Title: Classical and Quantum Chaos 2021/2022 - Lecture 9

Speakers: Meenu Kumari

Collection: Classical and Quantum Chaos 2021/2022

Date: March 18, 2022 - 9:00 AM

URL: https://pirsa.org/22030053

Perturbed Hamiltonian -

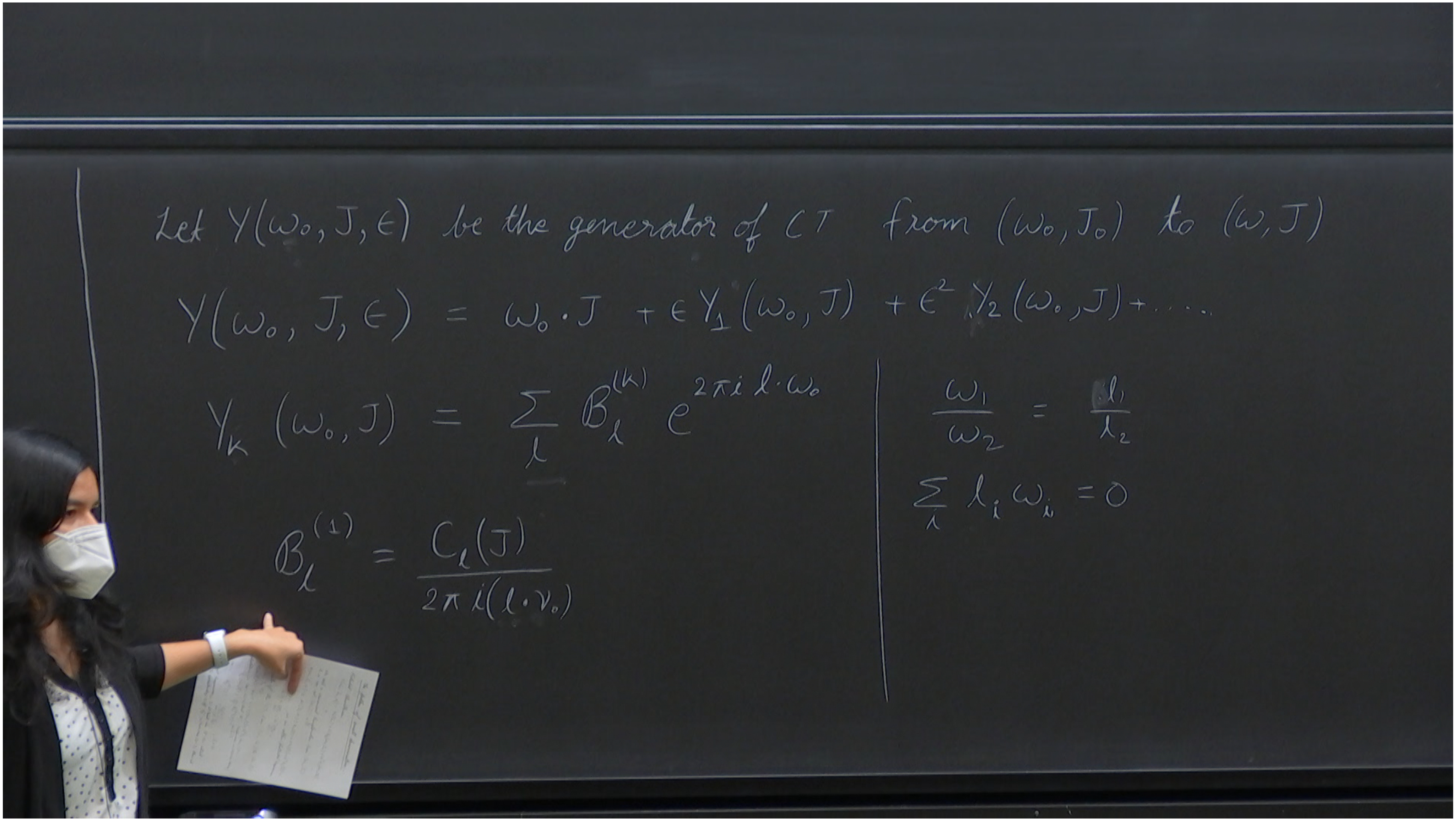
$$H(\omega_0, J_0, \epsilon) = H_0(J_0) + \epsilon H_1(\omega_0, J_0) + \epsilon^2 H_2(\omega_0, J_0) + \dots$$

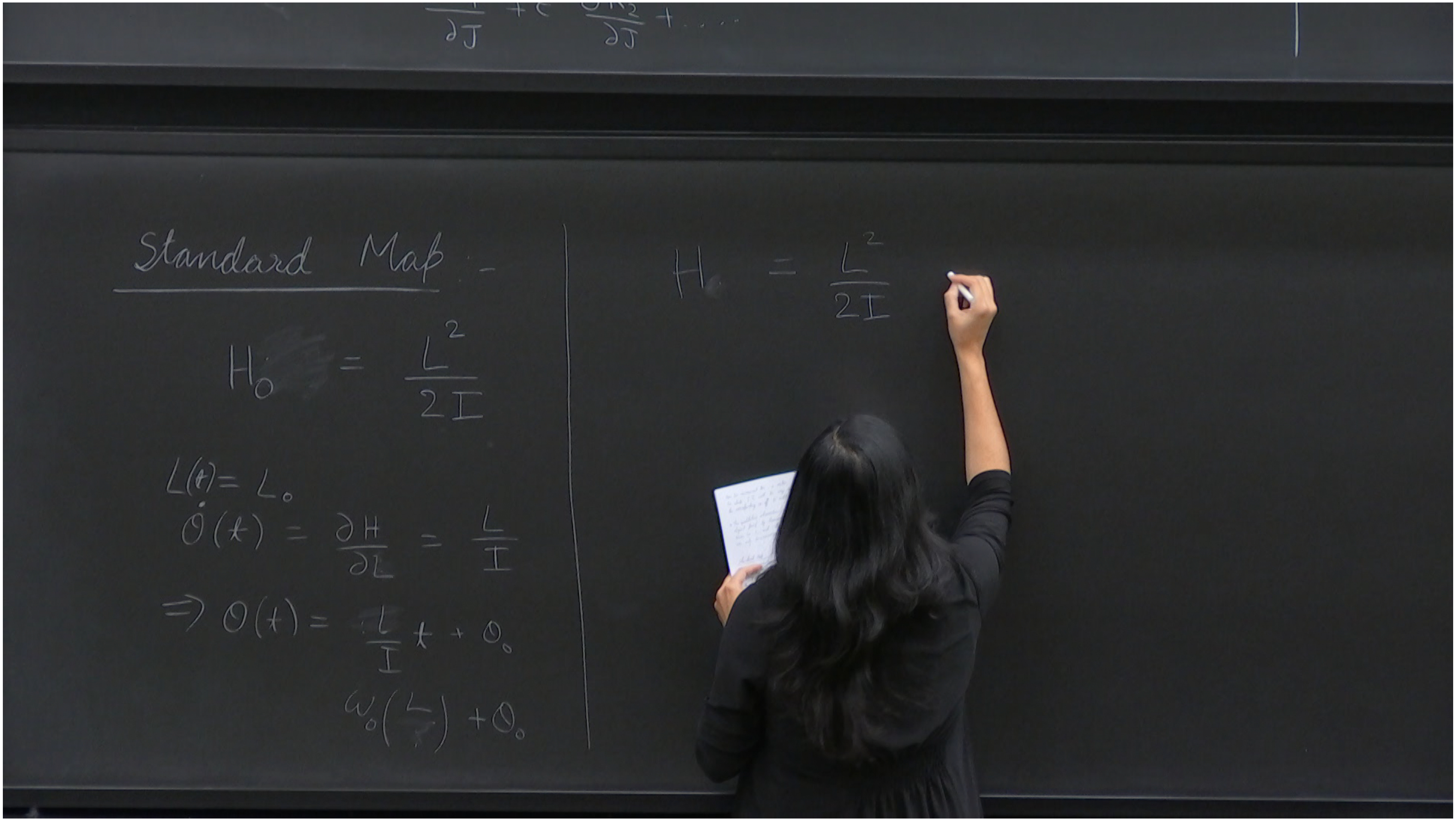
We seek a canonical transformation from (ω_0, J_0) to a new set (ω, J) such that

$$H(\omega_0, J_0, \epsilon) \equiv H(J, \epsilon) \equiv K_0(J) + \epsilon K_1(J) + \epsilon^2 K_2(J) + \dots$$

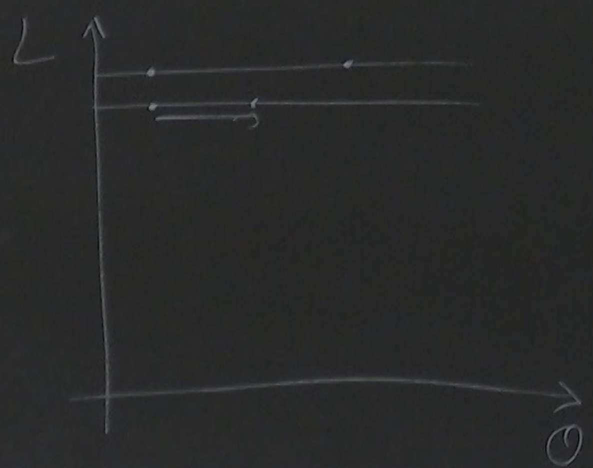
– Once we obtain $K_0, K_1, \dots$, we can obtain perturbed frequencies

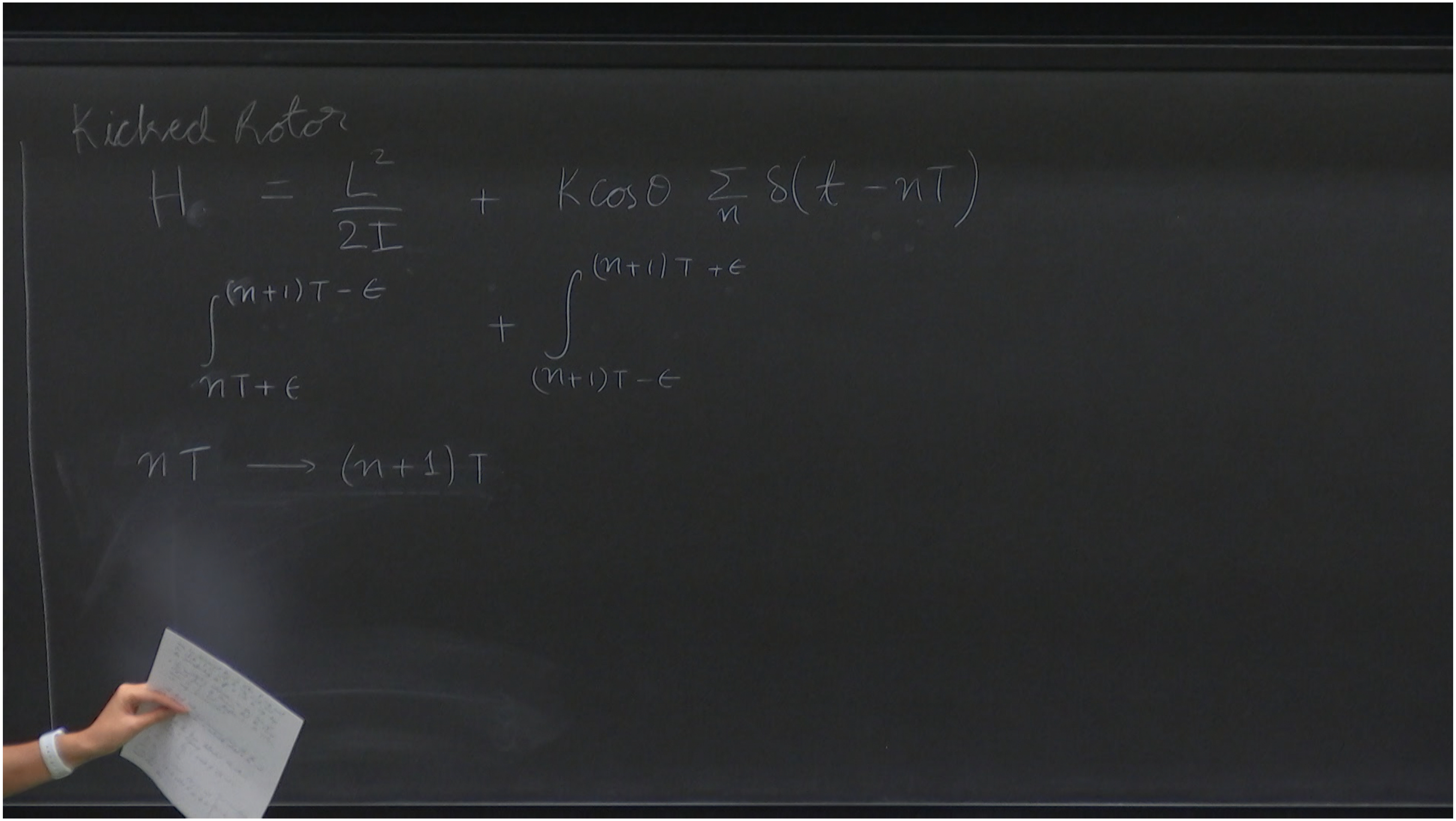
$$\nu = \nu_0 + \epsilon \frac{\partial K_1}{\partial J} + \epsilon^2 \frac{\partial K_2}{\partial J} + \dots$$


Let $Y(\omega_0, J, \epsilon)$ be the generator of CT from (ω_0, J_0) to (ω, J)
$Y(\omega_0, J, \epsilon) = \omega_0 \cdot J + \epsilon Y_1(\omega_0, J) + \epsilon^2 Y_2(\omega_0, J) + \ldots$
$Y_k(\omega_0, J) = \sum_l B_l^{(k)} e^{2\pi i l \cdot \omega_0}$
$B_l^{(1)} = \frac{C_l(J)}{2\pi i(l \cdot \nu_0)}$
$\frac{\omega_1}{\omega_2} = \frac{l_1}{l_2}$
$\sum_i l_i \omega_i = 0$


Standard Map
H₀ = L²/2I
L(t) = L₀
θ̇(t) = ∂H/∂L = L/I
⇒ θ(t) = (L/I) t + θ₀
ω₀(L) + θ₀
H = L²/2I

$$H = \frac{L^2}{2I} + K\cos\theta \sum_n \delta(t - nT)$$




Kicked Rotor
$H = \frac{L^2}{2I} + K\cos\theta \sum_n \delta(t - nT)$
$\int_{nT+\epsilon}^{(n+1)T-\epsilon} + \int_{(n+1)T-\epsilon}^{(n+1)T+\epsilon}$
$nT \longrightarrow (n+1)T$

Kicked Rotor

$$H = \frac{L^2}{2I} + K\cos\theta \sum_n \delta(t - nT)$$

$$\int_{nT+\epsilon}^{(n+1)T-\epsilon} + \int_{(n+1)T-\epsilon}^{(n+1)T+\epsilon}$$

$$L_{n+1} = L_n + K \sin\theta_n \quad , \mod 2\pi$$

$$\theta_{n+1} = \theta_n + L_{n+1} \quad , \mod 2\pi$$

Kicked Rotor

$$H(t) = \frac{L^2}{2I} + K \cos\theta \sum_n \delta(t - nT),$$

where θ is the angle of rotation and L is the angular momentum.

K = Kicking strength

Standard Map

$$L_{n+1} = L_n + K \sin\theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$

K=0.2, (L_0,θ_0)=(0.24,4.3)

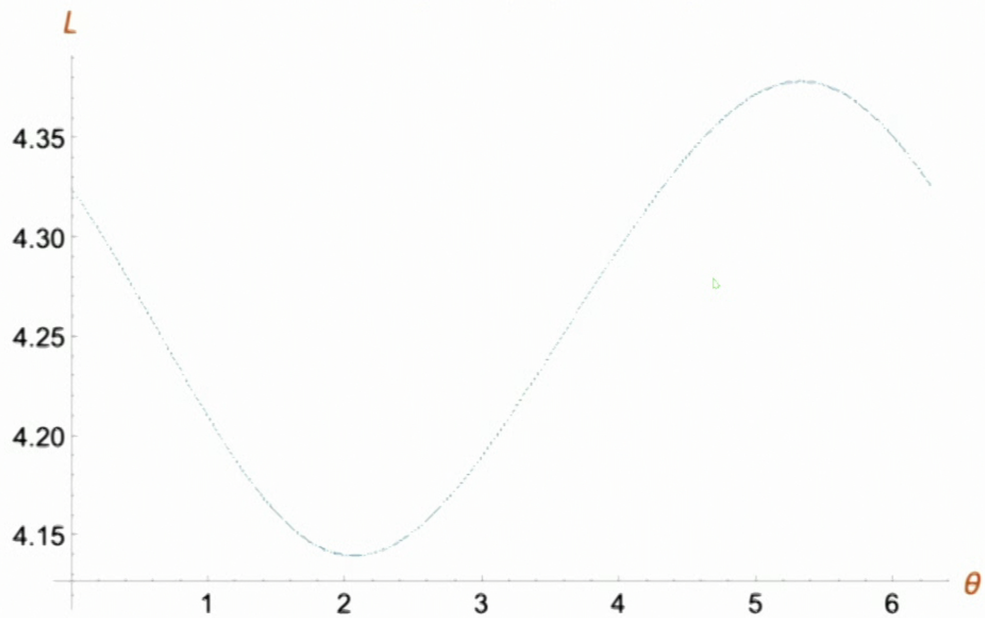


Figure: Evolution of a single initial condition

Standard Map

$$L_{n+1} = L_n + K \sin \theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$

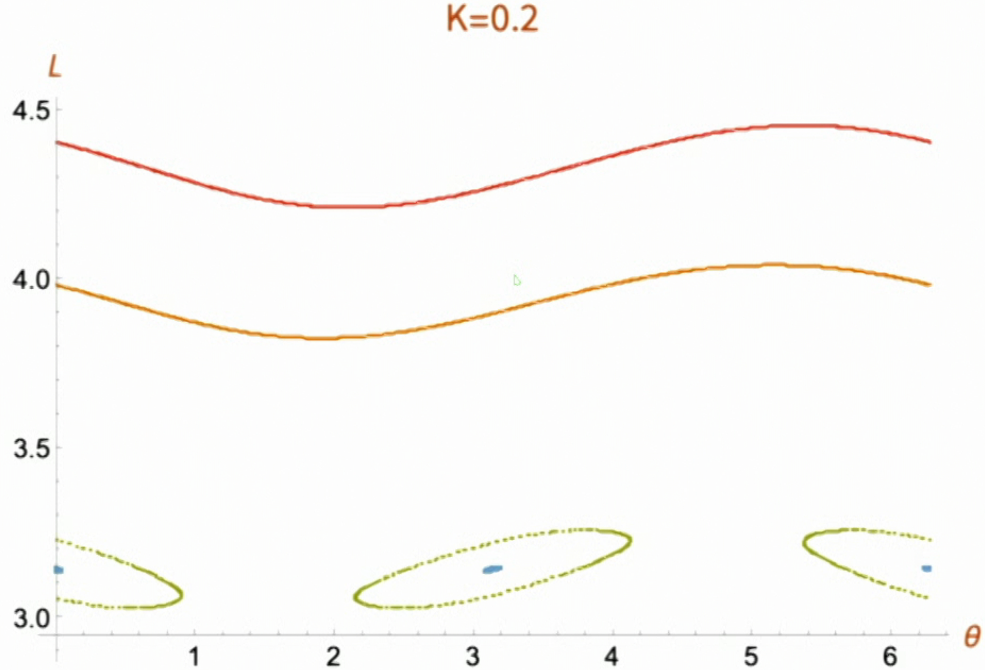
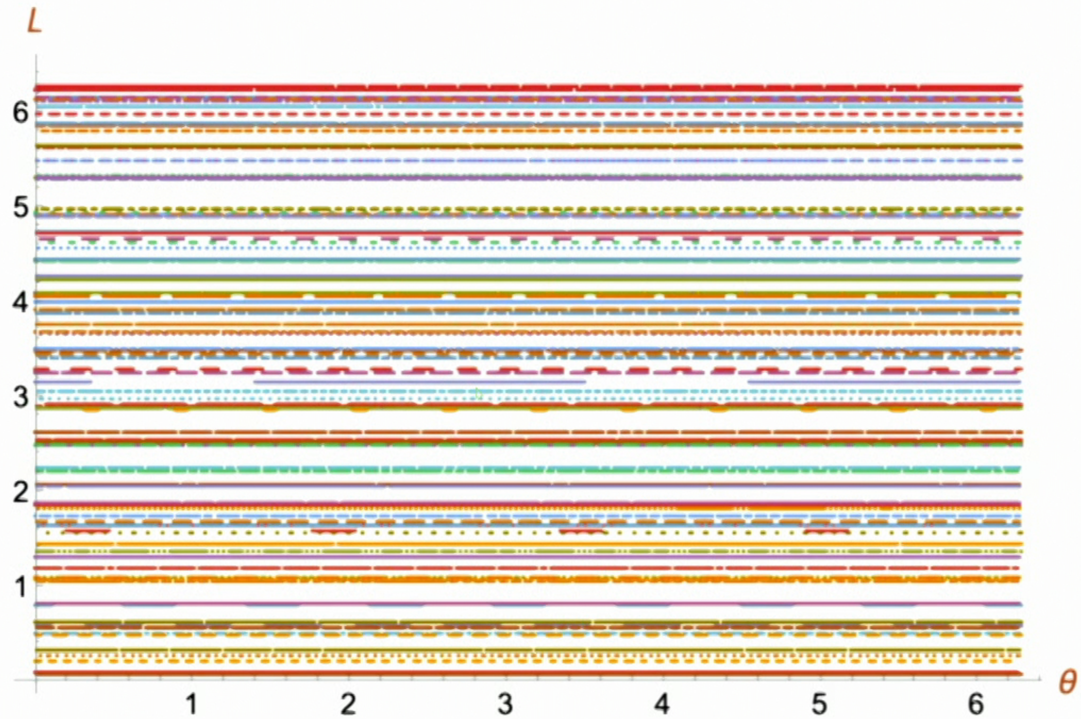


Figure: Evolution of 4 different initial conditions

Standard Map

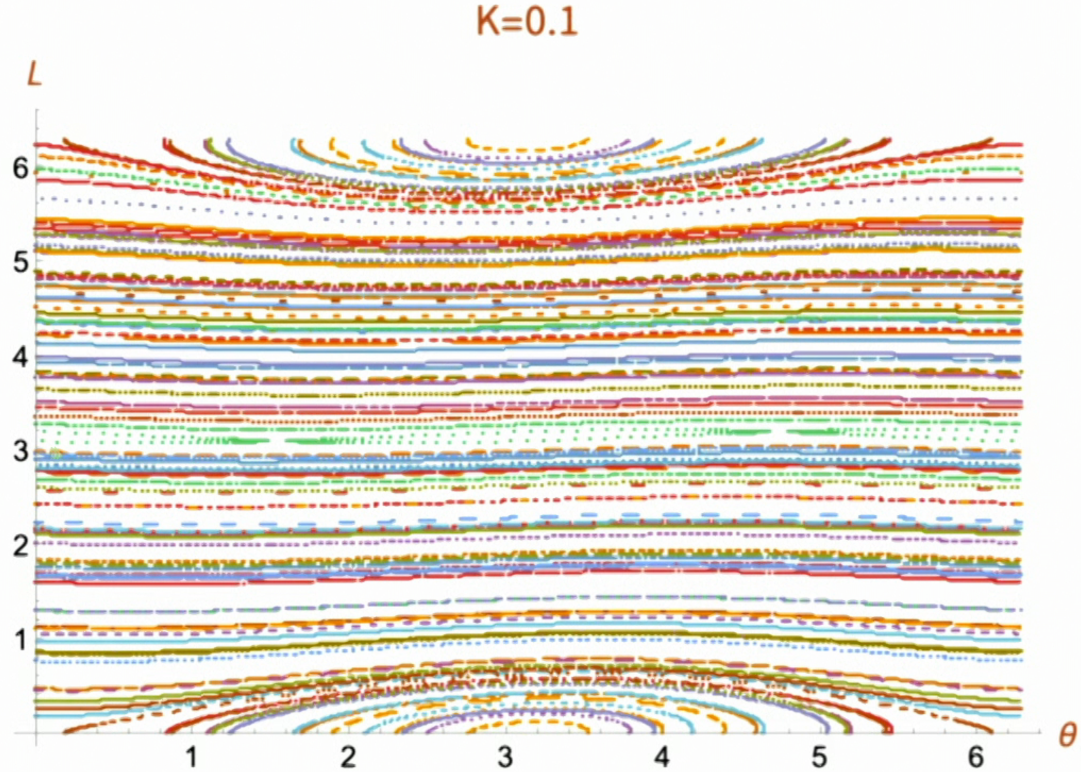
$$L_{n+1} = L_n + K \sin \theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$

K=0.



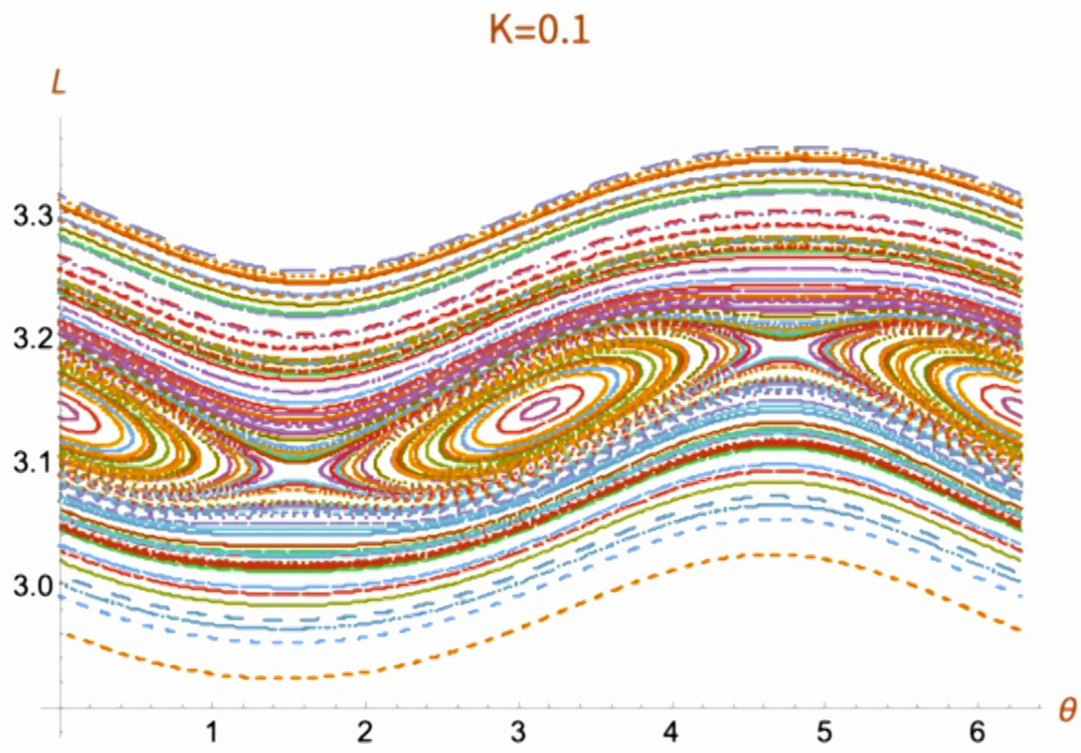
Standard Map

$$L_{n+1} = L_n + K \sin\theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$



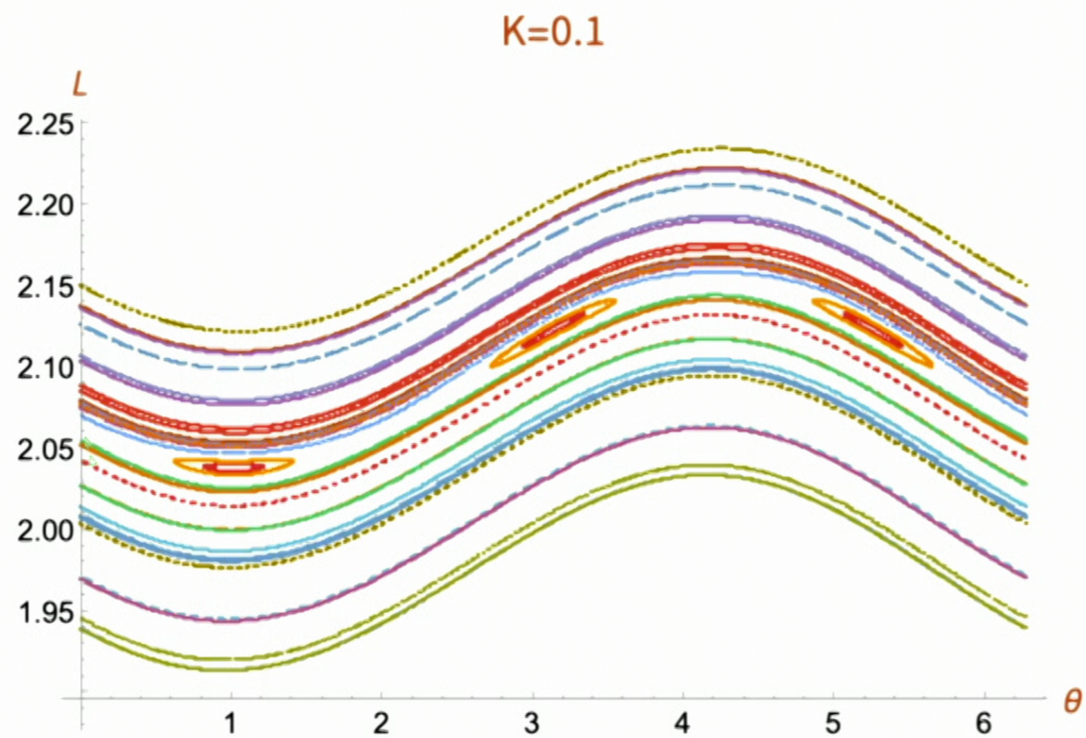
Standard Map

$$L_{n+1} = L_n + K \sin\theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$



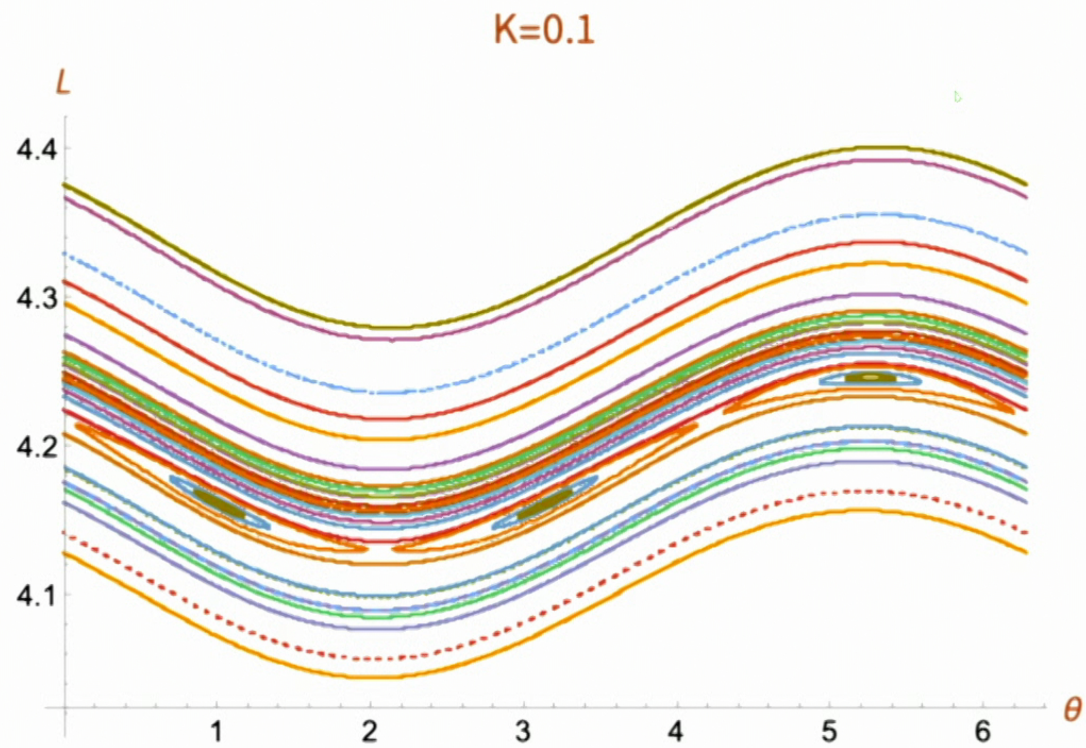
Standard Map

$$L_{n+1} = L_n + K \sin\theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$



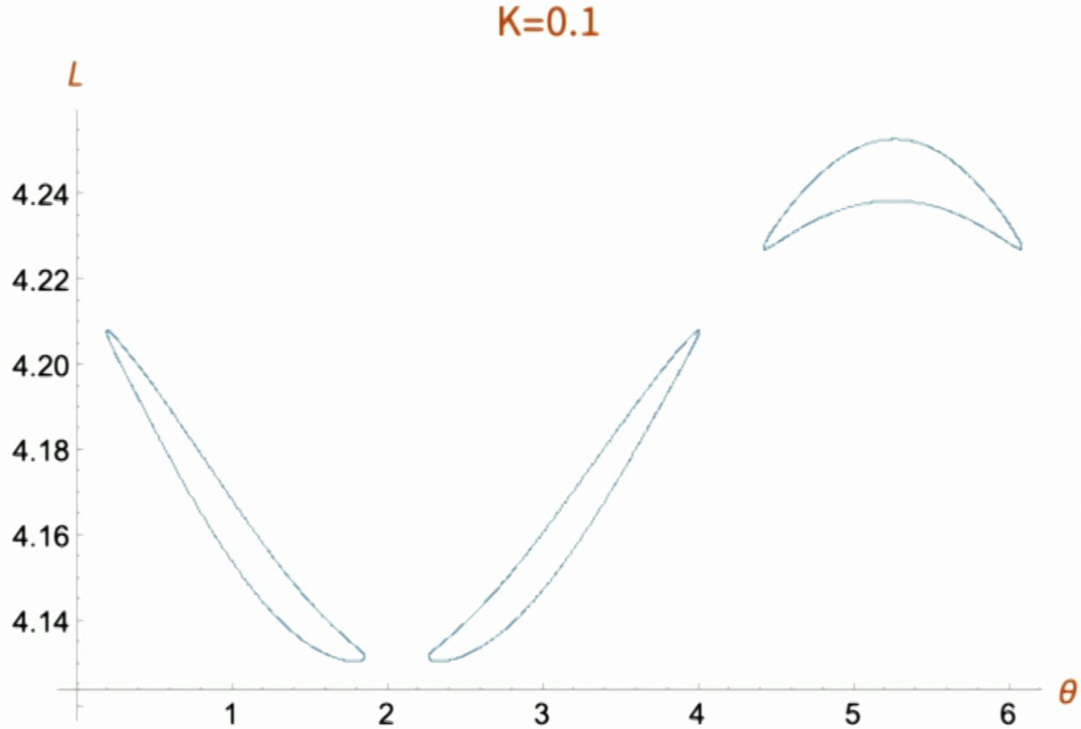
Standard Map

$$L_{n+1} = L_n + K \sin\theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$



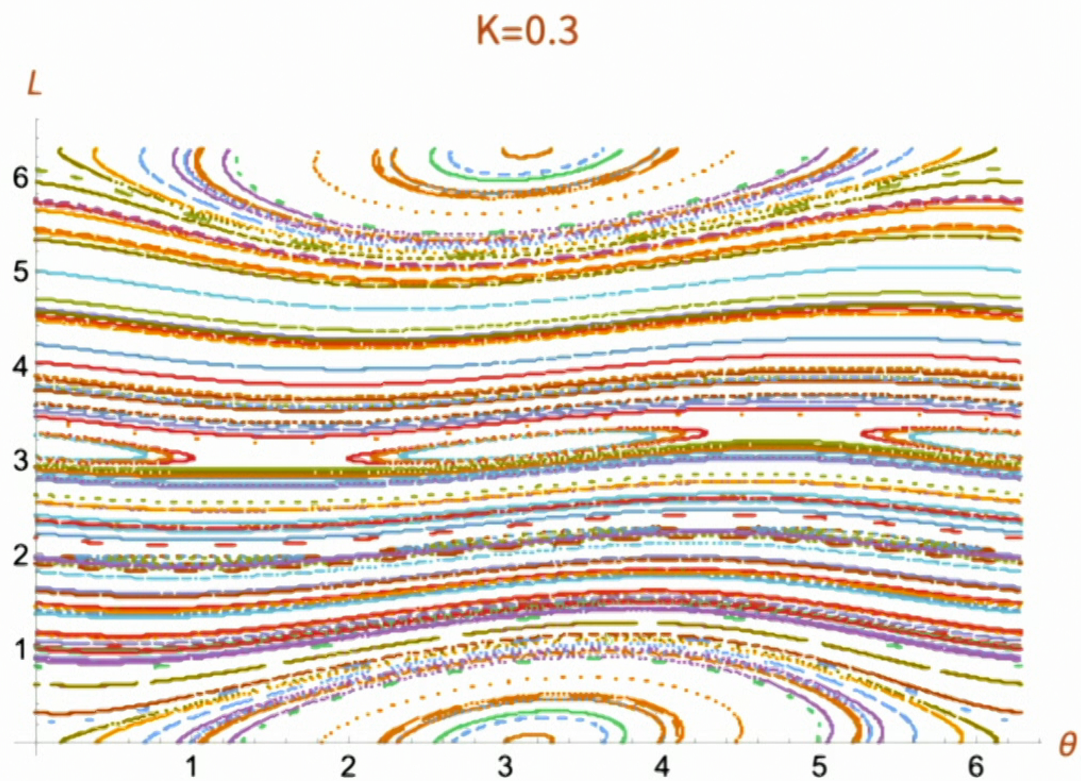
Standard Map

$$L_{n+1} = L_n + K \sin\theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$



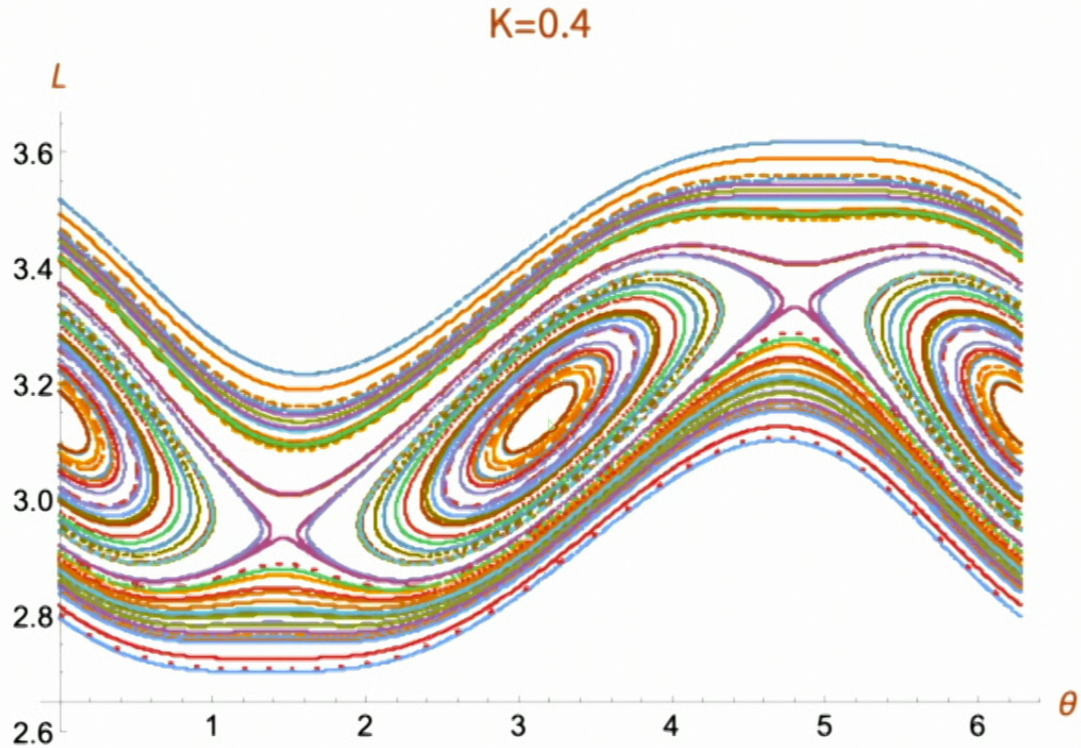
Standard Map

$$L_{n+1} = L_n + K \sin\theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$



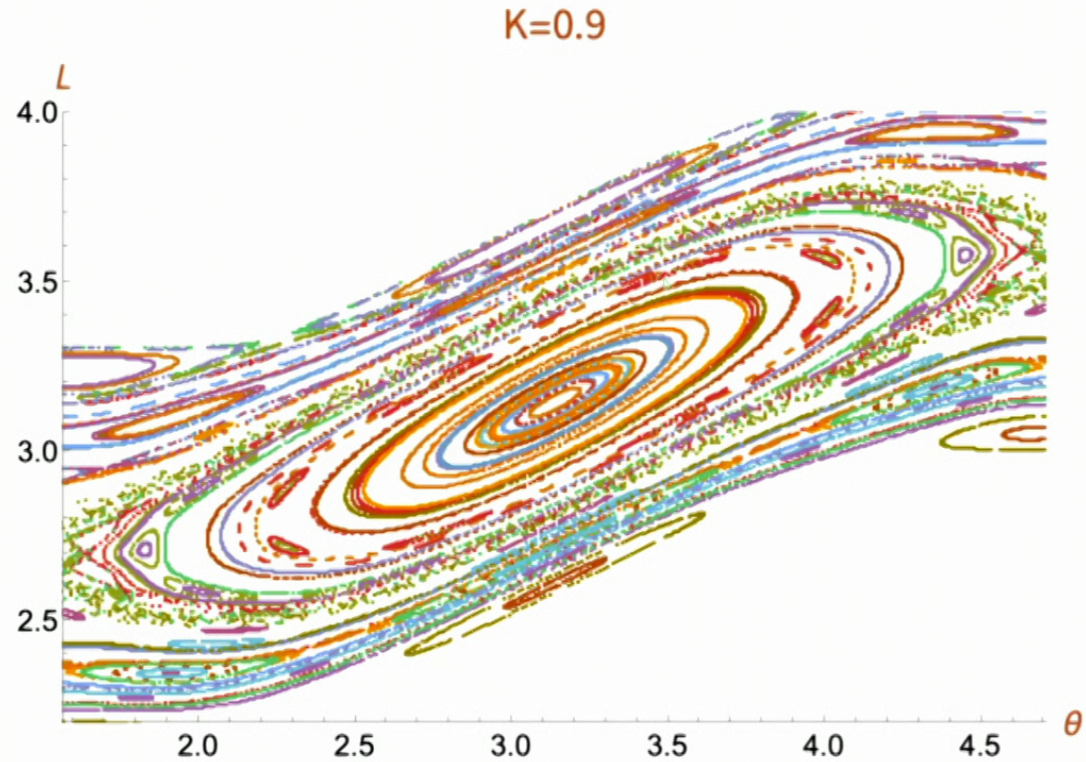
Standard Map

$$L_{n+1} = L_n + K \sin\theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$



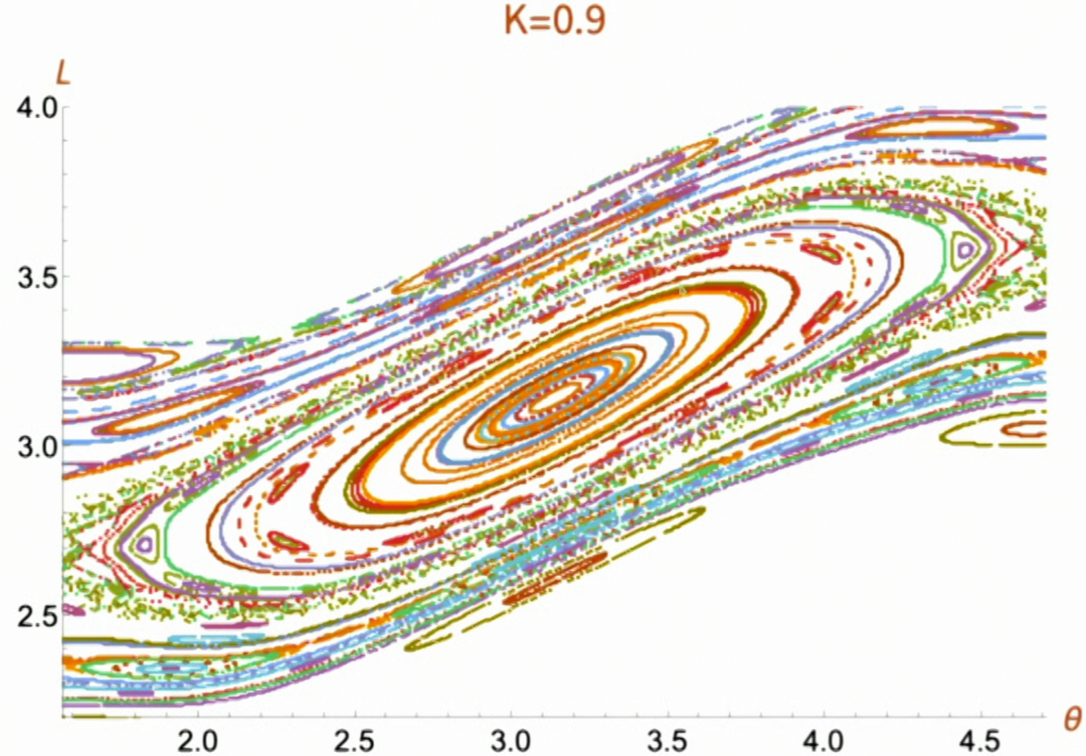
Standard Map

$$L_{n+1} = L_n + K \sin\theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$



Standard Map

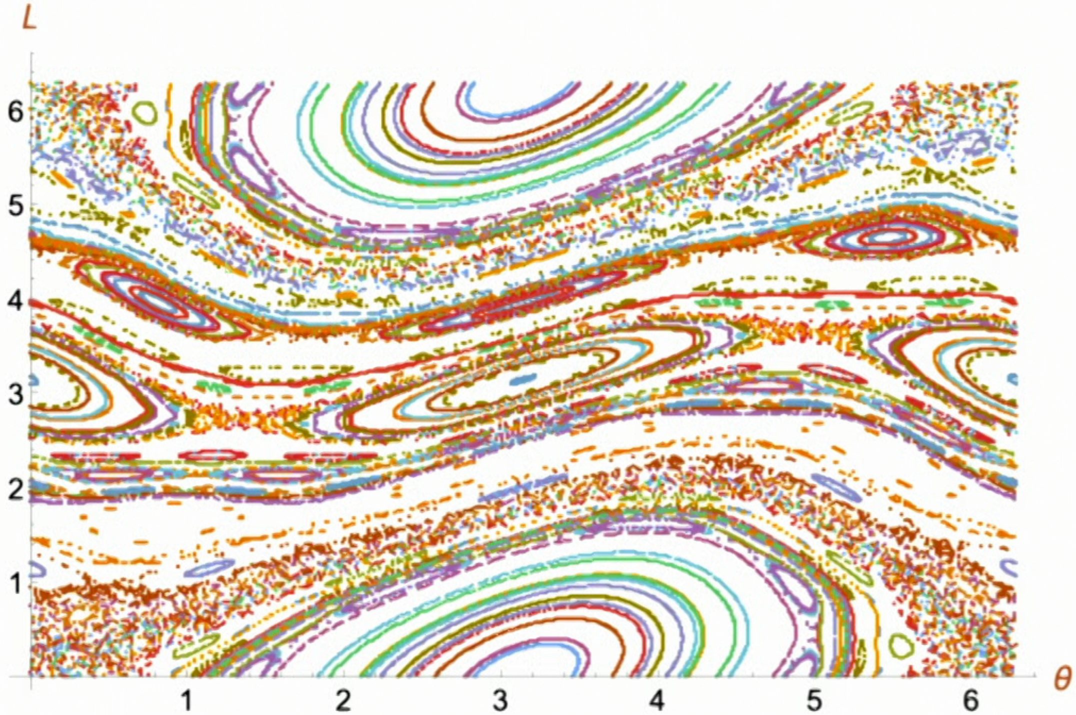
$$L_{n+1} = L_n + K \sin\theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$



Standard Map

$$L_{n+1} = L_n + K \sin \theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$

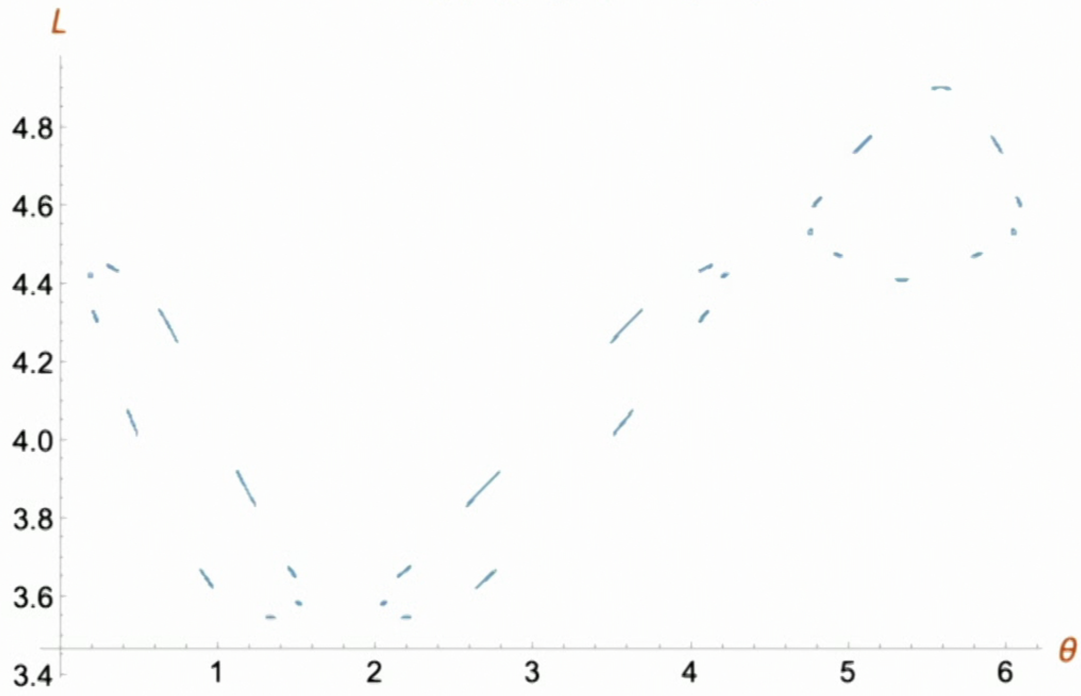
K=0.95



Standard Map

$$L_{n+1} = L_n + K \sin\theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$

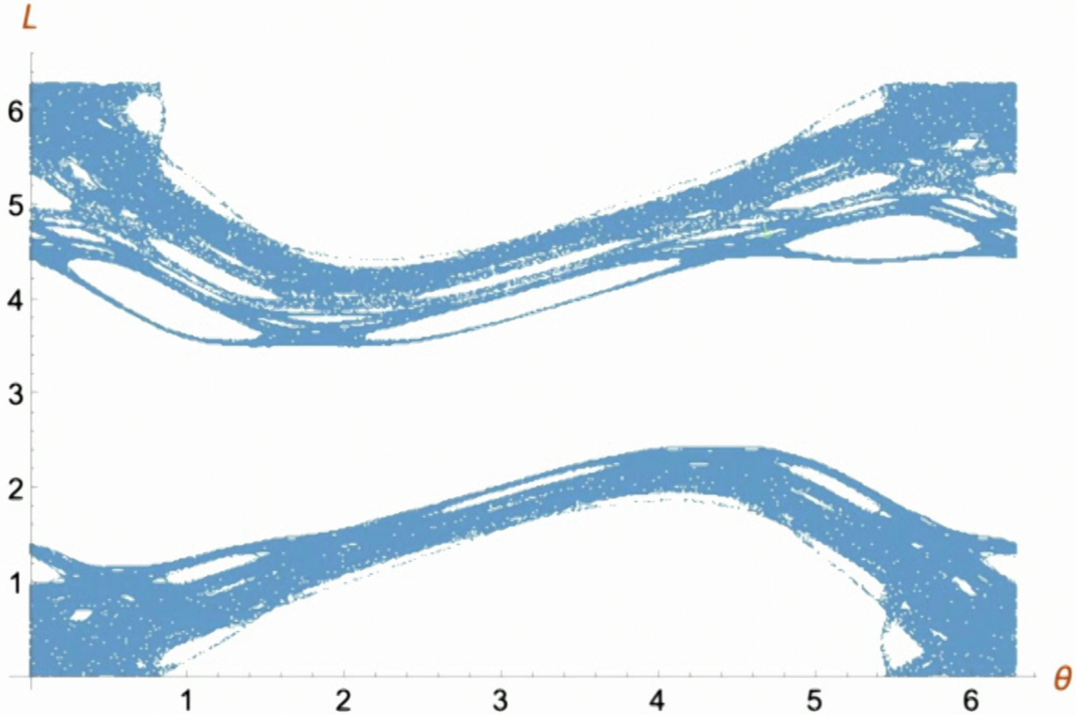
K=0.95, (L_0,θ_0)=(0.24,4.3)



Standard Map

$$L_{n+1} = L_n + K \sin\theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$

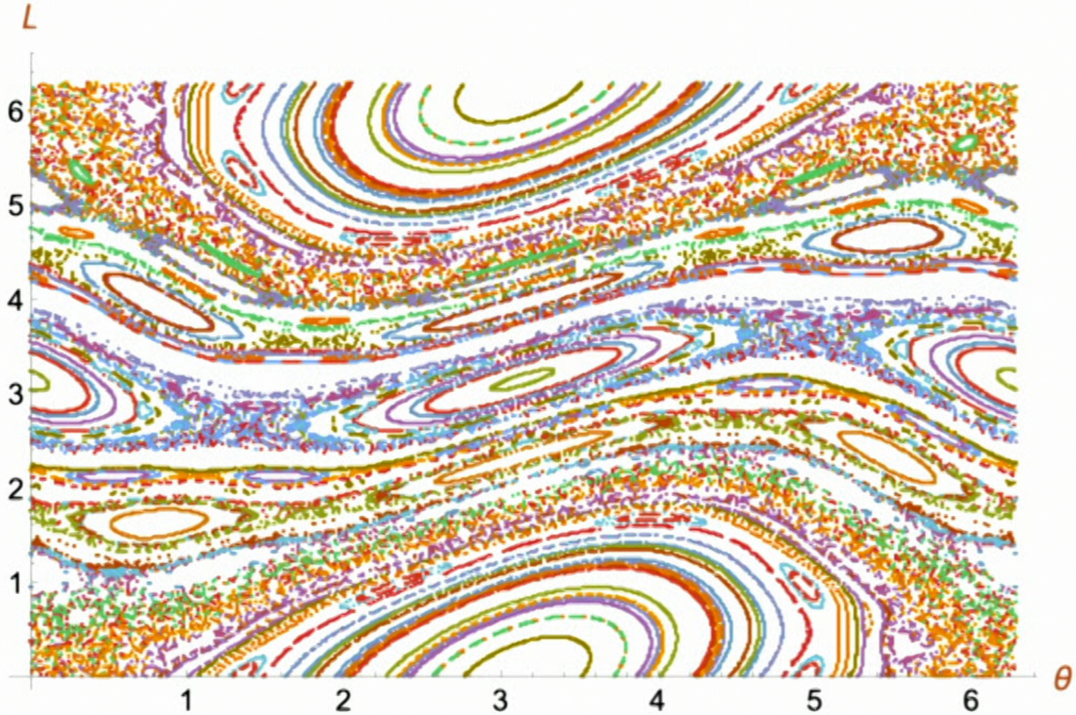
K=0.95, (L_0,θ_0)=(0.3,4.209)



Standard Map

$$L_{n+1} = L_n + K \sin \theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$

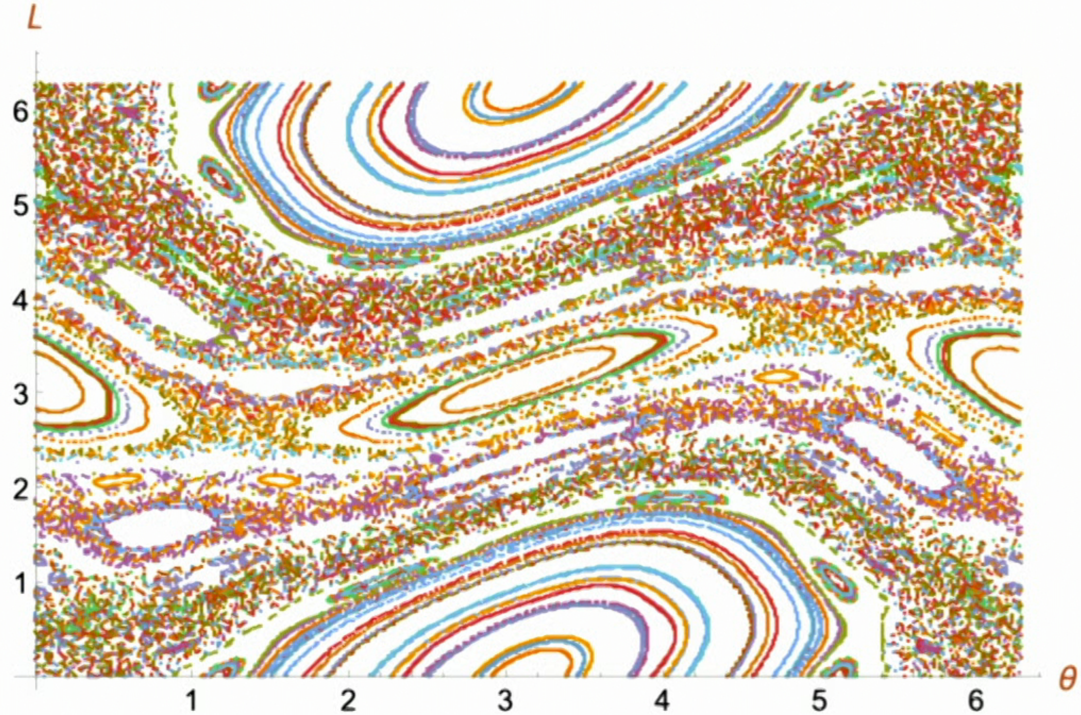
K=0.98



Standard Map

$$L_{n+1} = L_n + K \sin\theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$

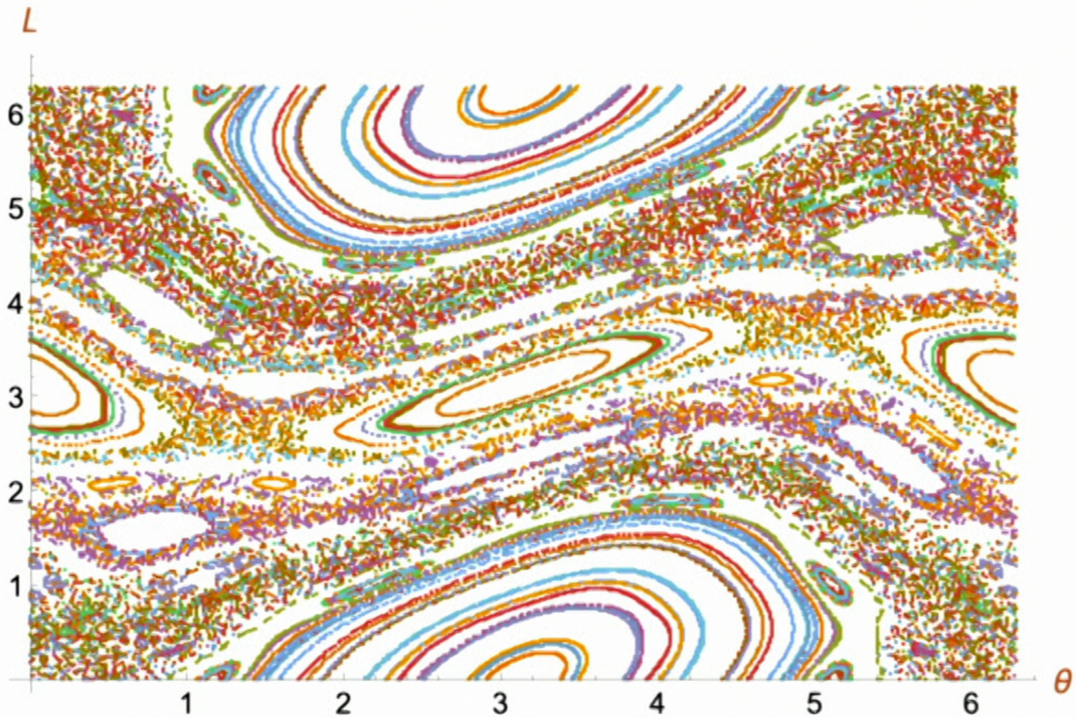
K=1.1



Standard Map

$$L_{n+1} = L_n + K \sin\theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$

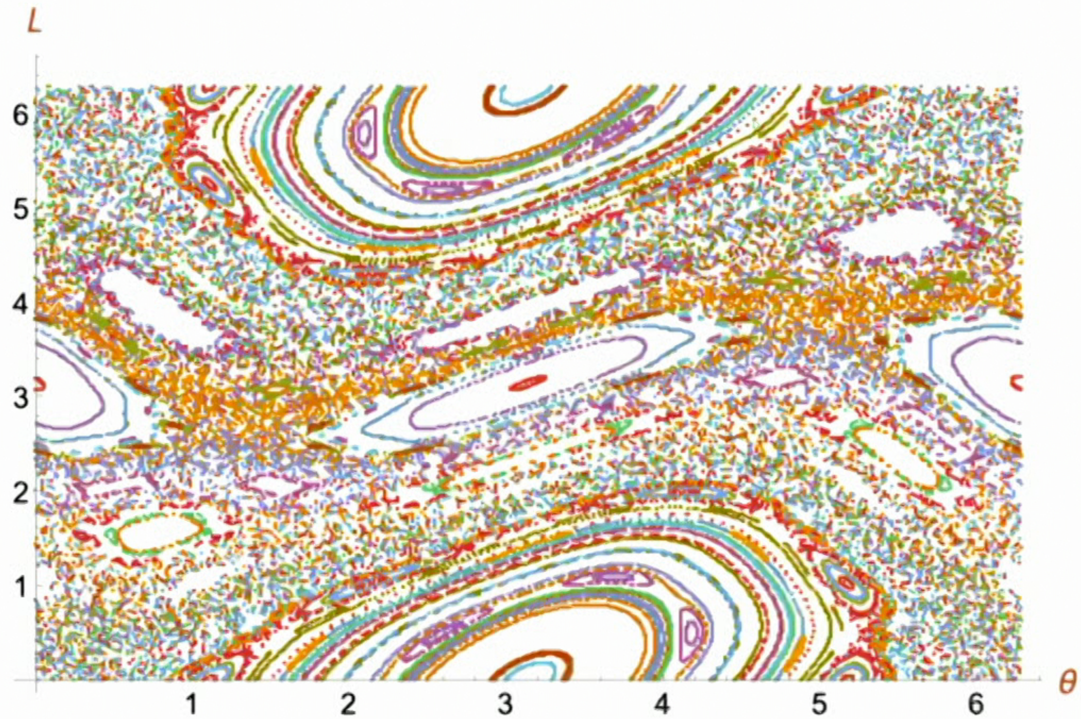
K=1.1



Standard Map

$$L_{n+1} = L_n + K \sin\theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$

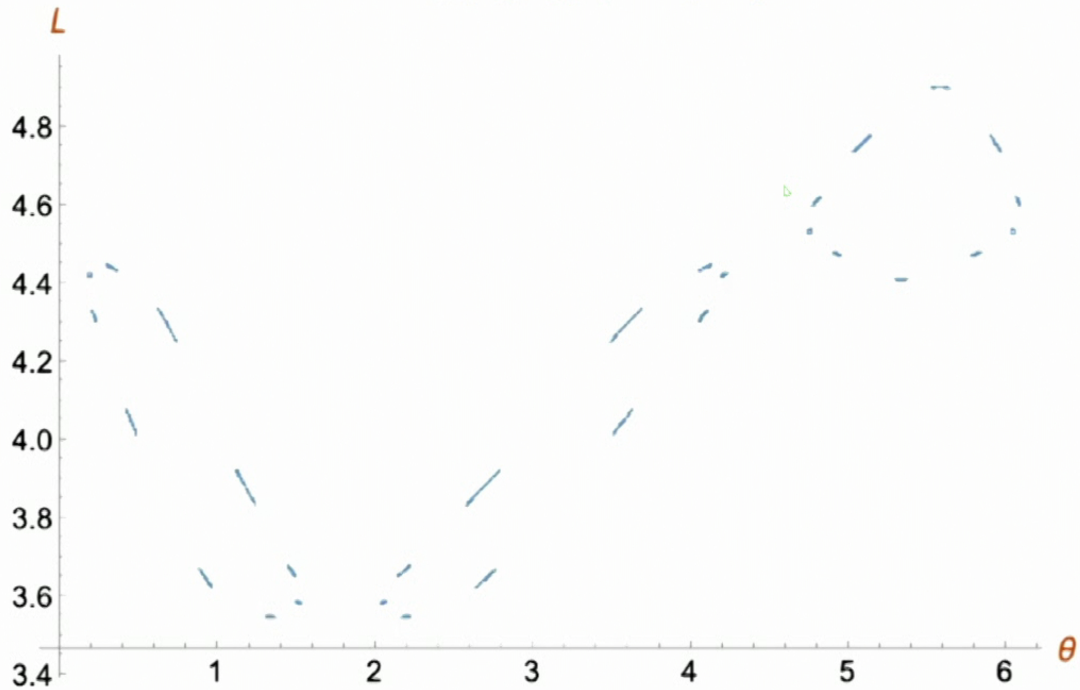
K=1.15



Standard Map

$$L_{n+1} = L_n + K \sin\theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$

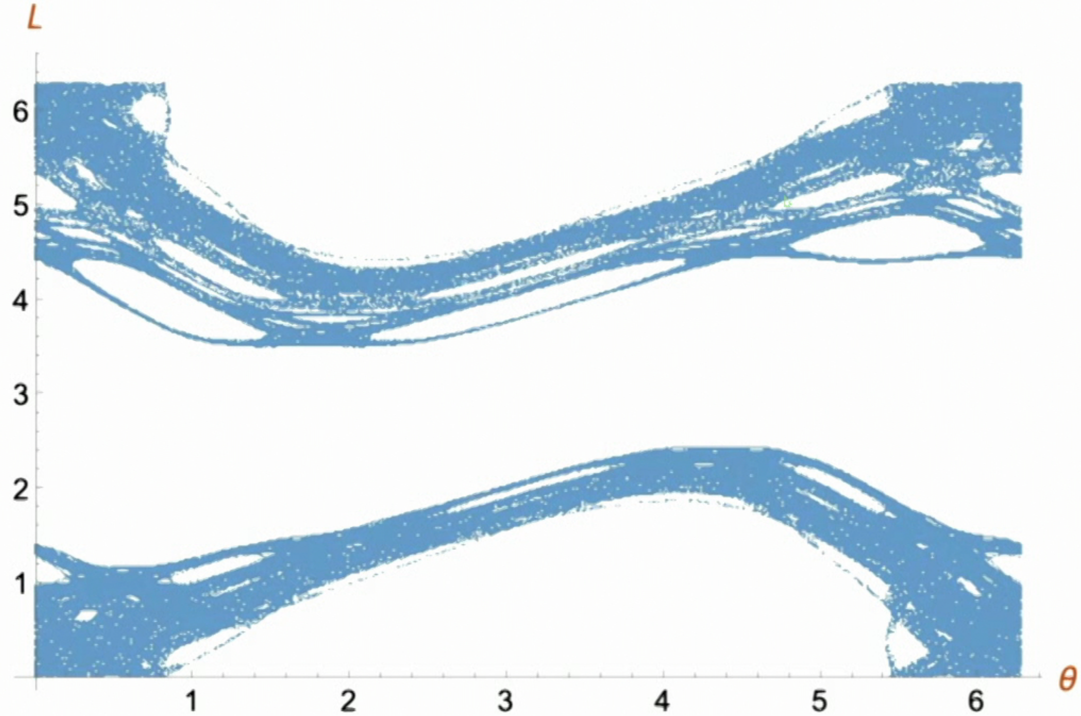
K=0.95, (L_0,θ_0)=(0.24,4.3)



Standard Map

$$L_{n+1} = L_n + K \sin \theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$

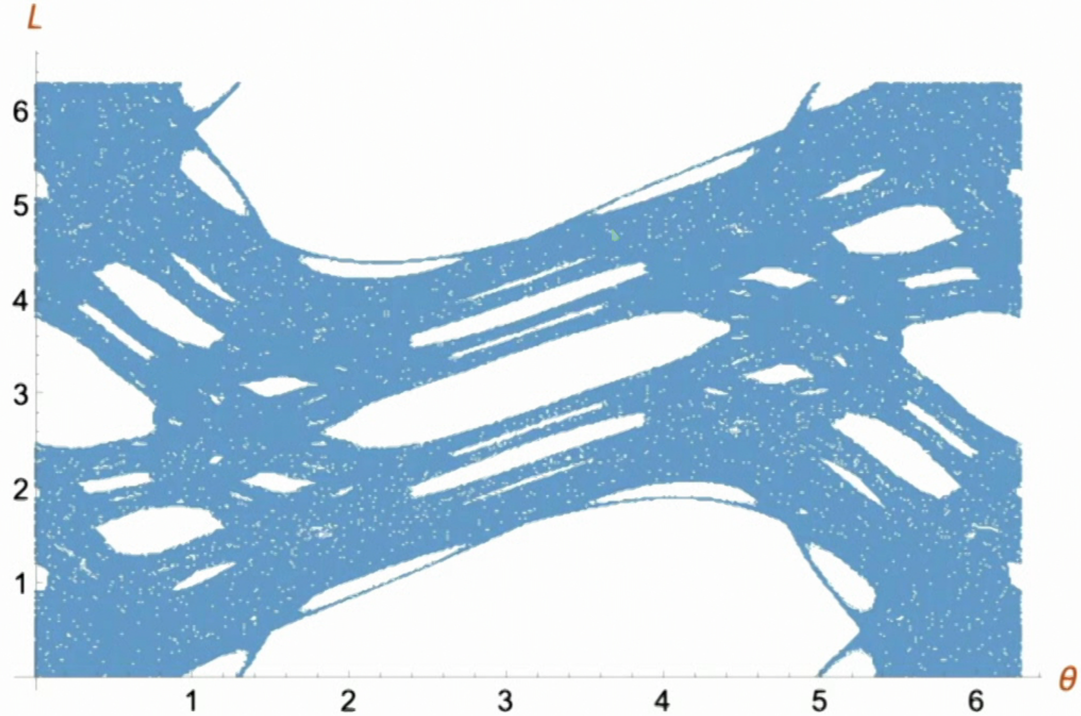
K=0.95, (L_0, θ_0)=(0.3,4.209)



Standard Map

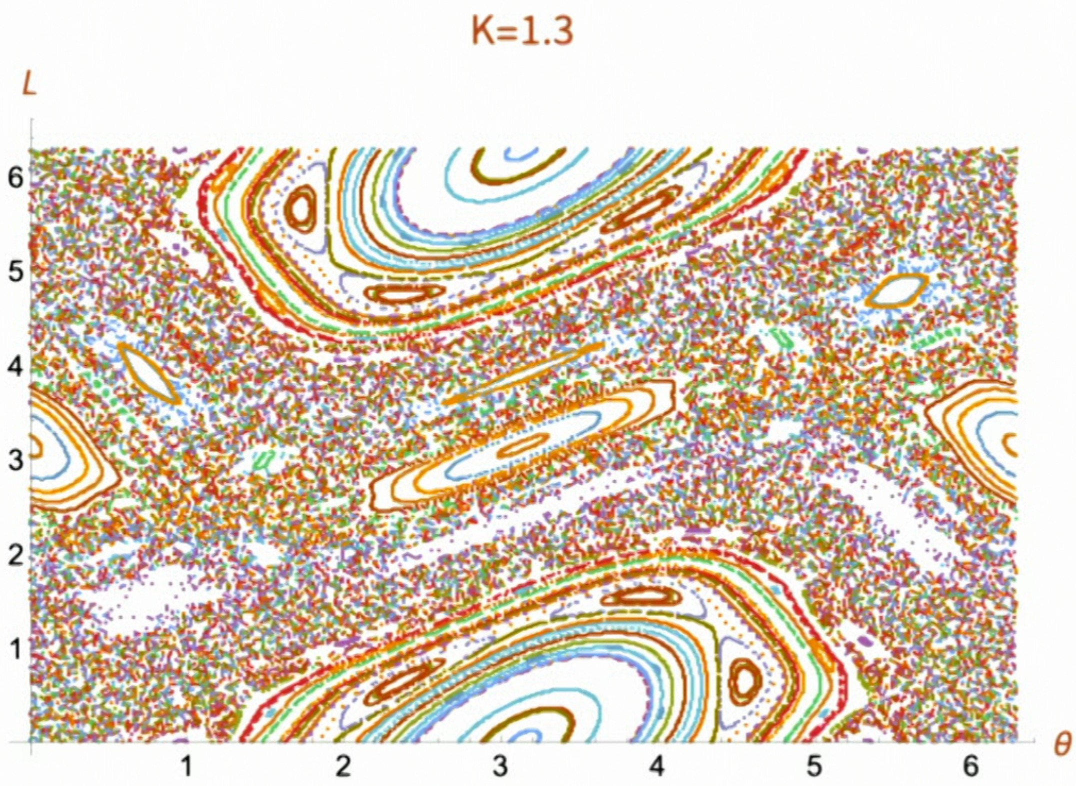
$$L_{n+1} = L_n + K \sin\theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$

K=1.15, (L_0,θ_0)=(0.24,4.3)



Standard Map

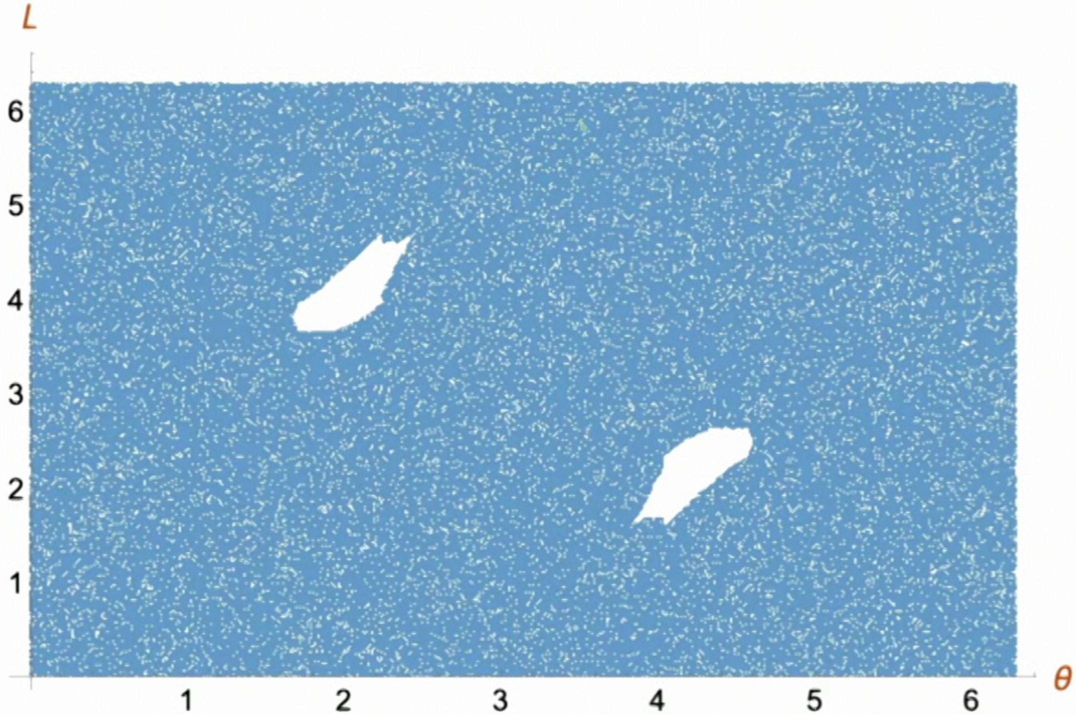
$$L_{n+1} = L_n + K \sin \theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$



Standard Map

$$L_{n+1} = L_n + K \sin\theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$

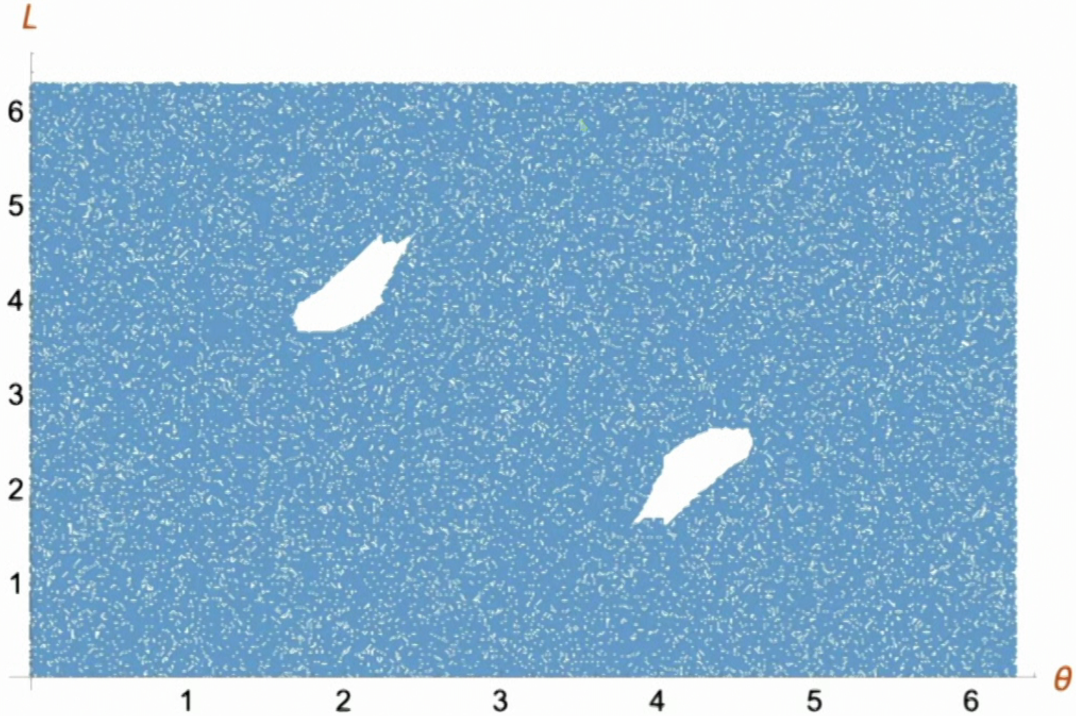
K=5, (L_0,θ_0)=(0.24,4.3)



Standard Map

$$L_{n+1} = L_n + K \sin\theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$

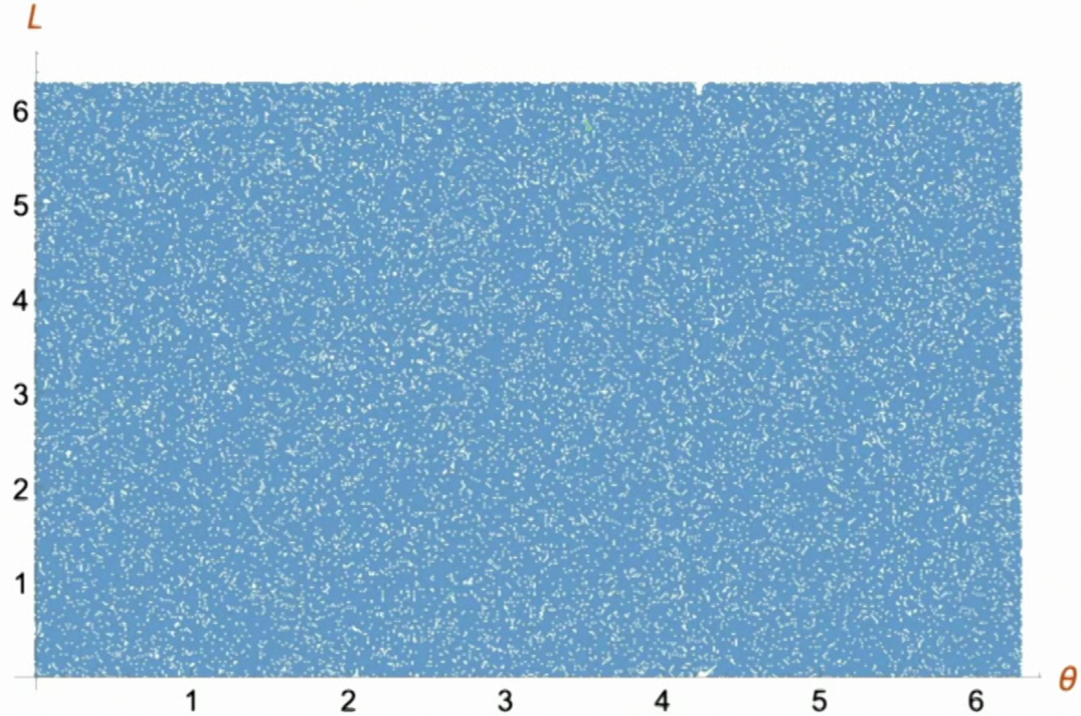
K=5, (L_0,θ_0)=(0.24,4.3)



Standard Map

$$L_{n+1} = L_n + K \sin\theta_n, \qquad \text{mod } 2\pi$$
$$\theta_{n+1} = \theta_n + L_{n+1}, \qquad \text{mod } 2\pi.$$

K=7, (L_0,θ_0)=(0.24,4.3)



Hénon-Heiles System

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y).$$

$$V(x, y) = \tfrac{1}{2}\left(x^2 + y^2 + 2x^2y - \tfrac{2}{3}y^3\right)$$

$V(x, y)$ is model potential for the motion of a star in a cylindrically symmetric and smooth galactic potential.

Hénon-Heiles System

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y).$$

$$V(x, y) = \tfrac{1}{2}\left(x^2 + y^2 + 2x^2 y - \tfrac{2}{3}y^3\right)$$

$V(x, y)$ is model potential for the motion of a star in a cylindrically symmetric and smooth galactic potential.

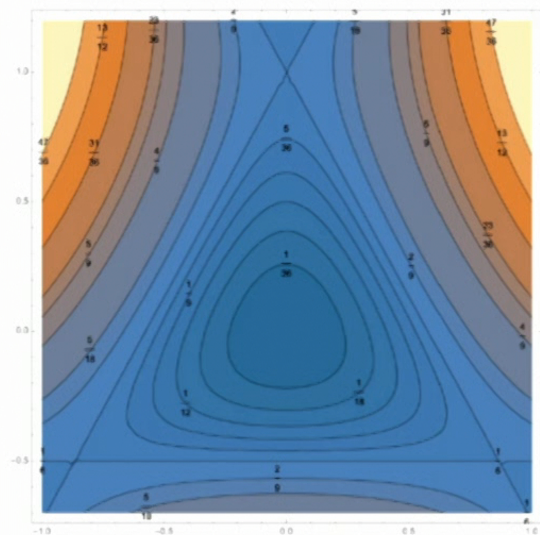


Figure: Equipotential lines for $V(x, y)$

Hénon-Heiles System

$$V(x,y) = \frac{1}{2}\left(x^2 + y^2 + 2x^2y - \frac{2}{3}y^3\right)$$

Poincare Map with surface of section:

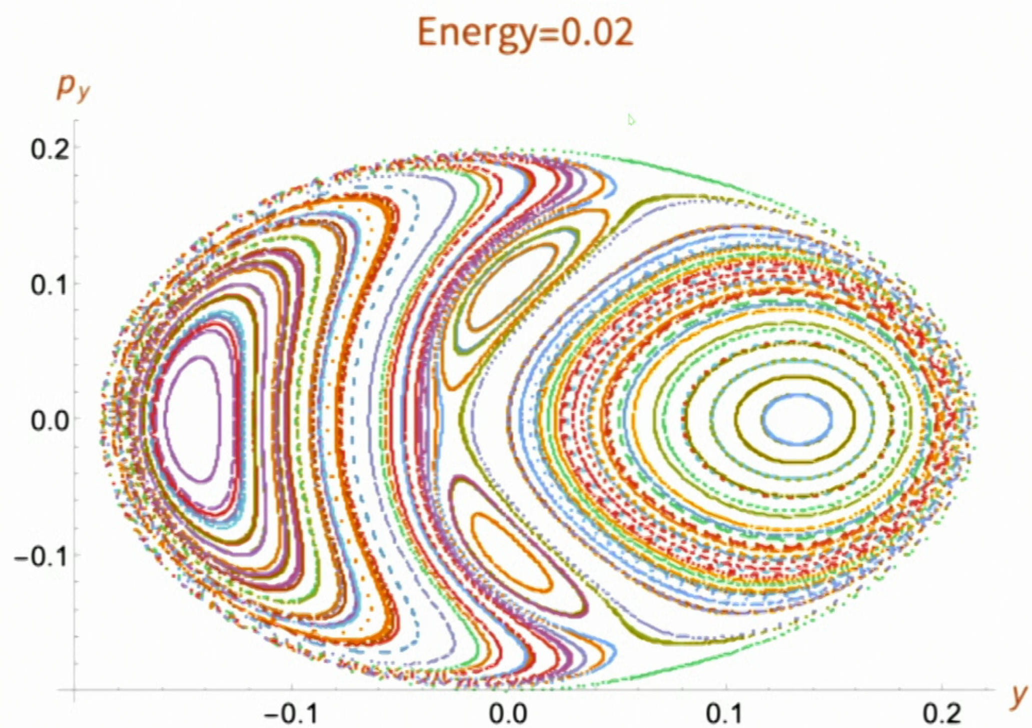
$$x = 0, \text{ and } \dot{x} \geq 0$$

Hénon-Heiles System

$$V(x, y) = \frac{1}{2}\left(x^2 + y^2 + 2x^2y - \frac{2}{3}y^3\right)$$

Poincare Map with surface of section:

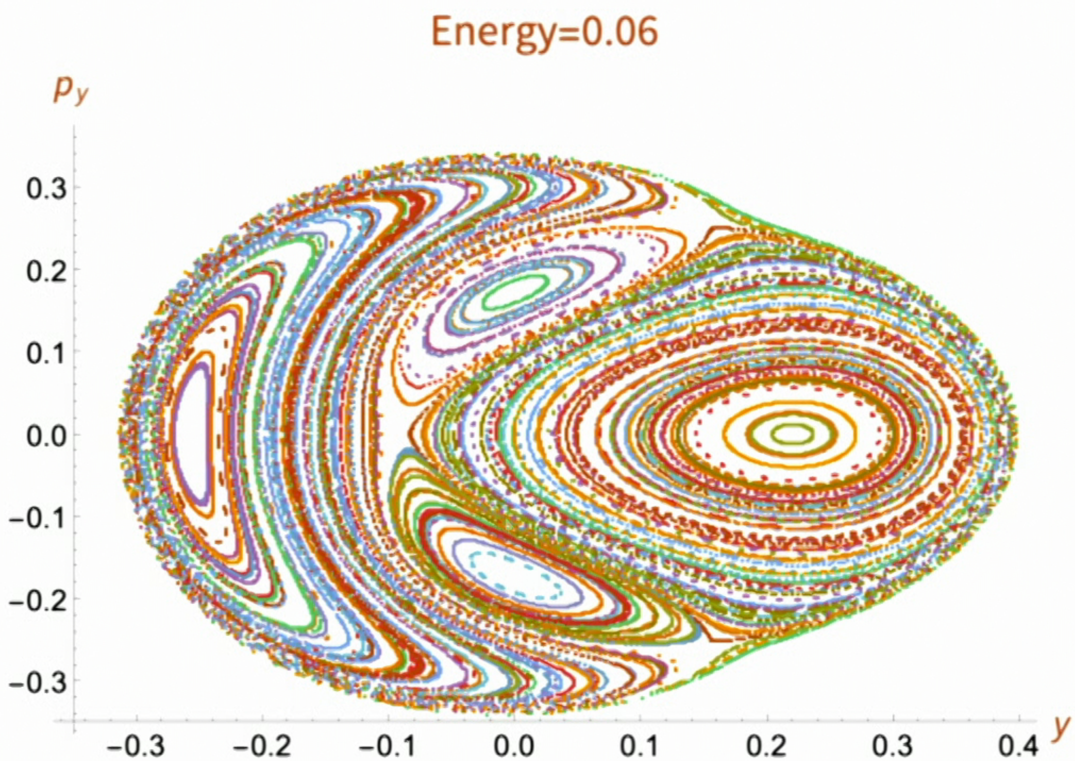
$$x = 0, \text{ and } \dot{x} \geq 0$$



Hénon-Heiles System

$$V(x, y) = \frac{1}{2}\left(x^2 + y^2 + 2x^2y - \frac{2}{3}y^3\right)$$

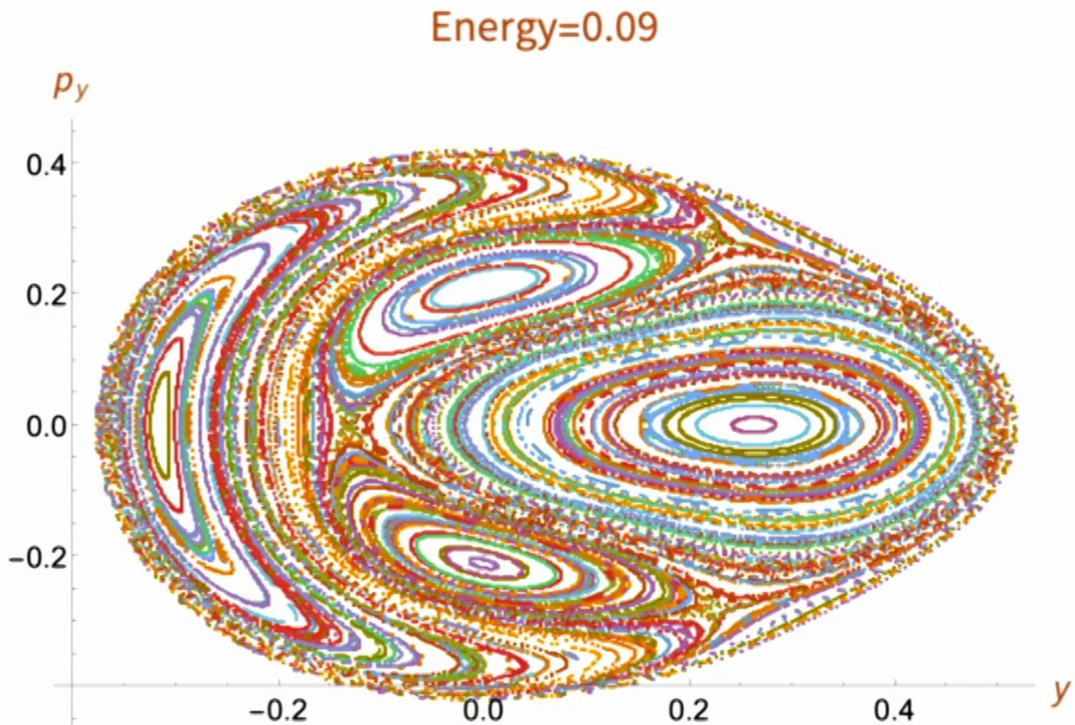
Poincare Map with surface of section: $x = 0$ and $\dot{x} \geq 0$



Hénon-Heiles System

$$V(x,y) = \frac{1}{2}\left(x^2 + y^2 + 2x^2y - \frac{2}{3}y^3\right)$$

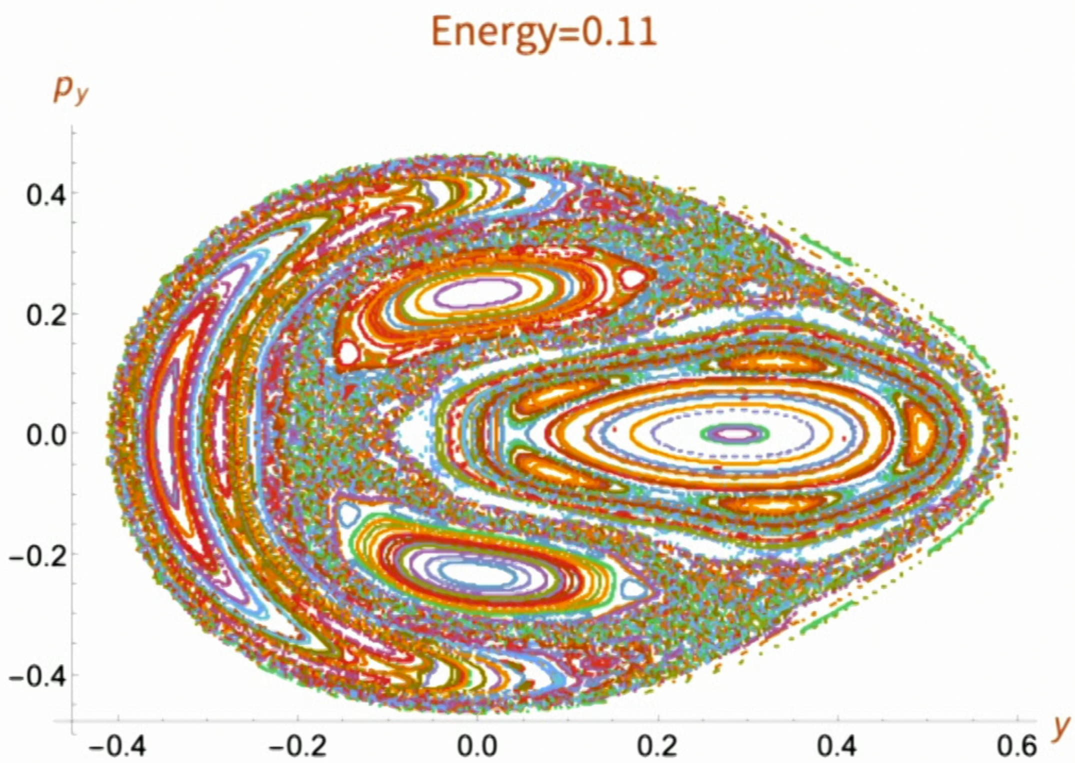
Poincare Map with surface of section: $x = 0$ and $\dot{x} \geq 0$



Hénon-Heiles System

$V(x,y) = \frac{1}{2}\left(x^2 + y^2 + 2x^2y - \frac{2}{3}y^3\right)$

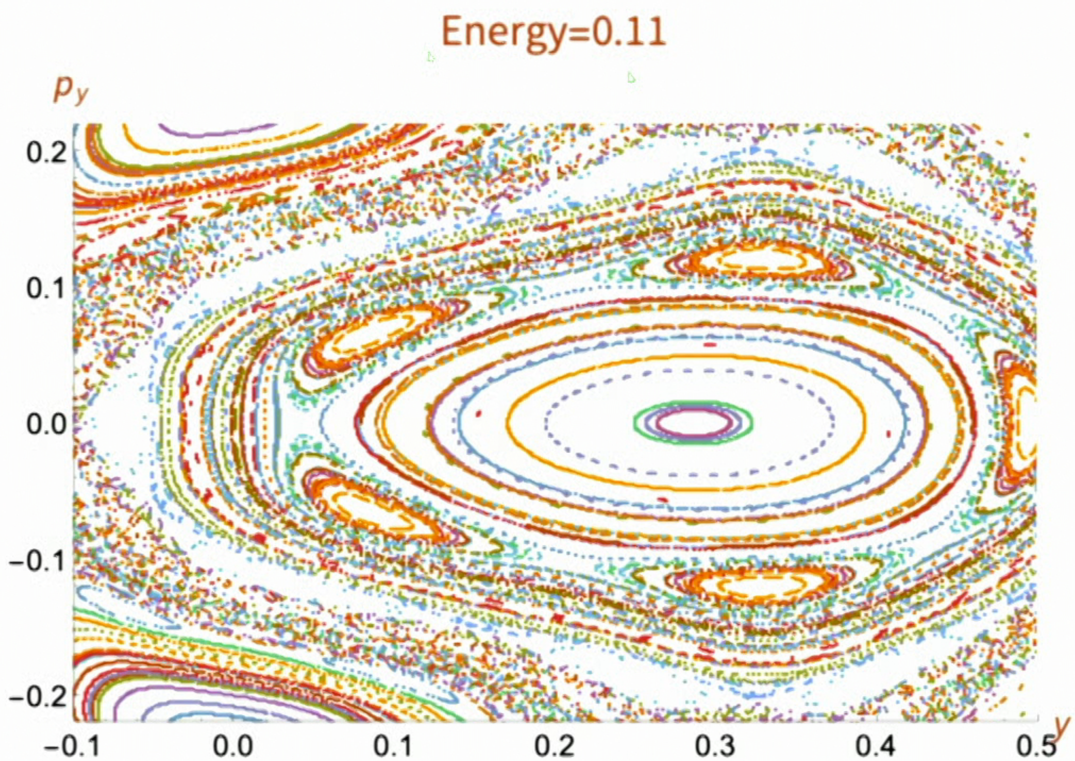
Poincare Map with surface of section: $x = 0$ and $\dot{x} \geq 0$



Hénon-Heiles System

$$V(x, y) = \frac{1}{2}\left(x^2 + y^2 + 2x^2 y - \frac{2}{3}y^3\right)$$

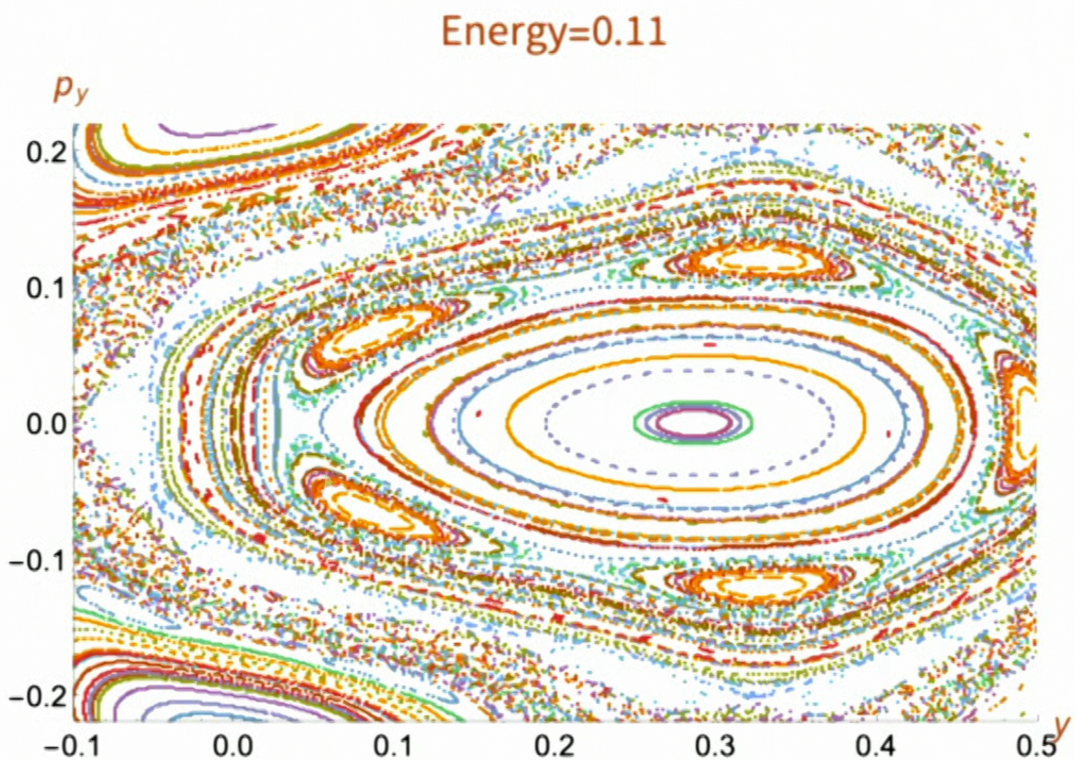
Poincare Map with surface of section: $x = 0$ and $\dot{x} \geq 0$



Hénon-Heiles System

$V(x,y) = \frac{1}{2}\left(x^2 + y^2 + 2x^2y - \frac{2}{3}y^3\right)$

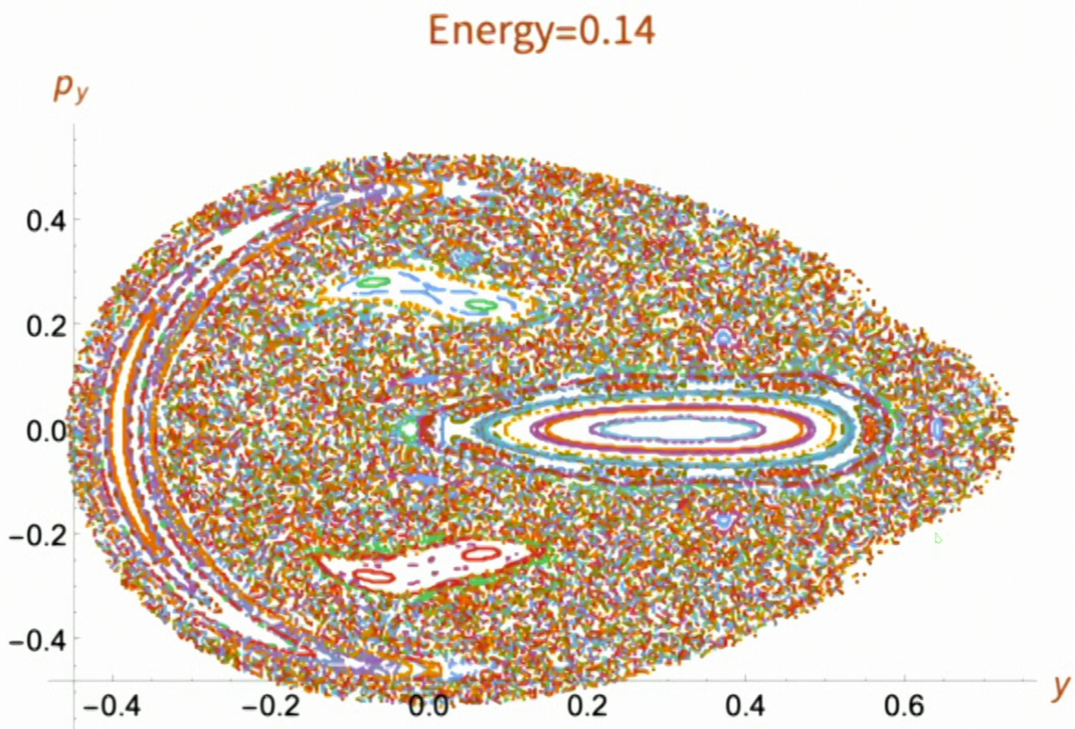
Poincare Map with surface of section: $x = 0$ and $\dot{x} \geq 0$



Hénon-Heiles System

$V(x, y) = \frac{1}{2}\left(x^2 + y^2 + 2x^2y - \frac{2}{3}y^3\right)$

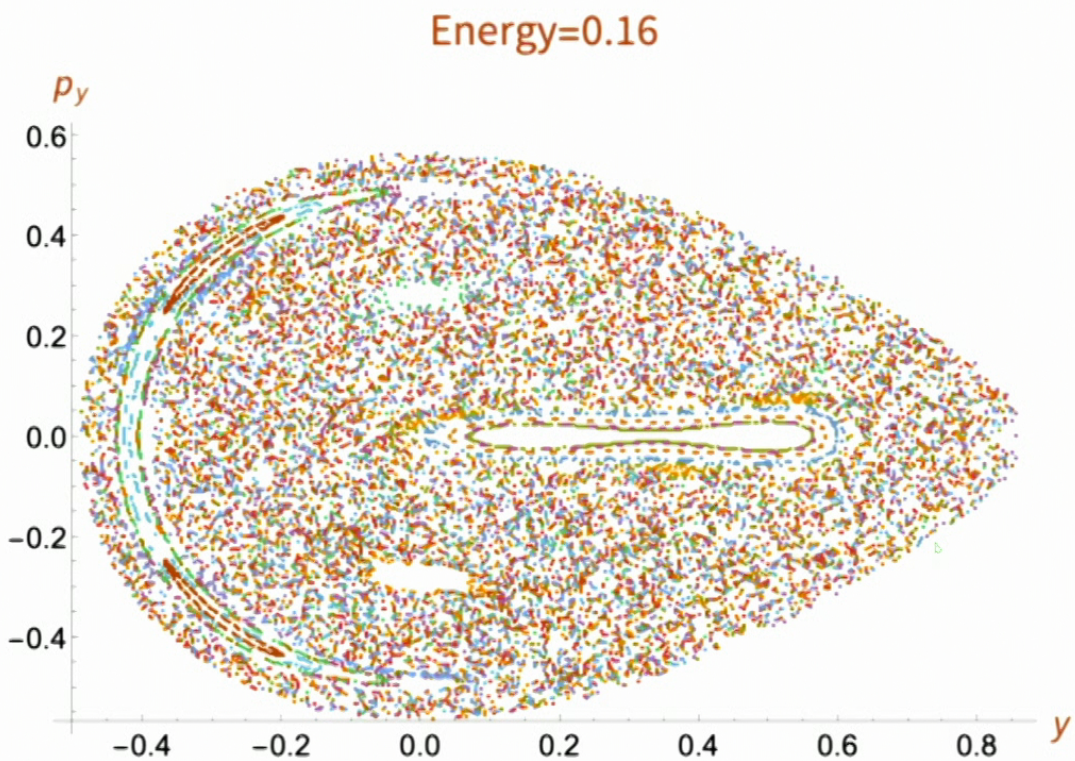
Poincare Map with surface of section: $x = 0$ and $\dot{x} \geq 0$



Hénon-Heiles System

$V(x, y) = \frac{1}{2}\left(x^2 + y^2 + 2x^2 y - \frac{2}{3}y^3\right)$

Poincare Map with surface of section: $x = 0$ and $\dot{x} \geq 0$



sumed valid when the number of oscillators is large) and is exemplified in the work of Fermi[5] and Peierls.[6]

A brief paper by Kolmogorov[7] enunciated a theorem which can perhaps provide a cornerstone for linking the two aforementioned divergent views on the effects of the weak perturbation V in Eq. (1). Kolmogorov did not present a detailed proof of his theorem; the missing proof, which is quite long and mathematically sophisticated, was supplied almost a decade later by Arnol'd[8] and independently by Moser.[9] As a consequence perhaps, the physical scientist has largely remained unaware of Kolmogorov's theorem and its implications. For details of the theory with applications, the reader is referred to the review article by Arnol'd.[10] However, in order to make this paper self-contained, we briefly present here those details of the theory relevant to this paper; in particular we may restrict our attention to systems with two degrees of freedom without significant loss of generality.

Introducing action-angle type variables (J_i, φ_i) for a two-oscillator system, Hamiltonian (1) may be written

$$H = H_0(J_1, J_2) + V(J_1, J_2, \varphi_1, \varphi_2) . \qquad (2)$$

If we set $V = 0$, then Hamiltonian (2) generates motion for which the J's are constant and $\varphi_i = \omega_i(J_1, J_2)t + \varphi_{io}$, where the unperturbed frequencies ω_i are given by $\omega_i = \partial H_0/\partial J_i$. Following Kolmogorov, we view the unperturbed system motion in phase space as lying on two-dimensional tori where (φ_1, φ_2) are the angle coordinates on the tori and (J_1, J_2) are the "radii" of the tori. By assuming that V is sufficiently small and by as-

perturbation V largely serves only to slightly shift the frequencies and introduce small nonlinear harmonics into the motion.[11] Nonetheless, the relatively small set of initial conditions leading to motion not on preserved tori is, from a physical point of view, pathologically interspersed between the preserved tori. Moreover, Arnol'd[10] conjectures that the system phase-space trajectory in regions of the destroyed tori is quite complicated indeed, perhaps ergodically filling the destroyed region. Thus, if Hamiltonian (2) is ever to provide generally ergodic motion best described in terms of statistical mechanics, the source of such behavior must lie in the reasons for the very existence of this relatively small set of destroyed tori. Hence we now investigate the properties of V which lead to the destruction of tori.

To this end, we expand the V of Hamiltonian (2) in a Fourier series and write

$$H = H_0(J_1, J_2) + f_{mn}(J_1, J_2)$$

$$\times \cos(m\varphi_1 + n\varphi_2) + \cdots , \qquad (3)$$

where we have explicitly written only one term in the series. The KAM formalism seeks to eliminate the angle-dependent terms using a convergent sequence of canonical transformations, each of which is close to the identity transformation, thus obtaining a Hamiltonian which is a function of the transformed action variables alone and which is close to the original Hamiltonian. If this can be accomplished in some general sense, then one immediately finds that the perturbed motion, for the show that they are sufficient to demonstrate a predictable instability. Section IV relates our results to the Henon-Heiles system, and Sec. V presents our conclusions.

II. ISOLATED RESONANCES

In this section, we illustrate the distortion of unperturbed tori caused by isolated angle-dependent resonant terms. The Hamiltonians we consider are of the form

$$H = H_0(J_1, J_2) + f_{mn}(J_1, J_2)\cos(m\varphi_1 + n\varphi_2), \tag{10}$$

where $\omega_i = \partial H_0/\partial J_i$ are both positive and where m and n are integers such that the inequality (7) can be satisfied. For brevity, an isolated perturbation of this type is called an m–n resonance, and the associated zone of highly distorted tori, loosely specified by inequality (7), is called an m–n resonance zone. Such perturbations are especially easy to analyze since they give rise to a

$$J_i = \tfrac{1}{2}(p_i^2 + q_i^2). \tag{15}$$

Thus the unperturbed level curves in the (q_2, p_2) plane, hereafter called the J_2 plane, are concentric circles centered on the origin since J_2 is a constant. Similarly points on the level curves in the (q_1, p_1) plane or J_1 plane, defined by $q_2 = 0$, $p_2 \geq 0$ (or equivalently $\varphi_2 = \frac{1}{2}3\pi$), also lie on concentric circles. These circular level curves in either plane are enclosed by a bounding level curve representing the intersection of the energy surface with each plane.

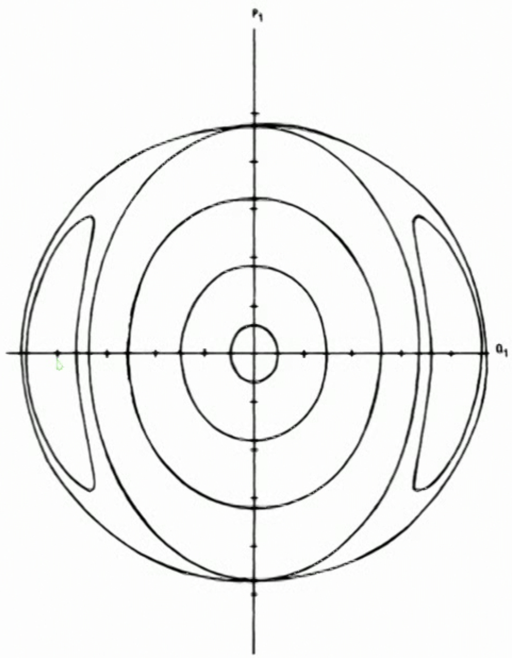
We now introduce a 2–2 resonance and write

$$H = H_0(J_1, J_2) + \alpha J_1 J_2 \cos(2\varphi_1 - 2\varphi_2). \tag{16}$$

Now this system has the additional constant of the motion

$$I = J_1 + J_2. \tag{17}$$

For all four of the above periodic orbits, we have $\dot{J}_1 = \dot{J}_2 = (\dot{\varphi}_1 - \dot{\varphi}_2) = 0$, where a dot denotes time differentiation. For the stable periodic orbits we find



$$J_2 = \tfrac{1}{13}\left[1 - (1 - \tfrac{13}{3}E)^{1/2}\right] , \qquad (22b)$$

as the values of J_1 and J_2 on the unperturbed 2-2 torus. Consequently, the unperturbed 2-2 torus and the perturbed 2-2 resonance zone exist for all allowed energies $0 \leq E \leq \frac{3}{13}$. As the energy increases from zero, the 2-2 resonance zone moves out from the origin and increases in width.

The closest (low-order) resonance to the 2-2 is the 3-2 or the 2-3. We investigate each. First consider

$$H = H_0(J_1, J_2) + \beta J_1^{3/2} J_2 \cos(3\varphi_1 - 2\varphi_2) . \qquad (23)$$

The additional constant of the motion is

$$I = 2J_1 + 3J_2 , \qquad (24)$$

and the level curves in the J_1 plane are given by

$$E = \tfrac{1}{3}I + \tfrac{1}{9}I^2 + (\tfrac{1}{3} - \tfrac{13}{9}I)J_1 + \tfrac{13}{9}J_1^2$$
$$- (\tfrac{1}{3}\beta)(IJ_1^{3/2} - 2J_1^{5/2})\cos 3\varphi_1 . \qquad (25)$$

Typical level curves for Eq. (25) are presented in Fig. 7. Here the points at the center of each of the three crescent regions do represent a single periodic solution, and thus the 3-2 resonance zone consists of a chain of three islands. Similarly, the three self-intersecting points on the separatrix represent a single unstable periodic solution.

Again setting $\dot{J}_1 = \dot{J}_2 = (3\dot{\varphi}_1 - 2\dot{\varphi}_2) = 0$ yields

$$J_2 = (1 + 2J_1^{3/2})/(13 + \tfrac{9}{12}J_1^{1/2}) , \qquad (26a)$$