Title: Classical and Quantum Chaos 2021/2022 - Lecture 13

Speakers: Meenu Kumari

Collection: Classical and Quantum Chaos 2021/2022

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Out-of-time-ordered Correlator (OTOC)

OTOC is defined as

$$F(t) = \mathrm{Tr}\left(\rho\left(\hat{W}^{\dagger}(t)\,\hat{V}^{\dagger}\,\hat{W}(t)\,\hat{V}\right)\right)$$

$$= \left\langle \hat{W}^{\dagger}(t)\,\hat{V}^{\dagger}\,\hat{W}(t)\,\hat{V} \right\rangle_{\rho}$$

where $\hat{W}(t) = e^{iHt}\,\hat{W}\,e^{-iHt}$

$$[\hat{W}(0), \hat{V}] = 0$$

* If $\hat{V}$ and $\hat{W}$ are unitary

$$Re\left(F(t)\right) = 1 - \frac{\left\langle [\hat{W}(t),\hat{V}]^{\dagger}[\hat{W}(t),\hat{V}] \right\rangle_{\rho}}{2}$$

$$\left([\hat{W}(t),\hat{V}]^{\dagger}[\hat{W}(t),\hat{V}] = 2\,\mathbb{1} - \left((\hat{W}^{\dagger}(t)\hat{V}^{\dagger}\hat{W}(t)\hat{V})^{\dagger} + W^{\dagger}(t)\hat{V}\hat{W}(t)\hat{V} \right) \right.$$

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Usually, $\rho = \frac{\mathbb{1}}{d}$ (infinite Temperature thermal state)

Scrambling $\equiv$ exponential growth of $\mathrm{Re}(F(t))$

* The rate of exponential growth is often referred as "quantum Lyapunov exponent"

Heuristic connection with classical LE

$$\hat{C}(t) = -[\hat{x}(t), \hat{p}(0)]^2$$

$$[\hat{x}(t), \hat{p}(0)] = i\hbar \frac{\partial \hat{x}(t)}{\partial \hat{x}(0)}$$

ent"

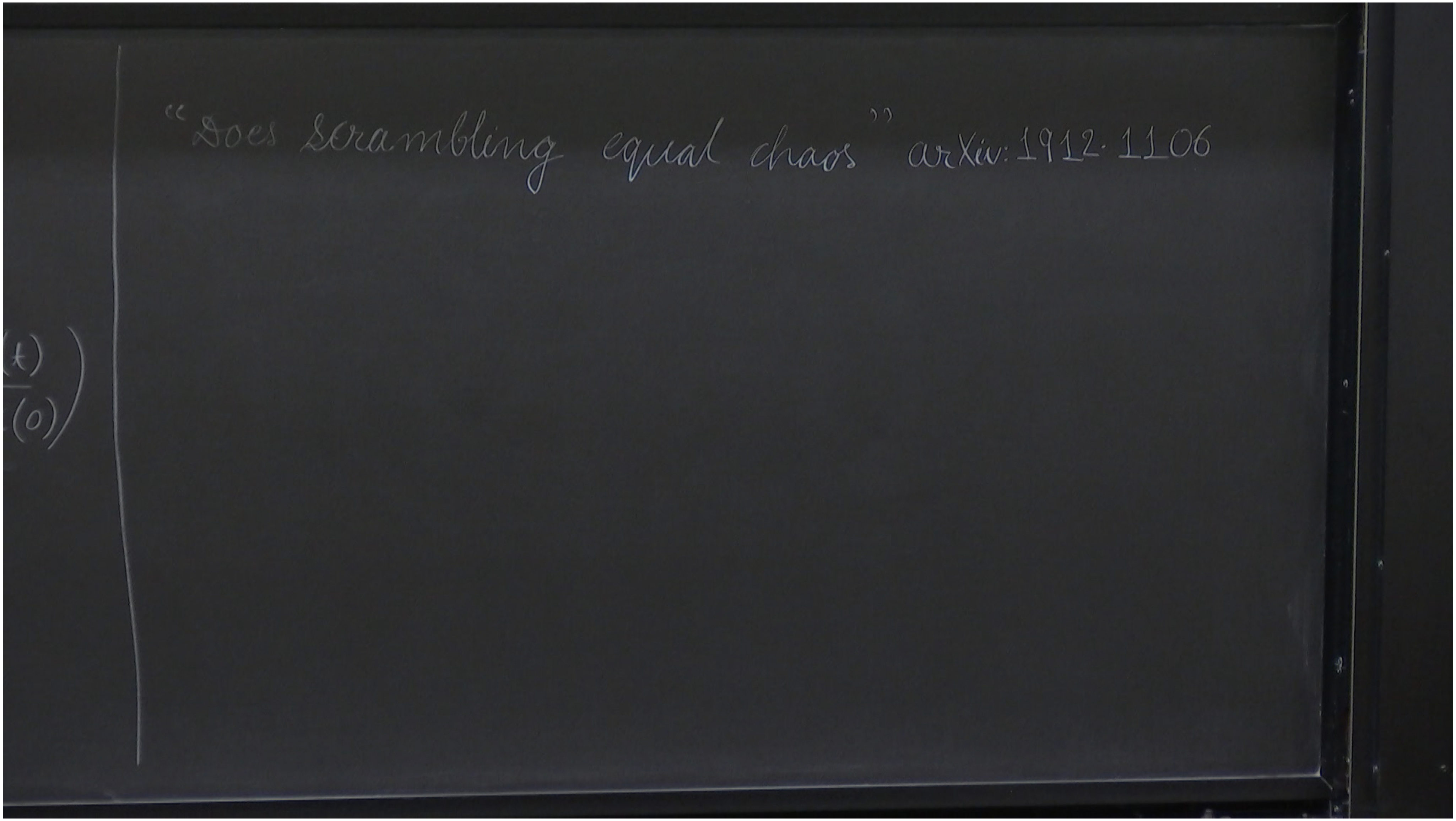
Exp. value w.r.t. $|\Psi\rangle$

$$C(t) = \hbar^2 \left\langle \left(\frac{\partial \hat{x}(t)}{\partial \hat{x}(0)} \right)^2 \right\rangle_{|\Psi\rangle}$$

Classically, $\left(\frac{\partial \hat{x}(t)}{\partial \hat{x}(0)} \right) \approx \left(\frac{\Delta x(t)}{\Delta x(0)} \right)$

$$\left(\frac{\Delta x(t)}{\Delta x(0)} \right)^2 \propto \exp(2\lambda t)$$

$$C(t) \propto \exp(2 \tilde{\lambda}_q t)$$


"Does scrambling equal chaos" arXiv:1912.11106

$$\frac{x(t)}{x(0)}\Bigg)$$

"Does scrambling equal chaos" arXiv: 1912.1106

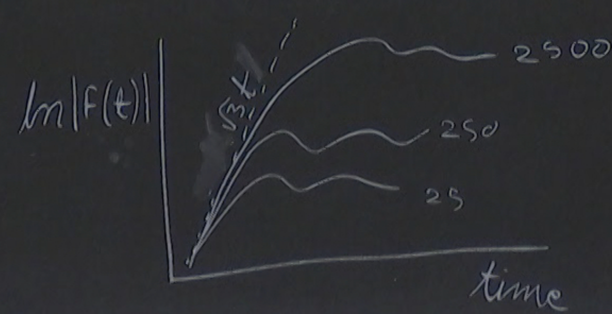
LMG model

$$H_{LMG} = -\frac{1}{2j}\left(\gamma_x J_x^2 + \gamma_y J_y^2\right) - h J_z$$

$$\lambda_{saddle} = \sqrt{3}$$

$$\lambda_{OTOC} = \sqrt{3}$$

$j = 2500, 250, 25$



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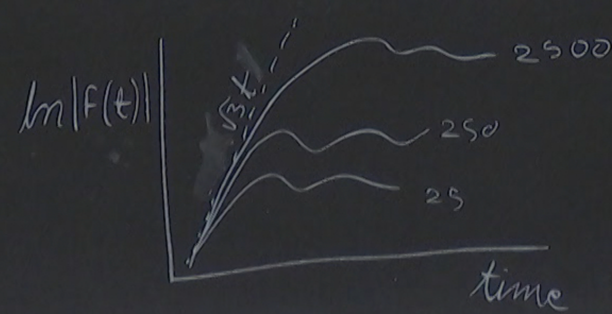


$\lambda_{OTOC} \geq \lambda_{saddle}$

Out-of-time-ordered correlators in quantum many-body systems

Cheng-Ju (Jacob) Lin
PSI lecture, Mar. 29

Reference: arXiv:2202.07060, Scrambling Dynamics and Out-of-Time Ordered Correlators in Quantum Many-Body Systems: a Tutorial, Shenglong Xu and Brian Swingle

Setup: qubit (spin-1/2) system

N qubits

Pauli $= \left\{ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

Starting with some local operator $\hat{W}_{r=0}(t=0) = \sigma_0^x$

Consider the Heisenberg evolution $\sigma_0^x(t) = U^\dagger(t)\sigma_0^x U(t)$

Schematically, $\hat{W}_0(t) = \sum_{\mathcal{S}} a_{\mathcal{S}}(t)\hat{\mathcal{S}}$

$\mathcal{S} = \ldots \otimes s_{j-1} \otimes s_j \otimes \ldots$ $\qquad s_j \in$ **Pauli**

$$\hat{W}_0(t) = \sum_{\mathcal{S}} a_{\mathcal{S}}(t)\hat{\mathcal{S}}$$

$$\hat{\mathcal{S}} = \ldots \otimes s_{j-1} \otimes s_j \otimes \ldots$$

$s_j \in$ **Pauli**

Pauli-strings are "orthonormal basis"

$$\frac{\textbf{Tr}[\mathcal{S}^\dagger \mathcal{S}']}{\textbf{Tr}[I_{2^N}]} = \delta_{\mathcal{S}\mathcal{S}'}$$

Assume $$\frac{\textbf{Tr}[W_0^\dagger W_0]]}{\textbf{Tr}[I_{2^N}]} = 1$$ **then** $$\sum_{\mathcal{S}} |a_{\mathcal{S}}(t)|^2 = 1$$

$a_{\mathcal{S}}(t)$ **: amplitudes of this "operator wavefunction"**

Out-of-time-ordered commutators/correlators

$$C_{WV}(r,t) = \mathbf{tr}(\rho\,|\,[\hat{W}_{r=0}(t), \hat{V}_r]\,|^2)$$

Let's consider the special case $\rho = \frac{I}{2^N}$ **and** $\hat{V}_r$ **being one-of the Pauli matrices**

$$\hat{W}_0(t) = \sum_{\mathcal{S}} a_{\mathcal{S}}(t)\hat{\mathcal{S}}$$

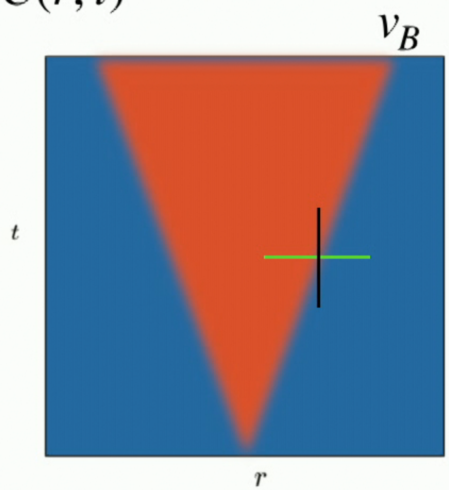
$$C_{WV}(r,t) = \sum_{\mathcal{S}:[\mathcal{S},V_r]\neq 0} |\,a_{\mathcal{S}}(t)\,|^2$$

A proxy of the "size" of the operator $\hat{W}_0(t)$

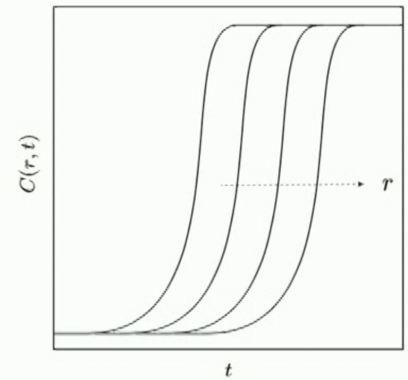
Average over Vr in Pauli, $C(r,t) = 2\sum_{\mathcal{S}:s_r\neq I} |\,a_{\mathcal{S}}(t)\,|^2$

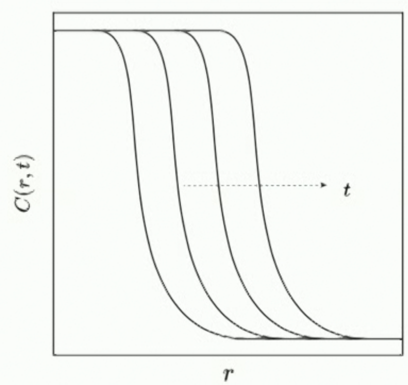
What to expect for a quantum many-body chaotic system, where the dynamics is generated with local Hamiltonian or local circuit?

$C(r,t)$



Blue 0 < - > red 1





1. The operator grows ballistically, with the butterfly velocity v_B .

2. Inside the "butterfly cone", C(r,t)~1.

A lot of the detailed (asymptotic) behavior will be model dependent.

Random unitary circuit model

A. Nahum, S. Vijay, and J. Haah, Operator spreading in random unitary circuits, Physical Review X 8, 021014 (2018).

C. Von Keyserlingk, T. Rakovszky, F. Pollmann, and S. L. Sondhi, Operator hydrodynamics, otocs, and entanglement growth in systems without conservation laws, Physical Review X 8, 021013 (2018).

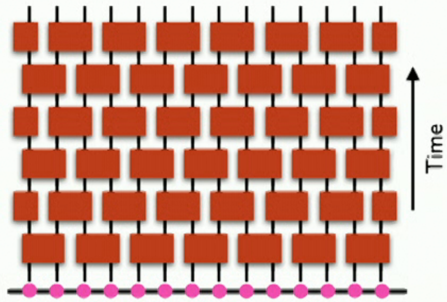
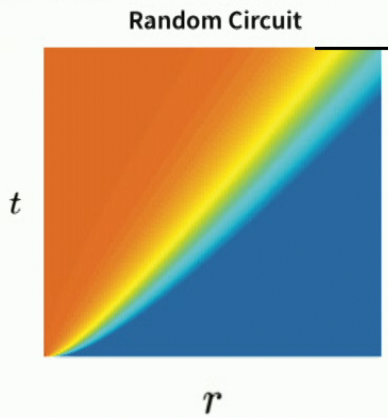


FIG. 1. Left: Random unitary circuit in $1+1$D. Each brick represents an independently Haar-random unitary, acting on the Hilbert space of two adjacent "spins" of local Hilbert space dimension q.

The dynamics of the "end-point density" of the operator can be described by a biased random walk equation.

$$\partial_t \overline{\rho(x,t)} = v_B(q)\partial_x \overline{\rho(x,t)} + D(q)\partial_x^2 \overline{\rho(x,t)}$$

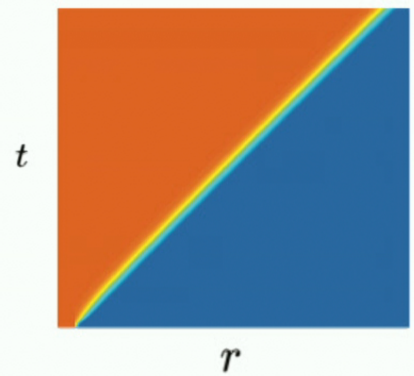
$$P_{\text{end}}(r,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{(r-v_B t)^2}{4Dt}\right)$$



Broadening operator wavefront

$$C(r,t) \approx 1 + \text{erf}\left(-\frac{(r-v_B t)}{\sqrt{4Dt}}\right)$$

Large-N/ semi-classical



SYK model/Coupled-SYK model

S. Sachdev and J. Ye, Gapless spin-fluid ground state in a random quantum Heisenberg magnet, Phys. Rev. Lett. 70, 3339 (1993)

A. Kitaev, A simple model of quantum holography, in KITP Program: Entanglement in Strongly-Correlated Quantum Matter (2015)

D. Chowdhury, A. Georges, O. Parcollet, and S. Sachdev, Sachdev-ye-kitaev models and beyond: A window into non-fermi liquids, arXiv preprint arXiv:2109.05037 (2021)

Interacting Fermionic model (implicitly large-N)

I. L. Aleiner, L. Faoro, and L. B. Ioffe, Microscopic model of quantum butterfly effect: out-of-time-order correlators and traveling combustion waves, Annals of Physics 375, 378 (2016).

O(N) model

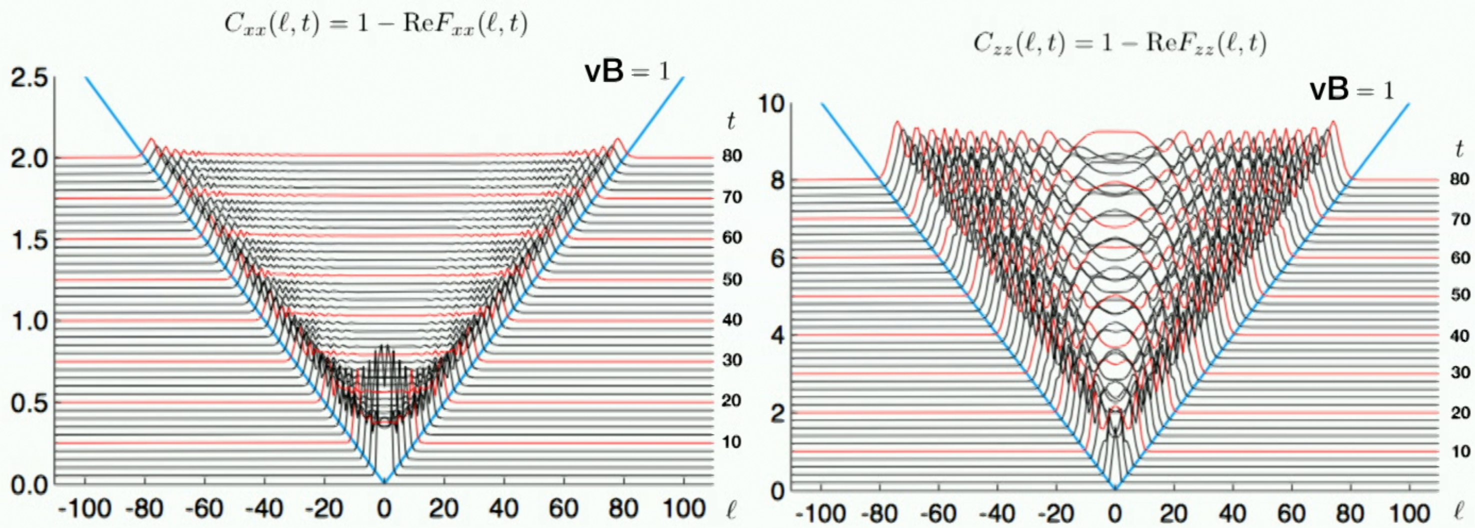
Onset of many-body chaos in the O(N) model, Debanjan Chowdhury and Brian Swingle, Phys. Rev. D **96**, 065005 (2017)

And more…..

- **Ballistic operator growth and non-broadening wavefront**

Quantum Ising model

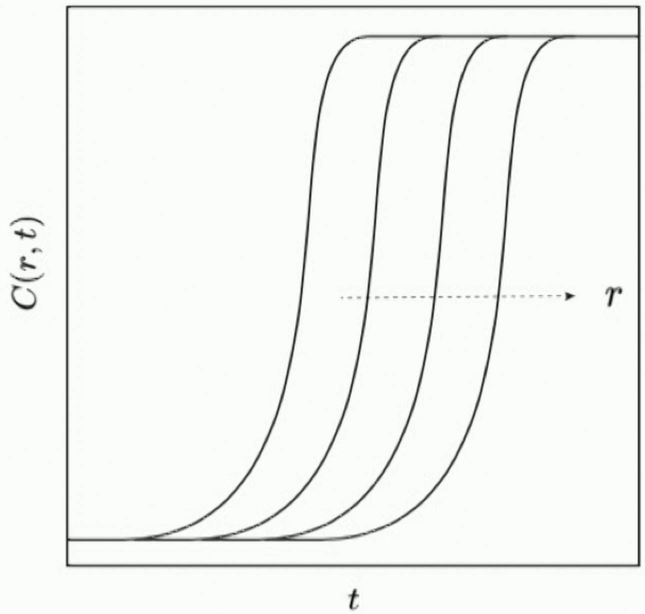
$$H = -\frac{J}{2}\left(\sum_{j=0}^{L-1} \sigma_j^z \sigma_{j+1}^z + g \sum_{j=0}^{L-1} \sigma_j^x\right)$$



Not all operator shows saturation inside the butterfly cone.

Broadening operator wavefront $\sim t^{1/3}$

Bounded/Unbounded local Hilbert space



SYK model

$$C(t) \sim \frac{1}{N} e^{\lambda_L t} \qquad 1 \ll Tt \ll \ln N$$

$$\lambda_L = 2\pi T$$

1. Need a small parameter
2. Need an asymptotically large time-window

Scrambling time $C(t_s) \approx 1$

$$t_s \approx \frac{\ln N}{\lambda_L}$$

Cannot find an asymptotically large time window to define exponential growth on lattice models with bounded local Hilbert space.

A not-very-rigorous relation to quantum information scrambling

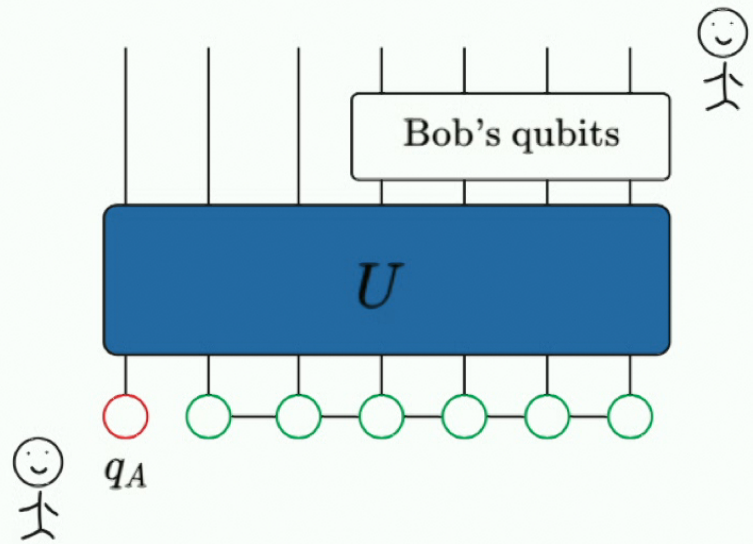


FIG. 1. Alice and Bob try to communicate through a strongly interacting system of N qubits. The time evolution of the system is described by an unitary operator U. Alice has full control of the first qubit q_A and Bob has access to a set of qubits in the system, but not all of them.

Alice want to send a bit to Bob

a = 0 Do-nothing
a = 1 Operate σ^x

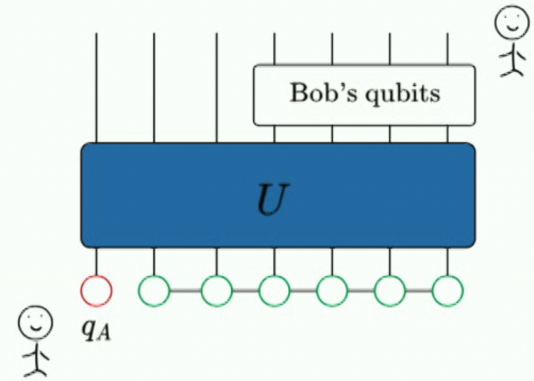
Bob needs to measure his qubits to determine if a=0 or a=1

a = 0 **Do-nothing**

$$\langle O_B \rangle_0 = \langle \psi | U^\dagger O_B U | \psi \rangle = \langle \psi(t) | O_B | \psi(t) \rangle$$

a = 1 **Operate** σ^x

$$\langle O_B \rangle_1 = \langle \psi | \sigma^x_{q_A} U^\dagger O_B U \sigma^x_{q_A} | \psi \rangle$$
$$= \langle \psi(t) | \sigma^x_{q_A}(-t) O_B \sigma^x_{q_A}(-t) | \psi(t) \rangle$$



$$\langle O_B \rangle_0 - \langle O_B \rangle_1 = \langle \psi(t) | \sigma^x(-t) [\sigma^x(-t), O_B] | \psi(t) \rangle$$

$$| \langle O_B \rangle_1 - \langle O_B \rangle_0 |^2 \le \langle \psi(t) | [\sigma^x_{q_A}(-t), O_B]^\dagger [\sigma^x_{q_A}(-t), O_B] | \psi(t) \rangle$$
$$\le \| [\sigma^x_{q_A}(-t), O_B] \|^2_\infty$$

OTOC measures when does the information reach.
Here the example is a bit "boring" since the information is classical.

- **The quantum problem: Bob needs to recover the quantum information by measuring his part of the qubits.**
- **OTOC is also a proxy of how "delocalized" the quantum information is (~how many qubits Bob has to have to recover the quantum info.) — quantum information scrambling.**

Chaos in quantum channels, JHEP, volume 2016, Article number: 4 (2016), Pavan Hosur, Xiao-Liang Qi, Daniel A. Roberts & Beni Yoshida

Holographic models

See for example,

- Localized shocks, Daniel A. Roberts, Douglas Stanford & Leonard Susskind, Journal of High Energy Physics **2015**, 51 (2015) (arXiv: 1409.8180)
- A bound on chaos, J Maldacena, SH Shenker, D Stanford - Journal of High Energy Physics, **2016**, 106 (2016) (arXiv: 1503.01409)
- A. Kitaev, A simple model of quantum holography, in KITP Program: Entanglement in Strongly-Correlated Quantum Matter (2015)

and references therein.

(Regularized) OTOC

$$F_{WV}(r,t) \equiv \mathbf{tr}(\rho^{1/2} W^\dagger(t) V^\dagger \rho^{1/2} W(t) V) \qquad \rho \sim e^{-\beta H}$$

Randomized measurement

- Probing Scrambling Using Statistical Correlations between Randomized Measurements, B. Vermersch, A. Elben, L.M. Sieberer, N.Y. Yao, and P. Zoller, Phys. Rev. X **9**, 021061
- X. Nie, Z. Zhang, X. Zhao, T. Xin, D. Lu, and J. Li, Detecting scrambling via statistical correlations between randomized measurements on an nmr quantum simula- tor, arXiv preprint arXiv:1903.12237 (2019).
- M. K. Joshi, A. Elben, B. Vermersch, T. Brydges, C. Maier, P. Zoller, R. Blatt, and C. F. Roos, Quantum information scrambling in a trapped-ion quantum simulator with tunable range interactions, Physical Review Letters 124, 240505 (2020).

$$F(\psi) = \langle\psi| W(t) |\psi\rangle \langle\psi| V^\dagger W(t) V |\psi\rangle$$

Average the initial state over Haar ensemble

$$\mathbb{E}_\psi F(\psi) = \frac{1}{2^N(2^N+1)}\mathrm{tr}(W(t))\mathrm{tr}(V^\dagger W(t)V) \;+\; \frac{1}{2^N(2^N+1)}\mathrm{tr}(W(t)V^\dagger W(t)V)$$

If choose traceless W, we get OTOC.