varphi_j^{-1}: W \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$$

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$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$  continuous maps

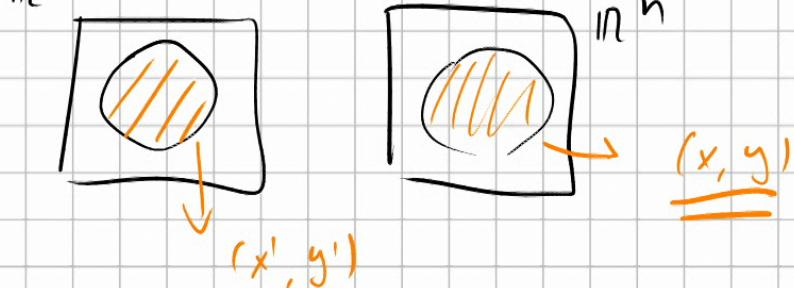
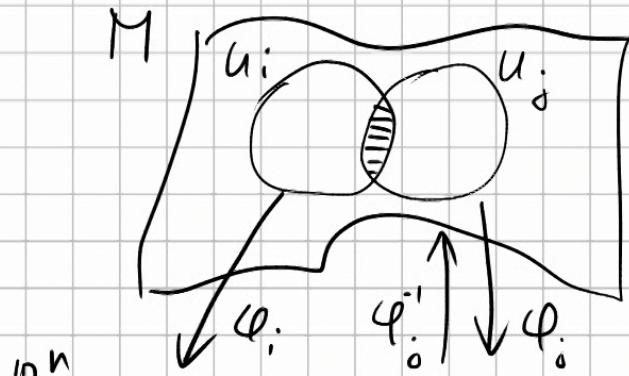
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$$W \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$$

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Atles is smooth if the transition maps are of class  $C^\infty$

Def A (topological) manifold is a set  $M$  together with

Page 6 of 8 Layer Layer 1

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Atles is smooth if the transition maps are of class  $C^\infty$

Def A (topological) manifold is a set  $M$  together with an  $n$ -dimensional atles  $\{A_M\}$

$\Rightarrow \dim(M) = n$

smooth atles  $\rightarrow$  smooth manifold

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an  $n$ -dimensional atlas  $\mathcal{A}_M$

$\Rightarrow \dim(M) = n$

smooth atlases  $\rightarrow$  smooth manifold

---

Topology on  $M$ ?    atlases induces topology on  $M$

Page 6 of 8 Layer Layer 1

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Examples!

①  $\mathbb{R}^n$  is a manifold ✓ } → on chart:  $\varphi = id|_U$

$U \subseteq \mathbb{R}^n$  is a manifold

② unit sphere  $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$

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charts:  $(U_{\pm}, \varphi_{\pm})$

$$U_{\pm} = S^2 \setminus \{(0, 0, \pm 1)\}$$

$$\varphi_{\pm}: (x, y, z) \in U_{\pm} \mapsto \left( \frac{x}{\sqrt{1-z^2}}, \frac{y}{\sqrt{1-z^2}} \right) \in \mathbb{R}^2$$

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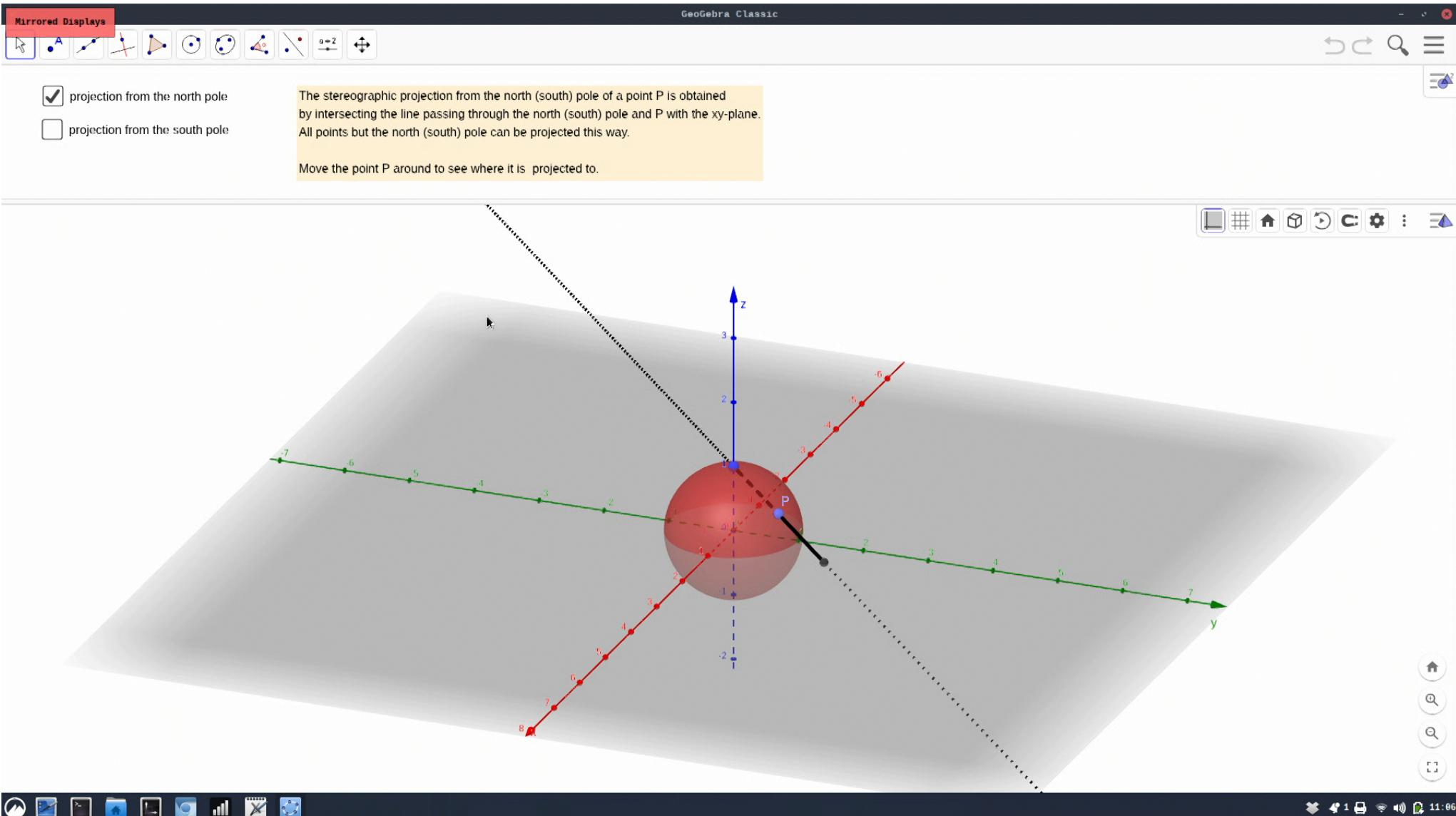
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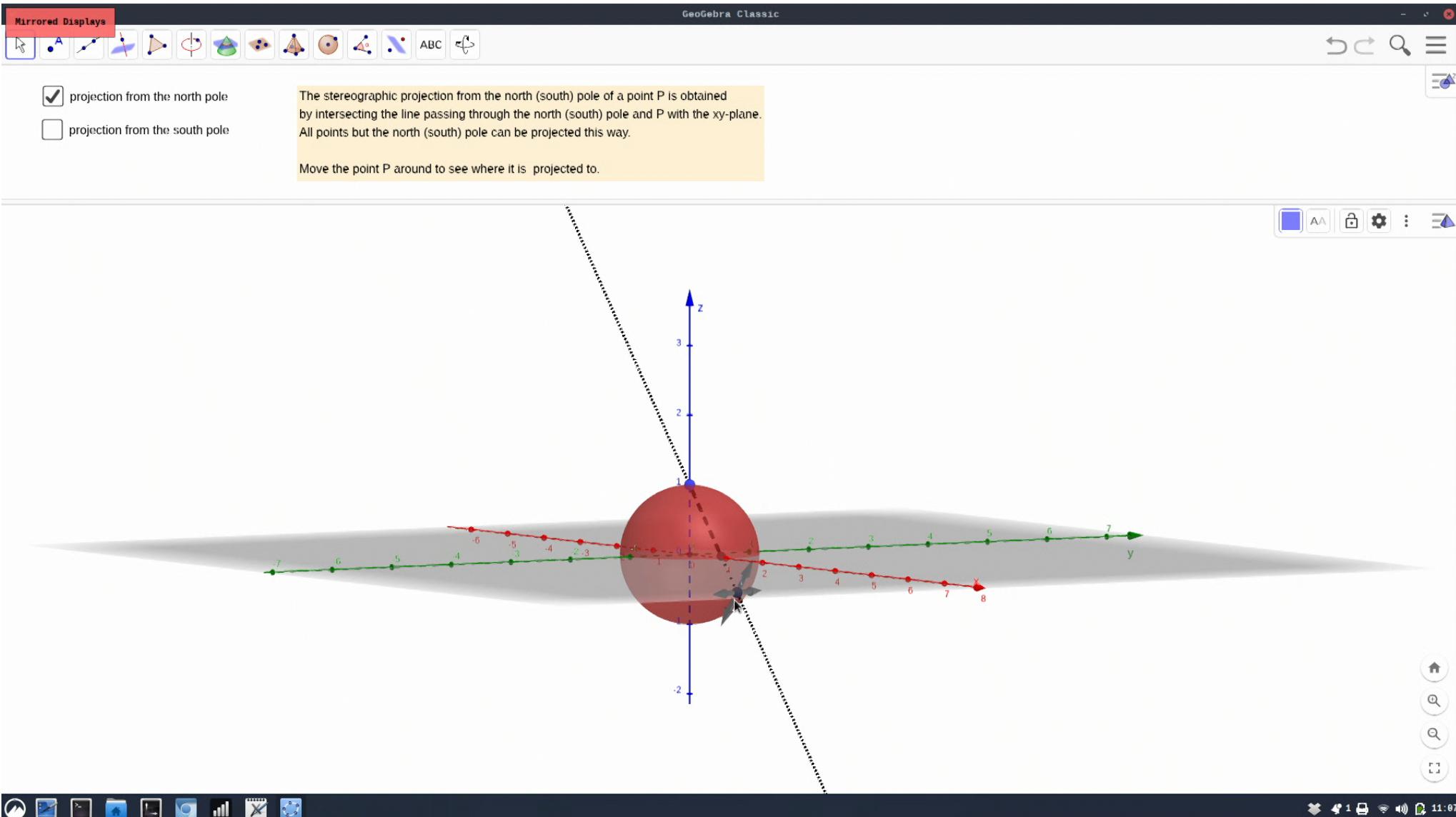
Stereographic projections

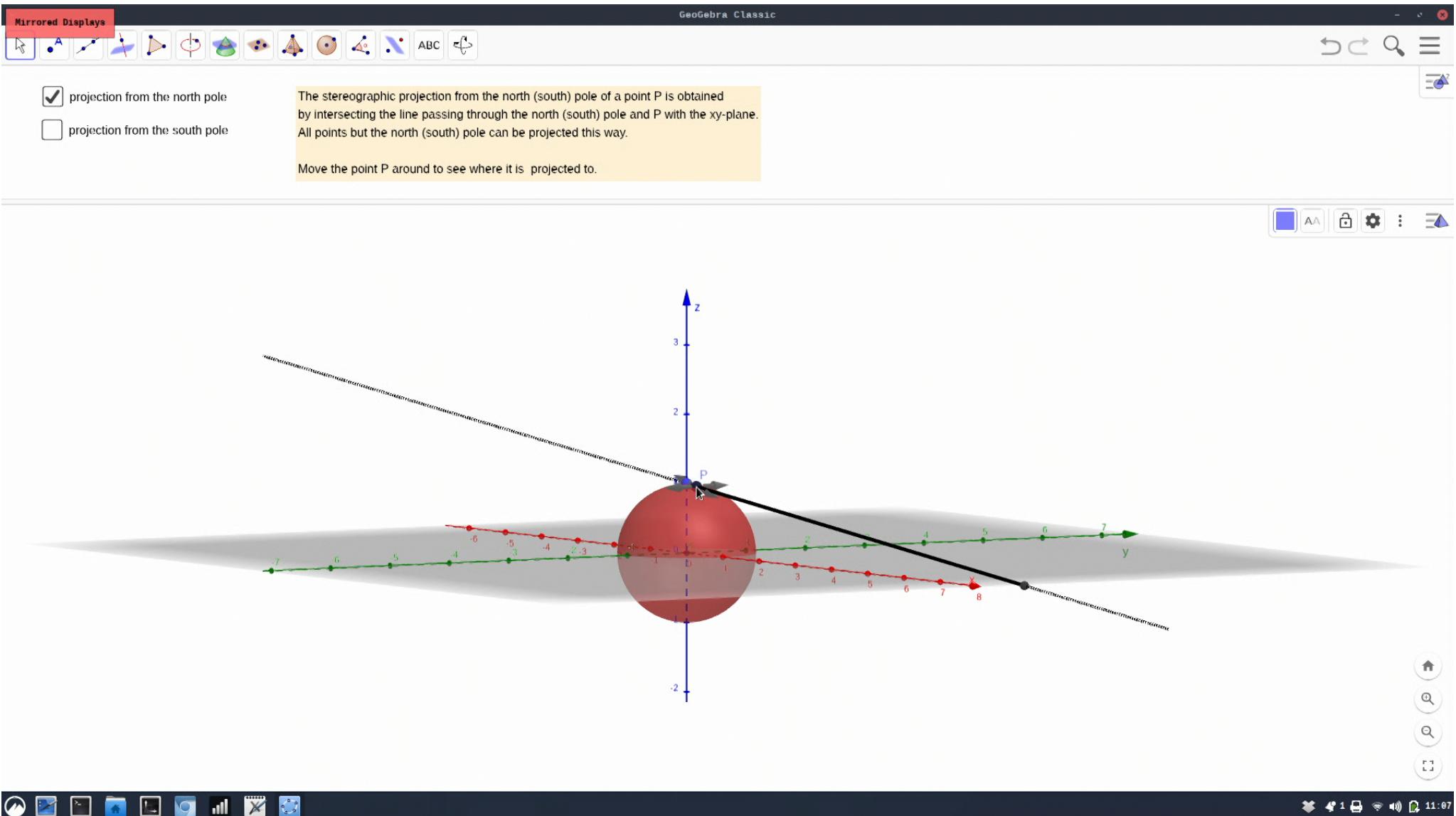
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Page 7 of 11 Layer Layer 1

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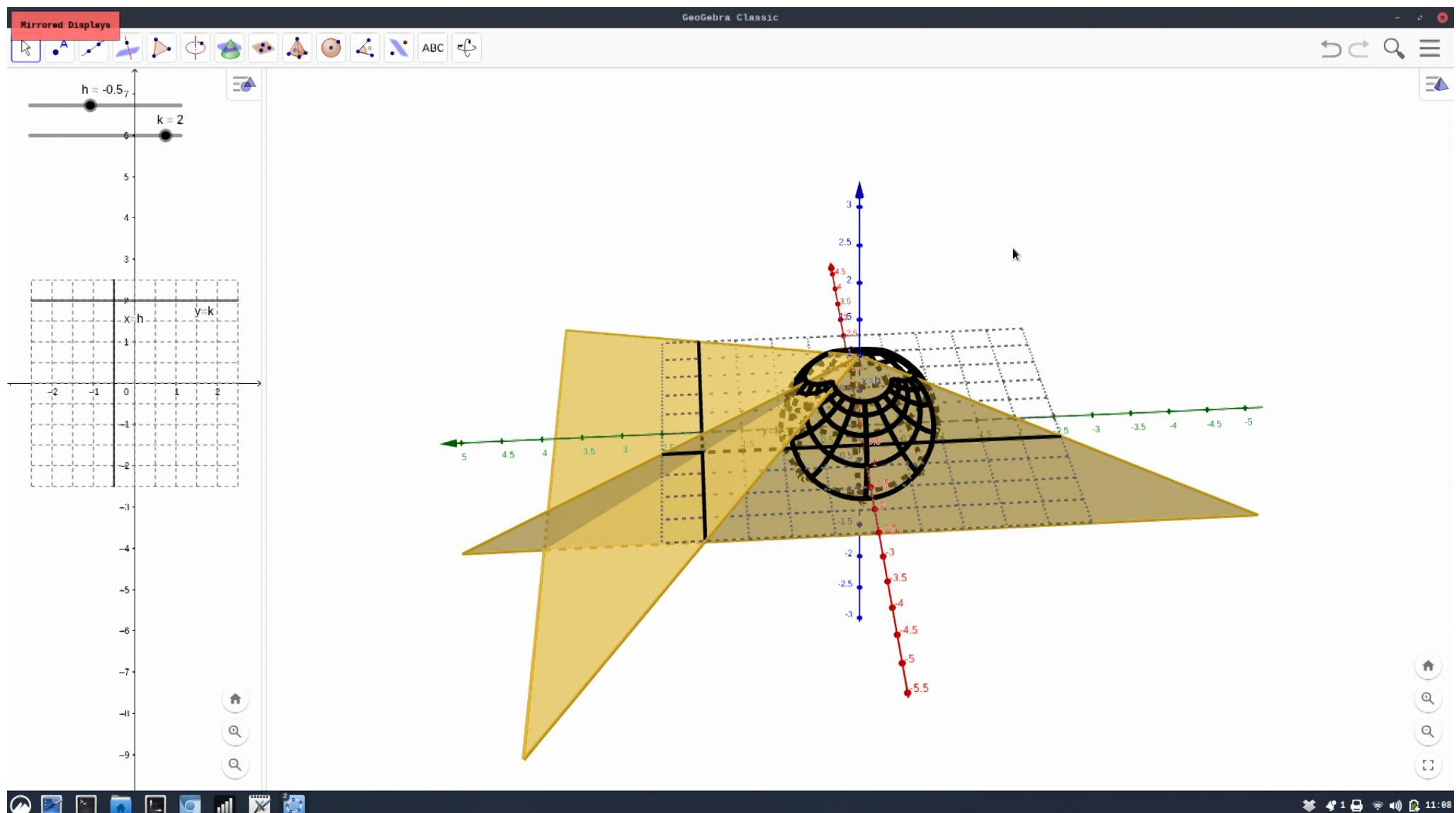
Stereographic projections

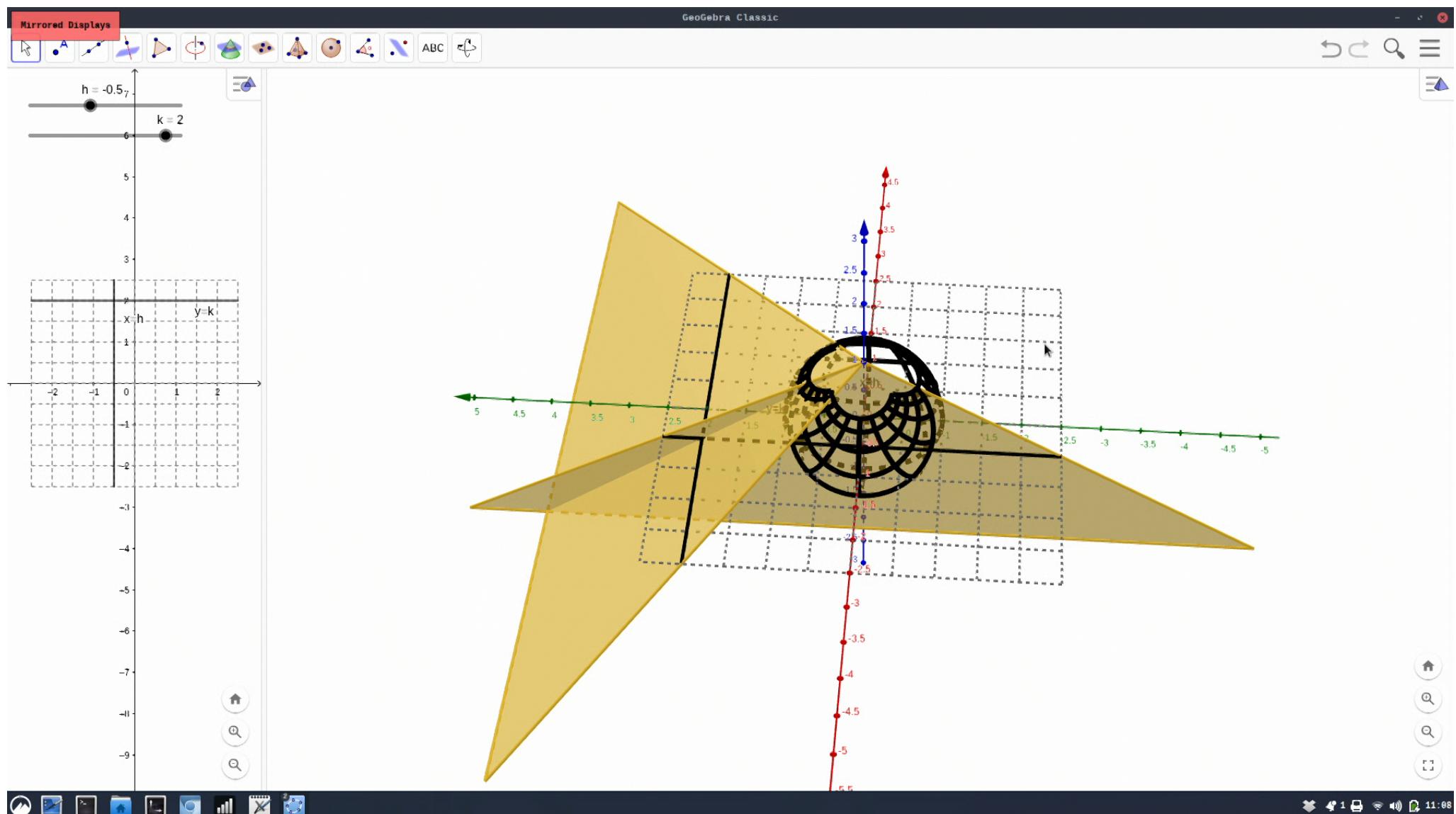
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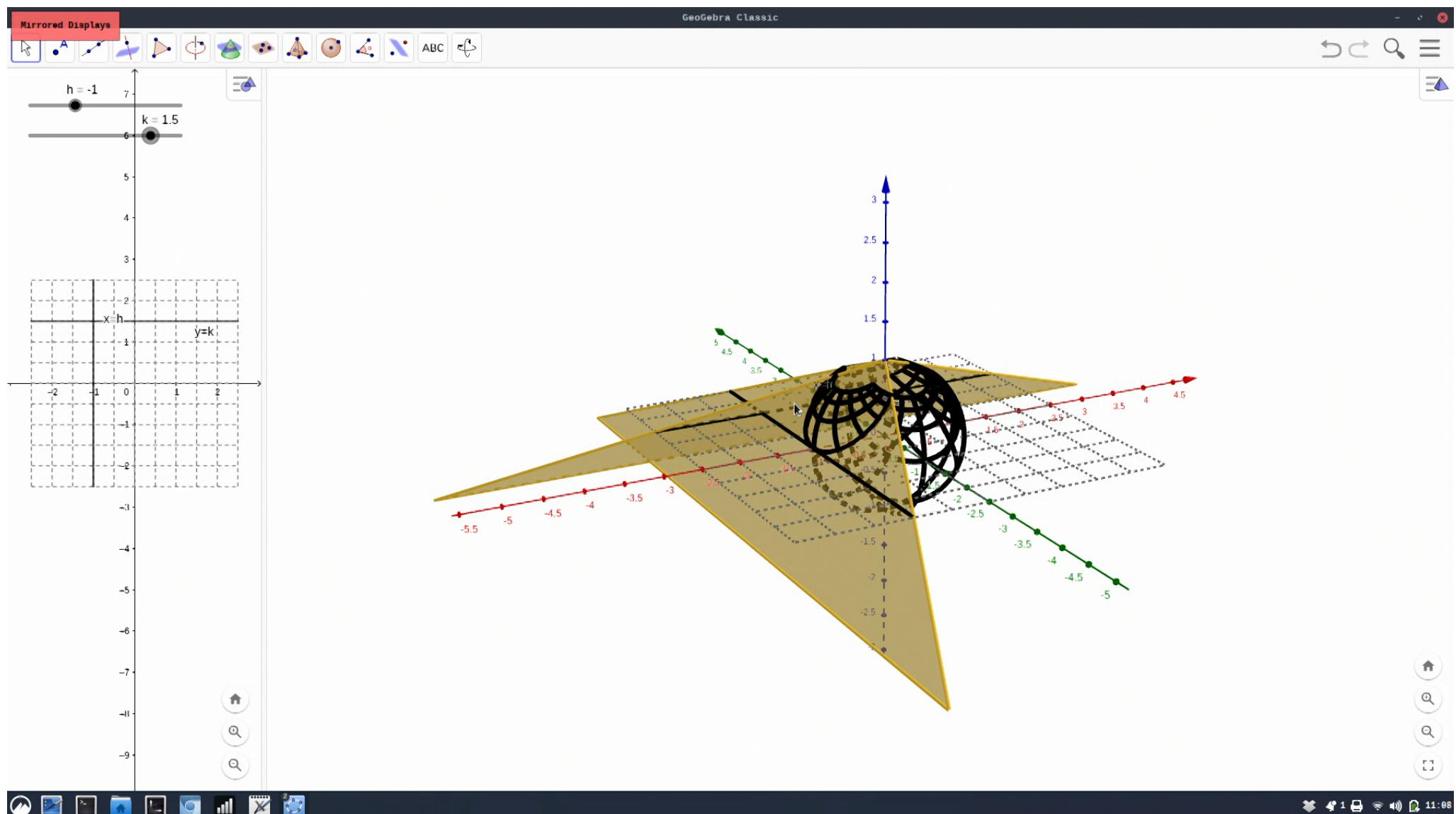
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Page 7 of 11 Layer Layer 1 312% 11:07







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③ Projective plane  $\rightarrow$  set of all lines through origin in  $\mathbb{R}^3$

$$\mathbb{P}_2(\mathbb{R}) = \left\{ [(x_1, x_2, x_3)] \mid (x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\} \right\}$$
$$\vec{x}' \sim \vec{x} \iff \vec{x}' = \lambda \vec{x}, \quad \lambda \neq 0$$

charts:  $(U_i, \varphi_i)$   $i=1, 2, 3$

Page 7 of 11 Layer Layer 1

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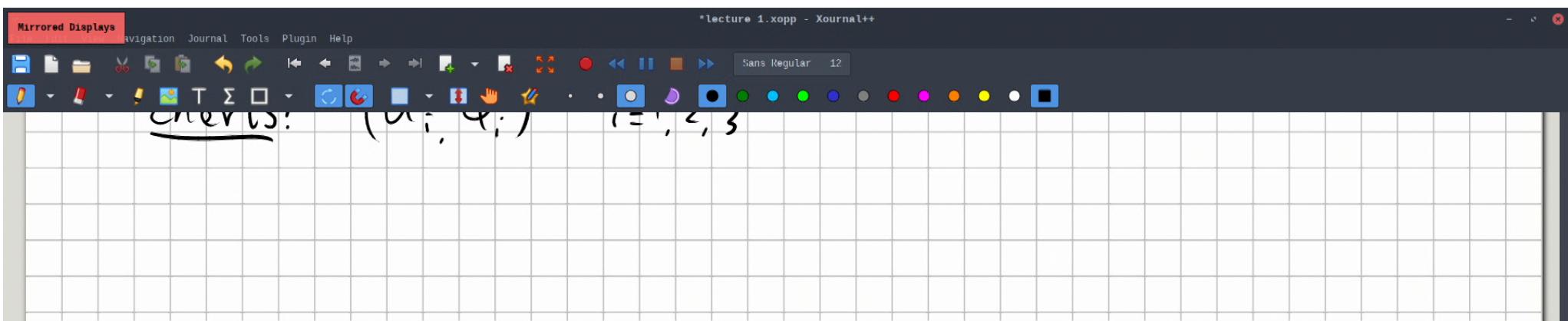
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Page 7 of 11 Layer Layer 1

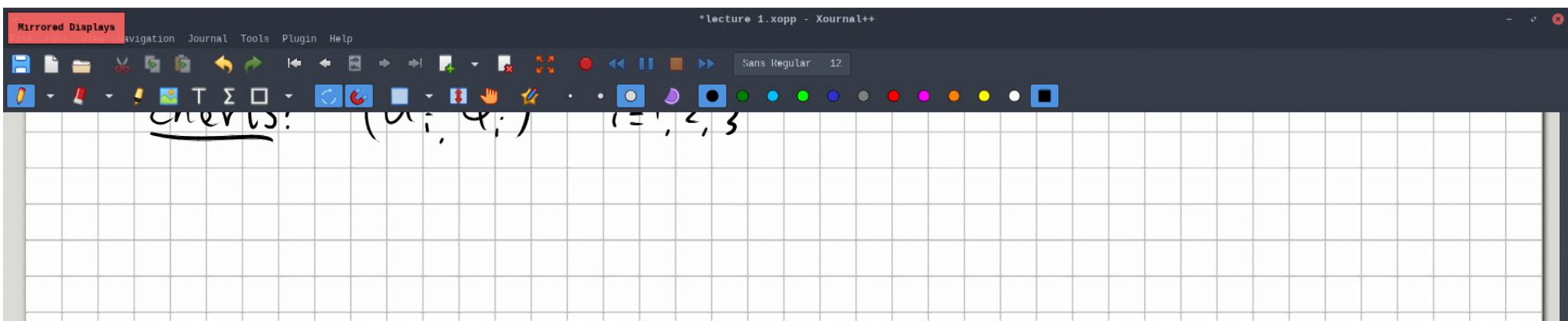
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$$U_i = \{ \bar{l}(x_1, x_2, x_3) \mid x_i \neq 0 \}$$

$$\varphi_i : [(x, y, z)] \mapsto \left( \frac{y}{x}, \frac{z}{x} \right)$$





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$$U_i = \{ [x_1, x_2, x_3] \mid x_i \neq 0 \}$$
$$\varphi_1: [(x, y, z)] \in U_1 \mapsto \left( \frac{y}{x}, \frac{z}{x} \right) \in \mathbb{R}^2$$
$$\varphi_2: [(x, y, z)] \in U_2 \mapsto \left( \frac{x}{y}, \frac{z}{y} \right) \in \mathbb{R}^2$$
$$\varphi_3: [(x, y, z)] \in U_3 \mapsto \left( \frac{x}{z}, \frac{y}{z} \right) \in \mathbb{R}^2$$

Page 11 of 11 Layer Layer 1

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